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Sborník prací Přírodovědecké fakulty University Palackého v Olomouci. Matematika, Vol. 17 (1978), No. 1, 109--114

Persistent URL: `http://dml.cz/dmlcz/120060`

1978 — ACTA UNIVERSITATIS PALACKIANAE OLOMUCENSIS
FACULTAS RERUM NATURALIUM: TOM 57

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ON THE SPECTRAL PROPERTIES OF ELEMENTS OF THE TYPE exp (ih) IN HERMITIAN BANACH ALGEBRAS

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(*Received January 31, 1977*)

In this paper it will be shown that a Banach algebra with involution is hermitian if and only if the spectra of all elements of the form exp (ih), where h is selfadjoint, are uniformly bounded.

1. INTRODUCTION

The starting point of the theory of Banach algebras with involution was the 1943 paper of I. M. Gelfand and M. A. Naimark [3]. The recent contributions to this theory have brought substantial results. One of the directions of investigation in this field is to study the hermitian algebras. It was V. Pták [4], [5], who fully characterized hermitian Banach algebras with involution in a general case (without assuming the continuity of involution) using the spectral norm and spectral properties of elements. V. Pták lists in his paper [5] fourteen conditions equivalent to the fact of algebra being hermitian. All of them are based on the spectral properties of elements from algebra. In this paper we concentrate our attention on the condition requiring the uniform boundedness of spectra for unitary elements. It is known and easily seen that in the case of continuous involution, the elements exp (ih) (h selfadjoint), are unitary. In this case it is not difficult to show that uniform boundedness of their spectra is equivalent to the fact of algebra being hermitian. The aim of the present paper is to show this fact in a general case i.e. without assuming the continuity of the involution.

2. PRELIMINARIES

Let A be an algebra over the complex field. A is said to be a topological algebra *if* it is also a topological space. An involution on A is a map $x \to x^*$ of A onto itself such that for each $x, y \in A$ and for each complex λ:

1. $(x^*)^* = x$
2. $(x + y)^* = x^* + y^*$
3. $(\lambda x)^* = \bar{\lambda} x^*$
4. $(xy)^* = y^* x^*$

A $*$-algebra (or algebra with involution) is an algebra endowed by an involution. A $*$-algebra which is also normed (respectively Banach) is called a normed (respectively Banach) $*$-algebra. Let A be a $*$-algebra. An element $h \in A$ is said to be selfadjoint if $h = h^*$. An element $u \in A$ is said to be unitary if $u^*u = uu^* = e$ (e denotes the unit of A if it does exist). An element $a \in A$ is said to be normal if $a^*a = aa^*$. The sets of all unitary, selfadjoint and normal elements of A will be denoted respectively by $U(A)$, $H(A)$, $N(A)$. Obviously $U(A) \subset N(A)$ and $H(A) \subset N(A)$. For any set $S \subset A$ let $S^* = \{x^*, x \in S\}$. If $S^* = S$, we say S is selfadjoint. If the elements of $S^* \cap S$ are pairwise commutative, we say S is normal. The set of all regular elements of A is denoted by $R(A)$. The spectrum of an element $x \in A$ will be denoted by $\sigma(x)$. If it is necessary to specify the algebra with respect to which the spectrum is taken, we use the notation $\sigma(x, A)$. The spectral radius of the element $x \in A$ is denoted by $| x |_\sigma$ and we recall its definition: $| x |_\sigma = \sup \{| \lambda | : \lambda \in \sigma(x)\}$. In $*$algebra A we define the spectral norm as follows: for $x \in A$ $p(x) = \sqrt{| xx^* |_\sigma}$. In this paper we shall deal only with algebras endowed by the unit e. The involution on A is called hermitian if the spectrum $\sigma(x)$ is real for each $x \in H(A)$. The star algebra A is called hermitian if its involution is hermitian.

We suppose the reader to be familiar with elementary properties of all notions introduced. A familiarity with the Gelfand representation theory of commutative Banach algebra is presupposed. The set of all multiplicative functionals of A will be denoted by $\mathfrak{M}(A)$. Now we recall some known but necessary theorems and facts.

2.1. Theorem: The following conditions are equivalent:

(i) A is a hermitian commutative Banach algebra.
(ii) for each $f \in \mathfrak{M}(A)$ and for each $x \in A$ $f(x^*) = \overline{f(x)}$.

Proof: See for M. A. Naimark [2]. For the possibility of using the Gelfand representation theory in some reduced extent, of course, also in non-commutative case it is usefull the following:

2.2. Theorem: Let A be a Banach algebra with involution. Every normal set $N \subset A$ is contained in a closed maximal commutative $*$-subalgebra C of A and $\sigma(x, A) = \sigma(x, C)$ for each $x \in C$.

Proof: See for [5].

2.3. Definition: Let A be a Banach algebra. For given $a \in A$ exp (a) is defined by

$$\exp(a) = e + \sum_{n=1}^{\infty} \frac{1}{n!} a^n. \tag{1}$$

For correctness of this definition and other facts concerning $\exp(a)$ see Bonsall [1]. We recall now the most important properties of the function exp.

2.4. Theorem: Let A be a Banach algebra and let $a, b \in A$ such that $ab = ba$. Then the following is true:

(i) $\exp(a + b) = \exp(a) + \exp(b)$

(ii) $\exp(a) \in R(A)$ and $(\exp(a))^{-1} = \exp(-a)$

(iii) $\sigma(\exp(a)) = \exp(\sigma(a))$

(iv) $\exp(a) = \lim\limits_{n} \left(e + \dfrac{1}{n} a\right)^n$.

Proof: See for Bonsall [1].

2.5. Note: It's easily seen in the case of a continuous involution on Banach algebra A that for each $x \in A$ and each $h \in H(A)$ the following holds:

(i) $(\exp(x))^* = \exp(x^*)$

(ii) $(\exp(ih))^* = \exp(\overline{(ih)}) = \exp(-ih) = (\exp(ih))^{-1}$.

Let's recall now the V. Pták's spectral characterization of hermitian Banach algebras with involution.

2.6. Theorem: Let A be a Banach algebra with involution. The following conditions are equivalent:

1. The involution is hermitian
2. $| x |_\sigma^2 \leqq | x^*x |_\sigma$ for each $x \in A$
3. $| x |_\sigma \leqq p(x)$ for each $x \in A$
4. $| x |_\sigma \leqq p(x)$ for each $x \in N(A)$
5. $| x |_\sigma^2 = | x^*x |_\sigma$ for each $x \in N(A)$
6. $| \frac{1}{2}(x^* + x) |_\sigma \leqq | x^*x |_\sigma^{\frac{1}{2}}$ for each $x \in A$
7. p is subadditive function on A
8. $| u |_\sigma = 1$ for each $u \in U(A)$
9. $| u |_\sigma \leqq 1$ for each $u \in U(A)$
10. There exists $\beta > 0$ such that for each $u \in U(A)$ is $| u |_\sigma \leqq \beta$
11. x^*x is non-negative (i.e. its spectrum is non-negative) for each $x \in A$
12. Re $\sigma(x^*x)$ is nonnegative for each $x \in A$
13. $\sigma(x^*x)$ does not contain negative numbers for each $x \in A$
14. $(e + x^*x)$ is invertible for each $x \in A$

Proof: Is completely contained in [5].

3. SPECTRAL PROPERTIES OF $\exp(ih(A))$ IN HERMITIAN BANACH ALGEBRAS

We begin with the following:

3.1. Lemma: Let A be a Banach algebra with a hermitian involution. Let $h \in H(A)$. Then $\sigma((\exp(ih))^* - \exp(-ih)) = \{0\}$.

Proof: Let C be the maximal commutative closed $*$-subalgebra of A containing elements h and e. The set $\{h, e\}$ is obviously normal and the existence of C follows by 2.2. Theorem. Further it holds for each $x \in C$ $\sigma(x, C) = \sigma(x, A)$. Using the Gelfand representation theory for commutative Banach algebras we get:

$$\sigma((\exp(ih))^* - \exp(-ih)) = \{f((\exp(ih))^* - \exp(-ih)), f \in \mathfrak{M}(C)\} \tag{2}$$

It is obvious that C is hermitian and so $\sigma(h)$ consists only of real numbers. Let $f \in \mathfrak{M}(C)$. Then f is continuous and by 2.1. Theorem we get

$$f((\exp(ih))^*) = \overline{f(\exp(ih))} = \overline{\exp(if(h))} \tag{3}$$

further we get

$$\overline{\exp(if(h))} = \exp(-if(h)) = (\exp(-ih)) \tag{4}$$

By (2), (3), (4) we have

$$\sigma((\exp(ih))^* - \exp(-ih)) = \{0\}$$ Q.E.D

Now we are able to state the main result:

3.2. Theorem: Let A be the Banach algebra with involution. The following conditions are equivalent:

1. The involution is hermitian.
2. For each $x \in A$ regular and normal and satisfying $\sigma(x^* - x^{-1}) = \{0\}$ is $|\sigma(x)| \leqq 1$
3. There exists $K > 0$ such that for each $x \in A$ which is regular, normal and satisfies $\sigma(x^* - x^{-1}) = \{0\}$ is $|\sigma(x)| \leqq K$
4. There exists $K > 0$ such that for each unitary $u \in A$ it is $|\sigma(u)| \leqq K$.
5. For each unitary $u \in A$ it is $|\sigma(u)| \leqq 1$.
6. For each selfadjoint $h \in A$ it holds $|\sigma(\exp(ih))| \leqq 1$.
7. There exists $K > 0$ such that for each $h \in A$, h selfadjoint it holds $|\sigma(\exp(ih))| \leqq K$.

Proof: We prove the following implications:

1. $\Rightarrow$ 2. $\Rightarrow$ 3. $\Rightarrow$ 4. $\Rightarrow$ 5. $\Rightarrow$ 1. $\Rightarrow$ 6. $\Rightarrow$ 7. $\Rightarrow$ 1.

1. $\Rightarrow$ 2.

Let $x \in A$ be normal, regular and $\sigma(x^* - x^{-1}) = \{0\}$.

We take C the maximal commutative closed $*$-subalgebra of A (by 2.2. Theorem) containing the normal set $\{e, x^*, x\}$. Applying the Gelfand representation theory we get for each $f \in \mathfrak{M}(C)$ $f(x^* - x^{-1}) = 0$ and so $f(x(x^* - x^{-1})) = f(x) \cdot f(x^* - x^{-1}) = f(x^* - x^{-1}) \cdot f(x) = f((x^* - x^{-1})x) = 0$. It follows immediately that $\sigma(x^*x - e) = \sigma(xx^* - e) = \{0\}$ and so $\sigma(x^*x) = \sigma(xx^*) = \sigma(e)$. By 2. of 2.6. Theorem we get that $|\sigma(x)| \leqq 1$

Q.E.D.

Implication 2. $\Rightarrow$ 3. is trivial.

Implication 3. $\Rightarrow$ 4. follows from the fact that each unitary $u \in A$ is normal and regular and it holds $\sigma(u^* - u^{-1}) = \sigma(0) = \{0\}$ Q.E.D.

4. $\Rightarrow$ 5. $\Rightarrow$ 1. follows immediately from 2.6. Theorem.

1. $\Rightarrow$ 6.

Let again suppose the involution being hermitian and let $h \in A$ be selfadjoint. By the preceding lemma we get

$$\sigma((\exp(ih))^* - \exp(-ih)) = \{0\}$$

and so by the implication 1. $\Rightarrow$ 2. we obtain that

$$|\sigma(\exp(ih))| \leqq 1 \qquad \text{what is desired.}$$

Q.E.D.

6. $\Rightarrow$ 7. being trivial we prove the last implication:

7. $\Rightarrow$ 1.

Let's suppose there exists a $K > 0$ such that for each $h \in H(A)$ the inequality $|\sigma(\exp(ih))| \leqq K$ holds. Suppose that a complex number $\alpha + \beta i \in \sigma(h)$ and $\beta \neq 0$. By standard technique we construct an element $h' \in H(A)$ such that $i \in \sigma(h')$.

By (iii) of 2.4. Theorem we obtain for each $\tau > 0$:

$$\sigma(\exp(-i\tau h')) = \exp(-i\tau\sigma(h')) \ni \exp(-i\tau i) = \exp(\tau).$$

We obtained that for each $\tau > 0$ by 6. it would be $\exp(\tau) < K$. This contradiction proves the implication. QE.D.

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SOUHRN

O SPEKTRÁLNÍCH VLASTNOSTECH PRVKŮ exp (ih) V HERMITEOVSKÉ BANACHOVĚ ALGEBŘE

DINA ŠTĚRBOVÁ

V práci se dokazuje, že nutnou a postačující podmínkou k tomu, aby Banachova algebra s involucí byla hermiteovskou, je, aby spektra všech prvků tvaru exp (ih), kde h je samoadjungovaný, byla stejně omezena. Nepředpokládá se přitom spojitost involuce.

РЕЗЮМЕ

О СПЕКТРАЛЬНЫХ СВОЙСТВАХ ЭЛЕМЕНТОВ exp (ih) В ВПОЛНЕ СИММЕТРИЧЕСКИХ БАНАХОВЫХ КОЛЬЦАХ

ДИНА ШТЕРБОВА

В настоящей работе показывается, что нужным и необходимым условием для того, чтобы польная нормированная алгебра с инволюцией была впольне симметрической, является следующее условие: Спектры всех элементов алгебры имеющих форму exp (ih), $h = h^*$, ограниченны в совокупности. При этом непрерывность инволюции не предполягается.