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ON 3-DIMENSIAL CR-MANIFOLDS

ALOIS ŠVEC

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In the following paper, I present the final and quite explicit version of the solution of the equivalence problem for real hypersurfaces of the space of two complex variables with respect to the pseudogroup of biholomorphic mappings. The first (not very precise) solution was given by E. Cartan [1]; his method was improved in [2]. My approach was presented earlier in [3]–[5].

1. Let M be a 3-dimensional differentiable manifold. At each point $m \in M$, be given two different tangent straight lines $t_1(m)$, $t_2(m) \subset T_m(M)$ such that the distribution of planes $\tau(m) = \{t_1(m), t_2(m)\}$ is non-integrable. The structure of this sort be called an *RR-structure* on M. Let v_1, v_2 be vector fields on M such that $v_1(m) \in t_1(m)$, $v_2(m) \in t_2(m)$ for each $m \in M$ (or in a neighbourhood of a fixed point $m_0 \in M$). Define

$$v_3 = [v_1, v_2] \tag{1.1}$$

the vector fields v_1, v_2, v_3 are then independent.Thus we are in the position to write

$$[v_1, v_3] = a_1 v_1 + a_2 v_2 + a_3 v_3, \qquad [v_2, v_3] = b_1 v_1 + b_2 v_2 + b_3 v_3. \tag{1.2}$$

From the Jacobi identity

$$[v_1, [v_2, v_3]] + [v_2, [v_3, v_1]] + [v_3, [v_1, v_2]] = 0$$

we get

$$\begin{aligned} v_1 b_1 - v_2 a_1 + a_1 b_3 - a_3 b_1 &= 0 \\ v_1 b_2 - v_2 a_2 + a_2 b_3 - a_3 b_2 &= 0 \\ v_1 b_3 - v_2 a_3 + a_1 + b_2 &= 0 \end{aligned} \tag{1.4}$$

Let $\mathcal{S}$ be an RR-structure on M. The couple (v_1, v_2) of tangent vector fields on M is called *special* if $v_1 \in t_1$, $v_2 \in t_2$ and $[v_1, [v_1, v_2]]$, $[v_2, [v_1, v_2]] \in \tau$ for each $m \in M$.

Lemma 1. *Let $\mathcal{S}$ be an RR-structure on M, $m_0 \in M$ a fixed point. Then there is, at least in a neighbourhood of m_0, a special couple (v_1^*, v_2^*) of tangent vector fields associated to $\mathcal{S}$.*

Proof. Let (v_1, v_2) be any couple associated to $\mathscr{S}$, and let

$$v_1^* = \alpha v_1, \qquad v_2^* = \beta v_2. \tag{1.5}$$

Then

$$\begin{aligned} v_3^* &:= [v_1^*, v_2^*] = -\beta v_2\alpha \,.\, v_1 + \alpha v_1\beta \,.\, v_2 + \alpha\beta v_3 \\ [v_1^*, v_3^*] &= (.)\, v_1 + (.)\, v_2 + \alpha(\beta v_1\alpha + 2\alpha v_1\beta)\, v_3 \\ [v_2^*, v_3^*] &= (.)\, v_1 + (.)\, v_2 + \beta(2\beta v_2\alpha + \alpha v_2\beta)\, v_3. \end{aligned} \tag{1.6}$$

Chosing α, β solutions of

$$\beta v_1\alpha + 2\alpha v_1\beta = 0, \qquad 2\beta v_2\alpha + \alpha v_2\beta = 0 \tag{1.7}$$

(v_1^*, v_2^*) is special. QED.

Now, let (v_1, v_2) be a special couple associated to $\mathscr{S}$. Then we have, from (1.1), (1.2) and (1.4),

$$[v_1, v_2] = v_3, \qquad [v_1, v_3] = av_1 + bv_2, \qquad [v_2, v_3] = cv_1 - av_2; \tag{1.8}$$

$$v_1a + v_2b = 0, \qquad v_2a - v_1c = 0. \tag{1.9}$$

Let (1.5) be any other special couple; then α, β satisfy (1.7). Thus there are functions P_1, P_2 such that

$$v_1\alpha = 2\alpha P_1, \qquad v_2\alpha = -\alpha P_2; \qquad v_1\beta = -\beta P_1, \qquad v_2\beta = 2\beta P_2. \tag{1.10}$$

The integrability conditions of $(1.10_{1,2})$ and $(1.10_{3,4})$ are

$$v_3\alpha = -\alpha(2v_2P_1 + v_1P_2), \qquad v_3\beta = \beta(v_2P_1 + 2v_1P_2). \tag{1.11}$$

Set

$$v_2P_1 = Q_1, \qquad v_1P_2 = Q_2; \tag{1.12}$$

then

$$v_3\alpha = -\alpha(2Q_1 + Q_2), \qquad v_3\beta = \beta(Q_1 + 2Q_2). \tag{1.13}$$

The integrability conditions of $(1.10_1) + (1.13_1)$, $(1.10_2) + (1.13_1)$, $(1.10_3) + (1.13_2)$ and $(1.10_0) + (1.13_2)$ are

$$\begin{aligned} 2v_3P_1 + 2v_1Q_1 + v_1Q_2 &= -2P_1a + P_2b, \qquad v_3P_2 - 2v_2Q_1 - v_2Q_2 = 2P_1c + P_2a, \\ v_3P_1 + v_1Q_1 + 2v_1Q_2 &= -P_1a + 2P_2b, \qquad 2v_3P_2 - v_2Q_1 - 2v_2Q_2 = P_1c + 2P_2a. \end{aligned} \tag{1.14}$$

Set

$$v_3P_1 = R_1, \qquad v_3P_2 = R_2; \tag{1.15}$$

then

$$\begin{aligned} v_1Q_1 &= -R_1 - P_1a, \qquad v_2Q_1 = -P_1c; \\ v_1Q_2 &= P_2b, \qquad v_2Q_2 = R_2 - P_2a. \end{aligned} \tag{1.16}$$

The integrability conditions of $(1.12_1) + (1.15_1)$ and $(1.12_2) + (1.15_2)$ are

$$v_2R_1 - v_3Q_1 - cv_1P_1 = -aQ_1, \qquad v_1R_2 - v_3Q_2 - bv_2P_2 = aQ_2. \quad (1.17)$$

Set

$$v_1P_1 = S_1, \quad v_2P_2 = S_2, \quad v_3Q_1 = S_3, \quad v_3Q_2 = S_4; \quad (1.18)$$

then

$$v_2R_1 = S_1c + S_3 - Q_1a, \qquad v_1R_2 = S_2b + S_1 + Q_2a. \quad (1.19)$$

The integrability conditions of $(1.18_1) + (1.12_1)$, $(1.18_1) + (1.15_1)$, $(1.12_2) + (1.18_2)$, $(1.12_2) + (1.15_2)$, $(1.16_1) + (1.18_3)$, $(1.16_2) + (1.18_3)$, $(1.16_3) + (1.18_4)$ and $(1.16_4) + (1.18_4)$ are

$$\begin{gathered} v_2S_1 = -2R_1 - P_1a, \qquad v_3S_1 - v_1R_1 = -S_1a - Q_1b, \\ v_1S_2 = 2R_2 - P_2a, \qquad v_3S_2 - v_2R_2 = S_2a - Q_2c, \\ v_1S_3 + v_3R_1 = -2R_1a - P_1(v_3a + a^2 + bc), \qquad v_2S_3 = -2R_1c - P_1v_3c, \\ v_1S_4 = 2R_2b + P_2v_3b, \qquad v_2S_4 - v_3R_2 = -2R_2a - P_2(v_3a - a^2 - bc). \end{gathered} \quad (1.20)$$

Set

$$v_1R_1 = T_1, \quad v_3R_1 = T_2, \quad v_2R_2 = T_3, \quad v_3R_2 = T_4 \quad (1.21)$$

and, furthermore,

$$v_1S_1 = U_1, \qquad v_2S_2 = U_2. \quad (1.22)$$

Then, in summary,

$$\begin{aligned} &v_1\alpha = 2\alpha P_1, && v_1\beta = -\beta P_1, \\ &v_2\alpha = -\alpha P_2, && v_2\beta = 2\beta P_2, \\ &v_3\alpha = -\alpha(2Q_1 + Q_2), && v_3\beta = \beta(Q_1 + 2Q_2), \end{aligned}$$

$$\begin{aligned} &v_1P_1 = S_1, && v_1P_2 = Q_2, \\ &v_2P_1 = Q_1, && v_2P_2 = S_2, \\ &v_3P_1 = R_1, && v_3P_2 = R_2, \end{aligned} \quad (1.23)$$

$$\begin{aligned} &v_1Q_1 = -R_1 - P_1a, && v_1Q_2 = P_2b, && v_1R_1 = T_1, \\ &v_2Q_1 = -P_1c && v_2Q_2 = R_2 - P_2a, && v_2R_1 = S_1c + S_3 - Q_1a, \\ &v_3Q_1 = S_3, && v_3Q_2 = S_4, && v_3R_1 = T_2, \end{aligned}$$

$$\begin{aligned} &v_1R_2 = S_2b + S_4 + Q_2a, && v_1S_1 = U_1, && v_1S_2 = 2R_2 - P_2a, \\ &v_2R_2 = T_3, && v_2S_1 = -2R_1 - P_1a, && v_2S_2 = U_2, \\ &v_3R_2 = T_4, && v_3S_1 = T_1 - S_1a - Q_1b, && v_3S_2 = T_3 + S_2a - Q_2c, \end{aligned}$$

$$\begin{aligned} &v_1S_3 = -T_2 - 2R_1a - P_1(v_3a + a^2 + bc), && v_1S_4 = 2R_2b + P_2v_3b, \\ &v_2S_3 = -2R_1c - P_1v_3c, && v_2S_4 = T_4 - 2R_2a - P_2(v_3a - a^2 - bc). \end{aligned}$$

Let (v_1, v_2), (v_1^*, v_2^*) be special couples associated to $\mathscr{S}$, let them be related by (1.5). Then (1.6_1) turns out to be

$$v_3^* = \alpha\beta(P_2 v_1 - P_1 v_2 + v_3); \tag{1.24}$$

further,

$$\begin{aligned}[v_1^*, v_3^*] &= \alpha\beta(2Q_1 + 2Q_2 - 2P_1P_2 + a)\, v_1^* + \alpha^2(-S_1 - P_1^2 + b)\, v_2^*, \\ [v_2^*, v_3^*] &= \beta^2(S_2 + P_2^2 + c)\, v_1^* - \alpha\beta(2Q_1 + 2Q_2 - 2P_1P_2 + a)\, v_2^*.\end{aligned} \tag{1.25}$$

Writing for v_1^*, v_2^*, v_3^* equations similar to (1.8), we have

$$\begin{aligned}a^* = \alpha, \beta(2Q_1 + 2Q_2 - 2P_1P_2 + a), \qquad b^* = \alpha^2(-S_1 - P_1^2 + b), \\ c^* = \beta^2(S_2 + P_2^2 + c).\end{aligned} \tag{1.26}$$

Lemma 2. *Let $\mathscr{S}$ be an RR-structure on M; let (v_1, v_2), (v_1^*, v_2^*) be two special couples associated to it and related by* (1.5). *Define*

$$\begin{aligned}R &= v_1 v_1 a - 2[v_1, v_2]\, b - 3ab, & S &= v_2 v_2 a - 2[v_1, v_2]\, c + 3ac, \\ R^* &= v_1^* v_1^* a^* - 2[v_1^*, v_2^*]\, b^* - 3a^* b^*, & S^* &= v_2^* v_2^* a^* - 2[v_1^*, v_2^*]\, c^* + 3a^* c^*.\end{aligned} \tag{1.27}$$

Then

$$R^* = \alpha^3\beta R, \qquad S^* = \alpha\beta^3 S. \tag{1.28}$$

Proof. Using (1.26) and (1.23), we get

$$\begin{aligned}
v_1^* a^* &= \alpha^2\beta(-2P_2S_1 - 2R_1 + 2P_1Q_1 - 2P_1^2P_2 - P_1 a + 2P_2 b + v_1 a), \\
v_1^* v_1^* a^* &= \alpha^3\beta\{-2P_2U_1 - 2T_1 + (2Q_1 - 2Q_2 - 10P_1P_2 - a)\, S_1 - 8P_1R_1 + 6P_1^2Q_1 - \\
&\quad -2(P_1^2 - b)\, Q_2 - 6P_1^3P_2 + 6P_1P_2 b - 5P_1^2 a + 2P_1 v_1 a + 2P_2 v_1 b + v_1 v_1 a\}, \\
v_3^* b^* &= \alpha^3\beta\{-P_2U_1 - T_1 + (4Q_1 + 2Q_2 - 8P_1P_2 + a)\, S_1 - 4P_1R_1 + 3(2P_1^2 - b)\, Q_1 + \\
&\quad + 2(P_1^2 - b)\, Q_2 - 6P_1^3P_2 - P_1^2 a + 6P_1P_2 b - P_1 v_2 b + P_2 v_1 b + v_3 b\}, \\
v_2^* a^* &= \alpha\beta^2(-2P_1S_2 + 2R_2 + 2P_2Q_2 - 2P_1P_2^2 - 2P_1 c - P_2 a + v_2 a), \\
v_2^* v_2^* a^* &= \alpha\beta^3\{-2P_1U_2 + 2T_3 + (2Q_2 - 2Q_1 - 10P_1P_2 - a)\, S_2 + 8P_2R_2 - \\
&\quad - 2(P_2^2 + c)\, Q_1 + 6P_2^2Q_2 - 6P_1P_2^3 - 6P_1P_2 c - 5P_2^2 a - 2P_1 v_2 c + 2P_2 v_2 a + v_2 v_2 a\}, \\
v_3^* c^* &= \alpha\beta^3\{-P_1U_2 + T_3 + (2Q_1 + 4Q_2 - 8P_1P_2 + a)\, S_2 + 4P_2P_2 + \\
&\quad + 2(P_2^2 + c)\, Q_1 + 3(2P_2^2 - c)\, Q_2 - 6P_1P_2^3 - 6P_1P_2 c - P_2^2 a - P_1 v_2 c + P_2 v_1 c + v_3 c\}
\end{aligned} \tag{1.29}$$

and (1.28) follows from (1.26) and (1.29).

The geometrical meaning of the relative invariants R, S is given by the following

Theorem 1. *Let $\mathscr{S}$ be an RR-structure on M; let* sgn $RS = \varepsilon = \pm 1$ *at $m \in M$. For any special couple (v_1, v_2) around m associated to $\mathscr{S}$ there are functions A_i, B_i, C_i such that*

$$[v_1, [v_1, v_2]] = A_1 v_1 + A_2 v_2, \qquad [v_2, [v_1, v_2]] = A_3 v_1 - A_1 v_2,$$
$$[[v_1, v_2], [v_1, [v_1, v_2]]] = B_1 v_1 + B_2 v_2,$$
$$[[v_1, v_2], [v_2, [v_1, v_2]]] = B_3 v_1 + B_4 v_2, \tag{1.30}$$
$$[v_1, [v_1, [v_1, [v_1, v_2]]]] = C_1 v_1 + C_2 v_2 + C_3 [v_1, v_2],$$
$$[v_2, [v_2, [v_2, [v_1, v_2]]]] = C_4 v_1 + C_5 v_2 + C_6 [v_1, v_2].$$

Now, there is (at least in a neighbourhood of m) a special couple (v_1, v_2) such that

$$A_1(m) = B_2(m) = B_3(m) = 0, \qquad C_1(m) = 1, \qquad C_5(m) = -\varepsilon. \tag{1.31}$$

Let (v_1^*, v_2^*) be any other special couple around m satisfying the conditions analogous to (1.31); *then*

$$v_1^*(m) = \varepsilon_0 v_1(m), \qquad v_2^*(m) = \varepsilon_0 v_2(m); \qquad e_0 = \pm 1. \tag{1.32}$$

Proof. Let (v_1, v_2), (v_1^*, v_2^*) be special couples associated to $\mathscr{S}$ let us have (1.5). From (1.24) and (1.23), we get

$$v_3^*\alpha = \alpha^2\beta(-2Q_1 - Q_2 + 3P_1P_2), \qquad v_3^*\beta = \alpha\beta^2(Q_1 + 2Q_2 - 3P_1P_2),$$
$$v_3^*P_1 = \alpha\beta(P_2S_1 + R_1 - P_1Q_1), \qquad v_3^*P_2 = \alpha\beta(-P_1S_2 + R_2 + P_2Q_2),$$
$$v_3^*Q_1 = \alpha\beta(S_3 - P_2R_1 + P_1^2c - P_1P_2a), \qquad v_3^*Q_2 = \alpha\beta(S_0 - P_1R_2 + P_2^2b + P_1P_2a),$$
$$v_3^*R_1 = \alpha\beta(P_2T_1 + T_2 - P_1S_1c - P_1S_3 + P_1Q_1a), \tag{1.33}$$
$$v_3^*R_2 = \alpha\beta(-P_1T_3 + T_4 + P_2S_2b + P_2S_4 + P_2Q_2a),$$
$$v_3^*S_1 = \alpha\beta(P_2U_1 + T_1 - S_1a + 2P_1R_1 - Q_1b + P_1^2a),$$
$$v_3^*S_2 = \alpha\beta(-P_1U_2 + T_3 + S_2a + 2P_2R_2 - Q_2c - P_2^2a).$$

It is just a matter of patience to compute

$$[v_1^*, [v_1^*, v_3^*]] = \alpha^2\beta(-2P_2S_1 - 2R_1 + 2P_1Q_1 - 2P_1^2P_2 - P_1a + 2P_2b + v_1a)\, v_1^* +$$
$$+ \alpha^3(-U_1 - 6P_1S_1 - 4P_1^3 + 4P_1b + v_1b)\, v_2^* + \alpha^2(-S_1 - P_1^2 + b)\, v_3^*,$$
$$[v_1^*, [v_2^*, v_3^*]] = [v_2^*, [v_1^*, v_3^*]] =$$
$$= \alpha\beta^2(-2P_1S_2 + 2R_2 + 2P_2Q_2 - 2P_1P_2^2 - 2P_1c - P_2a + v_2a)\, v_1^* +$$
$$+ \alpha^2\beta(2P_2S_1 + 2R_1 - 2P_1Q_1 + 2P_1^2P_2 + P_1a - 2P_2b + v_2b)\, v_2^* - \tag{1.34}$$
$$- \alpha\beta(2Q_1 + 2Q_2 - 2P_1P_2 + a)\, v_3^*,$$
$$[v_2^*, [v_2^*, v_3^*]] = \beta^3(U_2 + 6P_2S_2 + 4P_2^3 + 4P_2c + v_2c)\, v_1^* +$$
$$+ \alpha\beta^2(2P_1S_2 - 2R_2 - 2P_2Q_2 + 2P_1P_2^2 + 2P_1c + P_2a - v_2a)\, v_2^* -$$
$$- \beta^2(S_2 + P_2^2 + c)\, v_3^*;$$

$$[v_3^*, [v_1^*, v_3^*]] = \alpha^2\beta^2\{S_1S_2 - (P_2^2 - c)\, S_1 + (3P_1^2 - b)\, S_2 + \tag{1.35}$$
$$+ 2S_3 + 2S_4 - 4P_2R_1 - 4P_1R_2 - 6Q_1^2 - 8Q_1Q_2 - 2Q_2^2 +$$
$$+ (12P_1P_2 - 5a)\, Q_1 + (4P_1P_2 - 3a)\, Q_2 - 3P_1^2P_2^2 + 4P_1P_2a + 3P_1^2c +$$
$$+ P_2^2b - P_1v_2a + P_2v_1a + v_3a - a^2 - bc\}\, v_1^* + \alpha^3\beta\{-P_2U_1 - T_1 +$$

$$+ (4Q_1 + 2Q_2 - 8P_1P_2 + a) S_1 - 4P_1R_1 + 3(2P_1^2 - b) Q_1 +$$
$$+ 2(P_1^2 - b) Q_2 - 6P_1^3P_2 + 6P_1P_2b - P_1^2a - P_1v_2b + P_2v_1b + v_3b\} v_2^*,$$
$$[v_3^*, [v_2^*, v_3^*] = \alpha\beta^3\{-P_1U_2 + T_2 + (2Q_1 + 4Q_2 - 8P_1P_2 + a) S_2 +$$
$$+ 4P_2P_2 + 2(P_2^2 + c) Q_1 + 3(2P_2^2 + c) Q_2 - 6P_1P_2^3 - 6P_1P_2c -$$
$$- P_2^2a - P_1v_2c + P_2v_1c + v_3c\} v_1^* + \alpha^2\beta^2\{S_1S_2 + (3P_2^2 + c) S_1 -$$
$$- (P_1^2 + b) S_2 - 2S_3 - 2S_4 + 4P_2R_1 + 4P_1R_2 - 2Q_1^2 - -8Q_1Q_2 -$$
$$- 6Q_2^2 + (4P_1P_2 - 3a) Q_1 + (12P_1P_2 - 5a) Q_2 - 3P_2^2P_2^2 +$$
$$+ 4P_1P_2a - P_1^2a - 3P_2^2b + P_1v_2a - P_2v_1a - v_3a - a^2 - bc\} v_2^*,$$

$$[v_1^*, [v_1^*, [v_1^*, v_3^*]]] = \alpha^3\beta\{-2P_2U_1 - 2T_1 - 2(4P_1P_2 + 2_2 + Qa) S_1 -$$
$$- 8P_1R_1 + 2(2P_1^2 + b) Q_1 - 4(P_1^2 - b) Q_2 - 4P_1^3P_2 - 6P_1^2a + 4P_1P_2b +$$
$$+ 2P_1v_1a + 2P_2v_1b + v_1v_1a + ab\} v_1^* + \alpha^4\{v_1U_1 - 12P_1U_1 - 5S_1^2 -$$
$$- 2(23P_1^2 - b) S_1 - 23P_1^4 + 22P_1^2b + 10P_1v_1b + v_1v_1b + b^2\} v_2^* -$$
$$- 2\alpha^3(U_1 + 6P_1S_1 + 4P_1^3 - 4P_1b - v_1b) v_3^*,$$

$$[v_2^*, [v_2^*, [v_2^*, v_3^*]]] = \beta^4\{v_2U_2 + 12P_2U_2 + 5S_2^2 + 2(23P_2^2 + c) S_2 +$$
$$+ 23P_2^3 + 22P_2^2c + 10P_2v_2c + v_2v_2c - c^2\} v_1^* +$$
$$+ \alpha\beta^3\{2P_1U_2 - 2T_3 + 2(4P_1P_2 + 2Q_1 + a) S_2 - 8P_2R_2 + 4(P_2^2 + c) Q_1 -$$
$$- 2(2P_2^2 - c) Q_2 + 4P_1P_2^3 + 6P_2^2a + 4P_1P_2c + 2P_1v_2c - 2P_2v_2a -$$
$$- v_2v_2a + ac\} v_2^* - 2\beta^3(U_2 + 6P_2S_2 + 4P_2^3 + 4P_2c + v_2c) v_3^*.$$

Choosing at the point $m \in M$

$$2(Q_1 + Q_2) = 2P_1P_2 - a, \tag{1.36}$$
$$P_2U_1 + T_1 = (4Q_1 + 2Q_2 - 8P_1P_2 + a) S_1 - 4P_1R_1 + 3(2P_1^2 - b) Q_1 +$$
$$+ 2(P_1^2 - b) Q_2 - 6P_1^3P_2 + 6P_1P_2b - P_1^2a - P_1v_2b + P_2v_1b + v_3b,$$
$$P_1U_2 - T_2 = (2Q_1 + 4Q_2 - 8P_1P_2 + a) S_2 + 4P_2R_2 + 2(P_2^2 + c) Q_1 +$$
$$+ 3(2P_2^2 + c) Q_2 - 6P_1P_2^3 - 6P_1P_2c - P_2^2a - P_1v_2c + P_2v_1c + v_3c,$$

we get at m

$$[v_1^*, v_3^*] = (.)\, v_2^*, \qquad [v_2^*, v_3^*] = (.)\, v_1^* \tag{1.37}$$
$$[v_3^*, [v_1^*, v_3^*]] = (.)\, v_1^*, \qquad [v_3^*, [v_2^*, v_3^*]] = (.)\, v_2^*$$
$$[v_1^*, [v_1^*, [v_1^*, v_3^*]]] = \alpha^3\beta R v_1^* + (.)\, v_2^* + (.)\, v_3^*$$
$$[v_2^*, [v_2^*, [v_2^*, v_3^2]]] = (.)\, v_1^* - \alpha\beta^3 S v_2^* + (.)\, v_3^*$$

For

$$\alpha = (\varepsilon R^{-3}S)^{1/8}, \beta = (\varepsilon R S^{-3})^{1/8} \qquad \text{at } m \tag{1.38}$$

the couple (v_1^*, v_2^*) has the properties as described in (1.31). From $R = R^* = 1$, $S = S^* = \varepsilon$ at m, we get (1.32). QED.

The following theorem has been proved in [4]:

Theorem 2. *Let $\mathcal{S}$ be an RR-structure on M; let $R = S = 0$ on M. Let $m \in M$. Then there is a neighbourhood $U \subset M$ of m and, on U, a special couple (v_1^*, v_2^*) associated to $\mathcal{S}$ such that*

$$[v_1^*, [v_1^*, v_2^*]] = [v_2^*, [v_1^*, v_2^*]] = 0 \tag{1.39}$$

Let us suppose the general case as described in Theorem 1. In this case, there is a special couple (v_1, v_2) associated to $\mathcal{S}$ such that we have (1.8), (1.9) and

$$v_1v_1a - 2v_3b - 3ab = 1, \qquad v_2v_2a - 2v_3c + 3ac = \varepsilon. \tag{1.40}$$

Set

$$v_1a = p_1, \qquad v_2a = p_2, \qquad v_3b = p_3, \qquad v_3c = p_4. \tag{1.41}$$

The systems (1.9) + (1.40) may be rewritten as (1.41) and

$$v_2b = -p_1, \qquad v_1c = p_2, \qquad v_1p_1 = 2p_3 + 3ab + 1,$$
$$v_2p_2 = 2p_4 - 3ac + \varepsilon. \tag{1.42}$$

The integrability conditions of $(1.41_{1,2})$, $(1.42_1) + (1.41_3)$ and $(1.42_2) + (1.41_4)$ are

$$v_3a = v_1p_2 - v_2p_1, \qquad v_3p_1 + v_2p_3 - cv_1b = ap_1,$$
$$v_3p_2 - v_1p_4 + bv_2c = -ap_2. \tag{1.43}$$

Set

$$v_1b = q_1, \qquad v_2c = q_2, \qquad v_2p_1 = q_5, \tag{1.44}$$
$$v_3p_1 = q_3, \qquad v_1p_2 = q_6, \qquad v_3p_2 = q_4;$$

then (1.43) read

$$v_3a = q_6 - q_5, \qquad v_2p_3 = cq_1 - q_3 + ap_1, \qquad v_1p_4 = bq_2 + q_5 + ap_2. \tag{1.45}$$

The integrability conditions of $(1.41_1) + (1.45_1)$, $(1.41_2) + (1.45_1)$, $(1.44_1) + (1.42_1)$, $(1.44_1) + (1.41_3)$, $(1.42_2) + (1.44_2)$, $(1.44_2) + (1.44_4)$, $(1.42_1) + (1.44_3)$, $(1.42_1) + (1.44_4)$, $(1.44_3) + (1.44_4)$, $(1.44_5) + (1.42_4)$, $(1.44_5) + (1.44_6)$ and $(1.42_4) + (1.44_6)$ are

$$v_1q_5 - v_1q_6 = -q_3 - ap_1 - bp_2, \qquad v_2q_5 - v_2q_6 = -q_4 - cp_1 + ap_2,$$
$$v_2q_1 = -3p_3 - 3ab - 1, \qquad v_3q_1 - v_1p_3 = -aq_1 + bp_1,$$
$$v_1q_2 = 3p_4 - 3ac + \varepsilon, \qquad v_3q_2 - v_2p_4 = aq_2 - cp_2,$$
$$v_1q_5 = 2cp_1 - q_3 - ap_1 + 3bp_2,$$
$$v_1q_3 - 2v_3p_3 = -2bq_5 + 3bq_6 + 5ap_3 + 3a^2b + a,$$
$$v_2q_3 - v_3q_5 = -aq_5 + 2cp_3 + 3abc + c, \qquad v_2q_6 = 2bq_2 + q_4 - 3cp_1 - ap_2,$$
$$v_1q_4 - v_3q_6 = aq_6 + 2bp_4 - 3abc + \varepsilon b,$$
$$v_2q_4 - 2v_3p_4 = 3cq_5 - 2cq_6 - 5ap_4 + 3a^2c - \varepsilon a. \tag{1.46}$$

Set

$$v_1 p_3 = r_1, \quad v_3 p_3 = r_3, \quad v_2 p_4 = r_2, \qquad (1.47)$$
$$v_3 p_4 = r_4, \quad v_2 q_3 = r_5, \quad v_1 q_5 = r_6;$$

then (1.46) implies

$$v_2 q_1 = -3p_3 - 3ab - 1, \qquad v_3 q_1 = r_1 - aq_1 + bp_1,$$
$$v_1 q_2 = 3p_4 - 3ac + \varepsilon, \qquad v_3 q_2 = r_2 + ag_2 - cp_2,$$
$$v_1 q_3 = 2r_3 - 2bq_5 + 3bq_6 + 5ap_3 + 3a^2 b + a,$$
$$v_2 q_4 = 2r_4 + 3cq_5 - 2cq_6 - 5ap_4 + 3a^2 c - \varepsilon a,$$
$$v_1 q_5 = 2cq_1 - q_3 - ap_1 + 3bp_2, \qquad v_2 q_5 = 2bq_2 - 4cp_1, \qquad (1.48)$$
$$v_3 q_5 = r_5 + aq_5 - 2cp_3 - 3abc - c,$$
$$v_1 q_6 = 2cq_1 + 4bp_2, \qquad v_2 q_6 = 2bq_2 + q_4 - 3cp_1 - ap_2,$$
$$v_3 q_6 = r_6 - aq_6 - 2bp_4 + 3abc - \varepsilon b.$$

The integrability conditions of $(1.48_{7,8})$ and $(1.48_{10,11})$ reduce to

$$\varepsilon b + c = 0. \qquad (1.49)$$

Thus we get the following very important technical

Lemma 3. *Let $\mathscr{S}$ be an RR-structure on M; let there exist a special couple (v_1, v_2) associated to $\mathscr{S}$ such that we have* (1.8), (1.9) *and* (1.40). *Then* (1.49) *is valid.*

2. Let M be a 3-dimensional differentiable manifold. At each point $m \in M$, be given a tangent plane τ_m and an endomorphism $J_m : \tau_m \to \tau_m$ satisfying $J_m^2 = -\text{id.}$; let us suppose that the field of planes τ_m is non-integrable. Such a structure $\mathscr{C}$ on M is called a *CR-structure*.

The purpose of this paper is to prove the following

Theorem 3. *Let $\mathscr{C}$ be a CR-structure on M. Let us choose a vector field w on M such that $w(m) \in \tau_m$ for each $m \in M$ and*

$$[w, [w, Jw]] = pw + qJw, \qquad [Jw, [w, Jw]] = rw - pJw, \qquad (2.1)$$
$$wp + (Jw)\, q = 0, \qquad (Jw)\, p - wr = 0; \qquad (2.2)$$

such vector fields do exist. Consider the functions

$$K_1 = (ww - Jw \,.\, Jw)(r - q) + 8[w, Jw]\, p - 3(r^2 - q^2),$$
$$K_2 = (w \,.\, Jw + Jw \,.\, w)(r - q) + 4[w, Jw]\, p + 6p(r - q). \qquad (2.3)$$

(1) *If $K_1 = K_2 = 0$ on M, the vector field w may be chosen in such a way that*

$$[w, [w, Jw]] = 0, \qquad [Jw, [w, Jw]] = 0. \qquad (2.4)$$

(2) *If* $K_1^2 + K_2^2 > 0$ *on* M, *we may choose* w *such that*

$$[w, [w, Jw]] = qJw, \qquad [Jw, [w, Jw]] = rw, \tag{2.5}$$

$$(Jw)\, q = 0, \qquad wr = 0, \tag{2.6}$$

$$wwq + Jw \,.\, Jwr = 3q^2 - 3r^2 - 1, \qquad Jw \,.\, wq - w \,.\, Jwr = 0.$$

The conditions (2.5) + (2.6) determine w up to the sign.

Proof. Obviously, there is (at least locally) a tangent vector field w on M such that $w(m) \in \tau_m$ and (2.1), (2.2) are valid. Let us write

$$w_1 = w, \qquad w_2 = Jw, \qquad w_3 = [w, Jw]. \tag{2.7}$$

Consider the complexification $T^C(M)$ of the tangent bundle $T(M)$ of M and the vector fields

$$v_1 = w_1 + iw_2, \qquad v_2 = w_1 - i_2 w, \qquad v_3 = -2iw_3. \tag{2.8}$$

Then

$$[v_1, v_2] = v_3, \qquad [v_1, v_3] = av_1 + bv_2, \qquad [v_2, v_3] = cv_1 - av_2 \tag{2.9}$$

with

$$a = r - q, \qquad b = r + q - 2ip, \qquad c = -r - q - 2ip. \tag{2.10}$$

Let w^* be another vector field satisfying $w^*(m) \in \tau_m$ for each $m \in M$ and equations (2.1*), (2.2*) in the obvious notation. If

$$w^* = \varrho w - \sigma Jw, \tag{2.11}$$

we have

$$w_1^* = \varrho w_1 - \sigma w_2, \qquad w_2^* = \sigma w_1 + \varrho w_2 \tag{2.12}$$

and

$$v_1^* = \alpha v_1, \qquad v_2^* = \beta v_2 \qquad \text{with} \qquad \alpha = \varrho + i\sigma, \qquad \beta = \varrho - i\sigma.$$

It is easy to see that

$$\begin{aligned} R &= v_1 v_1 a - 2v_3 b - 3ab = K_1 + iK_2 \\ S &= v_2 v_2 a - 2v_3 c + 3ac = K_1 - iK_2 \end{aligned} \tag{2.14}$$

of course, (1.28) imply

$$\begin{aligned} K_1^* &= (\varrho^4 - \sigma^4)\, K_1 - 2\varrho\sigma(\varrho^2 + \sigma^2)\, K_2 \\ K_2^* &= 2\varrho\sigma(\varrho^2 + \sigma^2)\, K_1 + (\varrho^4 - \sigma^4)\, K_2 \end{aligned} \tag{2.15}$$

and, as a consequence,

$$K_1^{*2} + K_2^{*2} = (\varrho^2 + \sigma^2)^4\, (K_1^2 + K_2^2) \tag{2.16}$$

In the case $K_1 = K_2 = 0$, we have $R = S = 0$, and we may apply Theorem 2. Therefore, let us suppose $K_1^2 + K_2^2 > 0$. Let $K_2 \neq 0$. Take $\varrho = -K_1 + \sqrt{(K_1^2 + K_2^2)}$, $\sigma = K_2$; (2.15_2) implies $K_2^* = 0$. Thus w may be chosen in such a way that $K_2 = 0$.

Then (2.15) reduces to $K_1^* = (\varrho^4 - \sigma^4) K_1$, $0 = 2\varrho\sigma(\varrho^2 + \sigma^2) K_1$ and we are in the position to achieve $K_1^* = 1$. Thus there is a tangent vector field w such that $K_1 = 1$, $K_2 = 0$. But this implies $R = S = 1$. Thus $b + c = 0$ according to Lemma 3, i.e., $p = 0$, and our assertions follow. QED.

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Souhrn

O TROJROZMĚRNÝCH CR-VARIETÁCH

ALOIS ŠVEC

V následující práci předkládám konečnou versi řešení problému ekvivalence pro reálné nadplochy prostoru dvou komplexních proměnných vzhledem k pseudogrupě biholomorfních zobrazení. První řešení bylo dáno E. Cartanem [1]; viz též [2]—[5].

Резюме

О ТРЕХРАЗМЕРНЫХ ЦР-МНОГООБРАЗИЯХ

АЛОЙС ШВЕЦ

В следующей работе предлагается полное решение проблемы эквивалентности для действительных гиперповерхностей пространства двух комплексных переменных в отношении к псевдогруппе биголоморфных преобразований. Первое решение предложенил Э. Картан [1]; смотри тоже [2]—[5].