Svatoslav Staněk
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TWO-POINT BOUNDARY PROBLEM
IN A SECOND ORDER NONHOMOGENEOUS LINEAR DIFFERENTIAL EQUATION

SVATOSLAV STANĚK
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Dedicated to Academician O. Borůvka on his 80th birthday

1. In this paper we investigate an equation
\[ y'' - q(t, \lambda) y = r(t, \mu), \quad q \in C^0(D), \quad r \in C^0(D), \quad (1) \]
with \( D = j \times \mathbb{R}, \quad j = (a, b) \quad (\infty \leq a < b \leq \infty), \quad \mathbb{R} = (-\infty, \infty), \) comprising two parameters \( \lambda \) and \( \mu \). Let \((x_0, y_0)\) and \((x_1, y_1)\) be arbitrary points of \( D, \) \( x_0 < x_1. \) The object of this article is:

a) to find sufficient conditions for the existence of the solution \( y \) of (1) where \( y(x_0) = y_0, \ y(x_1) = y_1 \) and for the relative homogeneous equation
\[ y'' = q(t, \lambda) y, \quad q \in C^0(D), \quad (2) \]
to have a nontrivial solution \( v \) such that \( v(x_0) = v(x_1) = 0 \) and \( v(t) \neq 0 \) for \( t \in (x_0, x_1), \)

b) to find satisfactory conditions for the solution of the above problem, where instead of the solution \( y \) of (1) and of the solution \( v \) of (2) we consider the derivative of these solutions.

Besides, there is investigated the uniqueness of the solutions of both problems.

2. Basic definitions, relations and notation.
Let \( x \in j \) and \( u \) be a nontrivial solution of
\[ y'' = p(t) y, \quad p \in C^0(j), \quad (p) \]
\( u(x) = 0. \) Denote by \( \varphi(x) \) the first zero of the solution \( u \) (as far as such exists) lying to the right of the point \( x. \) The function \( \varphi \) is called the fundamental dispersion of the 1st kind of \((p)\).

Let \( p(t) < 0 \) for \( t \in j \) and let \( v \) be a nontrivial solution of \((p), \) \( v'(x) = 0. \) Denote by \( \psi(x) \) the first zero of the function \( v' \) (as far as such exists) lying to the right of the
point \( x \). The function \( \psi \) is called the fundamental dispersion of the 2nd kind of \( p(t) \).

Let \( p \in C^2(j) \), \( p(t) < 0 \) for \( t \in j \). We set \( p_1(t) := p(t) + \sqrt{|p(t)|} \left( \frac{1}{\sqrt{|p(t)|}} \right)' \), \( t \in j \). Then the 2nd kind fundamental dispersion \( u \) of \( p(t) \) is equal to the 1st kind fundamental dispersion \( u_1 \): \( u^* = p_1(t) u \). For more details see \([1, 2]\).

Throughout the functions \( \varphi(t) \) and \( \psi(t) \) \( (\varphi(t, \lambda) \) and \( \psi(t, \lambda) \) \) will denote the fundamental dispersions of the 1st and 2nd kinds of the equation \( p(t) \) (the equation \( (2, 3) \)), respectively.

If for any \( \lambda_1 \) and \( \lambda_2 \) holds that \( \varphi(t, \lambda_1) < \varphi(t, \lambda_2) \) for \( t \in j \), then we conclude from the Sturm comparison theorem that \( \varphi(t, \lambda_1) < \varphi(t, \lambda_2) \) for \( t \) from the interval of definition of the function \( \varphi(t, \lambda) \). This set may be also empty.

It follows from \([5, 6]\): Let \( x_0 \in j \), \( y_0, y_0' \) be arbitrary numbers. Let \( u_1 \) and \( u_2 \) be two different solutions of

\[
y'' - p(t) y = f(t), \quad p \in C^0(j), \quad f \in C^0(j), \quad f(t) \neq 0, \tag{3}
\]

satisfying the condition \( u_1(x_0) = u_2(x_0) = y_0 \), and the 1st kind fundamental dispersion \( \varphi \) of \( p(t) \) be defined at \( x_0 \). Then \( u_1(t) \neq u_2(t) \) for \( t \in (x_0, \varphi(x_0)) \) and \( u_1(\varphi(x_0)) = u_2(\varphi(x_0)) := y_1 \). In this case the points \( (x_0, y_0) \) and \( (\varphi(x_0), y_1) \) are called the neighbouring knots of the 1st kind relative to \( (3) \) and to the initial condition \( (x_0, y_0) \).

Throughout this article we use \( \sim \) to denote the derivative with respect to the independent variable \( t \) to shorten the writing even in case of functions examined being of two independent variables.

In what follows we will occasionally investigate the function \( q(t, \lambda) \) for which one of the following assumptions applies:

(i) \( q \in C^0(D) \), \( q(t, \lambda_1) < q(t, \lambda_2) \) for \( \lambda_1 < \lambda_2, \ t \in j \) and

\[
\lim_{\lambda \to -\infty} q(t, \lambda) = -\infty, \quad \lim_{\lambda \to \infty} q(t, \lambda) = \infty, \quad t \in j \tag{4}
\]

(ii) \( q(t, \lambda) \equiv \lambda p(t), \ p \in C^0(j) \) and \( p(t) \neq 0 \) on every interval \( \subset j \);

(iii) \( q \in C^0(D) \), \( q(t, \lambda) \in C^0(D), \ q(t, \lambda) < 0 \) for \( (t, \lambda) \in D \),

\[
\lim_{\lambda \to -\infty} \left( q(t, \lambda) + \sqrt{|q(t, \lambda)|} \left( \frac{1}{\sqrt{|q(t, \lambda)|}} \right)' \right)' = -\infty \tag{5}
\]

\[
\lim_{\lambda \to \infty} \left( q(t, \lambda) + \sqrt{|q(t, \lambda)|} \left( \frac{1}{\sqrt{|q(t, \lambda)|}} \right)' \right)' = \infty, \quad t \in j
\]
and
\[
q(t, \lambda_1) + \sqrt{|q(t, \lambda_1)|} \left( \frac{1}{\sqrt{|q(t, \lambda_1)|}} \right)^n < q(t, \lambda_2) + \left( \frac{1}{\sqrt{|q(t, \lambda_2)|}} \right)^n
\]
for \( \lambda_1 < \lambda_2, t \in j; \)

(iv) \( q \in C^0(D), q''(t, \lambda) \in C^0(D), q(t, \lambda) < 0 \) for \((t, \lambda) \in D, \)
\[
q(t, y) + \sqrt{|q(t, \lambda)|} \left( \frac{1}{\sqrt{|q(t, \lambda)|}} \right)^n \equiv \lambda p(t)
\]
and \( p(t) \not\equiv 0 \) on every interval \((a, b). \)

**Lemma 1.** Let \( x_0 \in j, x_1 \in j \) be arbitrary numbers, \( x_0 < x_1. \) If the function \( q \) satisfies the assumption (i), then there exists exactly one number \( \lambda_0 : \varphi(x_0, \lambda_0) = x_1. \) If the function \( q \) satisfies the assumption (ii) and \( \inf \{x; x \in j, x > x_0, p(x) > 0\} < x_1 \) \( \inf \{x; x \in j, x > x_0, p(x) < 0\} < x_1 \), then there exists exactly one \( \lambda_1 > 0 \) \( (\lambda_2 < 0) \) with \( \varphi(x_0, \lambda_1) = x_1. \)

**Proof.** Following Lemma 1 \([3]\) the function \( \varphi(x_0, \lambda) \) is continuous on its interval of definition. If the function \( q \) satisfies the assumption (i), then \( \varphi(x_0, \lambda) \) is an increasing function mapping the interval of definition onto \((x_0, b). \) Hence, there exists exactly one number \( \lambda_0 : \varphi(x_0, \lambda_0) = x_1. \) Let \( q \) satisfy the assumption (ii) and \( \inf \{x; x \in j, x > x_0, p(x) > 0\} < x_1 \) \( \inf \{x; x \in j, x > x_0, p(x) < 0\} < x_1 \). It follows from Theorem 1 \([7]\) and from its proof that \( \varphi(x_0, \lambda) \) is for \( \lambda > 0 \) for which \( \varphi(x_0, \lambda) \) is defined (it is for \( \lambda < 0 \) for which \( \varphi(x_0, \lambda) \) is defined) a decreasing (an increasing) function. The rest of the statement of the Lemma follows from Corollary 5.1. \([4, p. 408]\) and from Corollary 1 \([7]\).

**Remark 1.** Let the function \( q \) satisfy the assumption (ii). Then it follows from Lemma 1 that there always exists at least one number \( \lambda_0 : \varphi(x_0, \lambda_0) = x_1. \)

**Corollary 1.** Let \( x_0 \in j, x_1 \in j \) be arbitrary numbers, \( x_0 < x_1. \) If the function \( q \) satisfies the assumption (iii), then there exists exactly one number \( \lambda_0 : \psi(x_0, \lambda_0) = x_1. \) If the function \( q \) satisfies the assumption (iv) and \( \inf \{x; x \in j, x > x_0, p(x) < 0\} < x_1 \) \( \inf \{x; x \in j, x > x_0, p(x) > 0\} < x_1 \) then there exists exactly one \( \lambda_1 > 0 \) \( (\lambda_2 < 0) \) with \( \psi(x_0, \lambda_1) = x_1. \)

The proof follows from Lemma 1 and from the fact that the 2nd kind fundamental dispersion of \((2)\) is equal to the 1st kind fundamental dispersion of \( y'' - \left( q(t, \lambda) + \sqrt{|q(t, \lambda)|} \right)^n \) \( y = 0. \)
Lemma 2. Let $x_0 \in I, y_0$ be arbitrary numbers, $r \in C^0(D)$, let the function $\varphi(t)$ be defined at $x_0$ and let

$$
\lim_{\mu \to -\infty} r(t, \mu) = -\infty, \quad \lim_{\mu \to \infty} r(t, \mu) = \infty
$$

(6)

uniformly on the interval $(x_0, \varphi(x_0))$ $(= j)$. Let $u(t, \mu)$ be a solution of

$$
y'' - p(t) y = r(t, \mu), \quad p \in C^0(j), \ r \in C^0(D)
$$

(7)

satisfying the condition $u(x_0, \mu) = y_0$. Setting

$$
\mathcal{M} := \{u(\varphi(x_0), \mu); \mu \in \mathbb{R}\},
$$

(8)

then

$$
\mathcal{M} = \mathbb{R}
$$

Proof. Let $\mathcal{M}$ be the set defined by (8). It follows from the continuous dependence of solutions on the parameter that $\mathcal{M}$ is a convex set. To prove Lemma 2 it suffices to show that $\inf \mathcal{M} = -\infty$, $\sup \mathcal{M} = \infty$. We prove the second of the given equalities (the proof of the first one proceeds similarly). Let $\mathcal{M} = L < \infty$. Let $y_1, y_2$ be solutions of (p) satisfying the initial conditions $y_1(x_0) = y_2(x_0) = 0$, $y_1'(x_0) = y_2(x_0) = 1$. Then $y_1(\varphi(x_0)) = 0$, $y_2(\varphi(x_0)) < 0$. We set

$$
k := -\frac{L + 1 - y_0 \cdot y_2(\varphi(x_0))}{y_2(\varphi(x_0))} \left( \int_{x_0}^{\varphi(x_0)} y_1(t) \, dt \right)^{-1}.
$$

According to the assumption there holds (6) uniformly on the interval $(x_0, \varphi(x_0))$ and consequently there exists $\mu_0 \in \mathbb{R}$ such that $r(t, \mu_0) > k$ for $t \in (x_0, \varphi(x_0))$. Let $v$ be a solution of the equation $y'' - p(t) y = k$, $v(x_0) = y_0$, $v'(x_0) = u'(x_0, \mu_0) := y_0'$. Then

$$
v(t) = y_0 y_2(t) + y_0' y_1(t) + k \int_{x_0}^{t} (y_1(t) y_2(s) - y_1(s) y_2(t)) \, ds.
$$

Setting $w(t) := u(t, \mu_0) - v(t), \ \ t \in I$, then $w'' - p(t) w = r(t, \mu_0) - k$, $w(x_0) = w'(x_0) = 0$. Hence, by Theorem 1.1 [6] and its proof, we have $w(\varphi(x_0)) > 0$ and therefore $u(\varphi(x_0), \mu_0) > v(\varphi(x_0))$. We have next

$$
v(\varphi(x_0)) = y_0 y_2(\varphi(x_0)) - k y_2(\varphi(x_0)) \int_{x_0}^{\varphi(x_0)} y_1(s) \, ds = L + 1.
$$

Thus $u(\varphi(x_0), \mu_0) > v(\varphi(x_0)) = L + 1$ contrary to the assumption $u(\varphi(x_0), \mu_0) \leq L$.

Lemma 3. Let $x_0 \in I, y_0$ be arbitrary numbers, let the function $\varphi(t)$ be defined at $x_0$ and let the function $r \in C^0(D)$ satisfy (6) for $t \in (x_0, \varphi(x_0))$ and

$$
r(t, \mu_1) < r(t, \mu_2) \quad \text{for} \ \mu_1 < \mu_2 \ \text{and} \ t \in (x_0, \varphi(x_0))
$$

(9)
Let $u(t, \mu)$ be a solution of (7) satisfying the condition $u(x_0, \mu) = y_0$. Then the function
\begin{equation}
\alpha(\mu) := u(\phi(x_0), \mu), \quad \mu \in \mathbb{R}
\end{equation}
is an increasing function on $\mathbb{R}$ and $\alpha(\mathbb{R}) = \mathbb{R}$.

**Remark 2.** As stated before, it follows from [5, 6], that all solutions $u(t, \mu)$ of (7) satisfying the condition $u(x_0, \mu) = y_0$ have equal values at the point $(\phi(x_0), \mu)$. Evidently this implies that the function $\alpha$ is by relation (10) correctly defined.

The proof of Lemma 3. From assumptions (6) and (9) laid on the function then follows the uniformly convergence of (6) on the interval $(x_0, \phi(x_0))$ which implies by Lemma 2 that $\alpha(\mathbb{R}) = \mathbb{R}$, where the function $\alpha$ is defined by (10). Let $\mu_1, \mu_2$ be arbitrary numbers $\mu_1 < \mu_2$. Let $y_1, y_2$ be solutions of (p), $y_1(x_0) = y_2(x_0) = 0$ and $y_1'(x_0) = y_2(x_0) = 1$. Then
\begin{align*}
\alpha(\mu_2) - \alpha(\mu_1) &= u(\phi(x_0), \mu_2) - u(\phi(x_0), \mu_1) = y_2(\phi(x_0)) \int_{x_0}^{\phi(x_0)} y'_1(s) (r(s, \mu_2) - r(s, \mu_1)) ds > 0.
\end{align*}
Consequently the function $\alpha$ is increasing on $\mathbb{R}$.

**Remark 3.** The function $r(t, \mu) = f(t) + \mu, f \in C^0(j)$ satisfies the assumptions of Lemma 3.

**Lemma 4.** Let $x_0 \in j, y_0$ be arbitrary numbers, $f \in C^0(j)$ and let the function $\phi(t)$ be defined at $x_0$. Next let
\begin{equation}
\phi(x_0) \int_{x_0}^{\phi(x_0)} f(t) y'_1(t) dt = 0,
\end{equation}
where $y_1$ is a solution of (p), $y_1(x_0) = 0, y_1'(x_0) = 1$. Let $u(t, \mu)$ be a solution of
\begin{equation}
y'' - p(t) y = \mu f(t)
\end{equation}
satisfying the condition $u(x_0, \mu) = y_0$. Then the function $\alpha(\mu)$ defined by (10) is strictly monotone on $\mathbb{R}$, $\alpha(\mathbb{R}) = \mathbb{R}$, $\alpha' \neq 0$.

**Proof.** Let $y_2$ be a solution of (p), $y_2(x_0) = 1, y_2'(x_0) = 0$ and $y_1, u$ be the function defined in Lemma 4. Then
\begin{align*}
\alpha(\mu_2) - \alpha(\mu_1) &= u(\phi(x_0), \mu_2) - u(\phi(x_0), \mu_1) = y_2(\phi(x_0)) \int_{x_0}^{\phi(x_0)} y'_1(s) (r(s, \mu_2) - r(s, \mu_1)) ds > 0.
\end{align*}

Consequently the function $\alpha$ is increasing on $\mathbb{R}$.
and from this we obtain

\[ \alpha(\mu) = \mu y_0 y_2(\varphi(x_0)) - \mu y_2(\varphi(x_0)) \int_{x_0}^{\varphi(x_0)} y_1(s) f(s) \, ds, \]

\[ \alpha'(\mu) = -y_2(\varphi(x_0)) \int_{x_0}^{\varphi(x_0)} y_1(s) f(s) \, ds = k \neq 0, \]

\[ \alpha(\mu) = k\mu + c \quad (c = \text{a constant}) \]

from which the statement of the Lemma results.

**Lemma 5.** Let \( p \in C^2(j), p(t) \neq 0 \) for \( t \in j \) and \( f \in C^1(j) \). Then for every solution \( y \) of the equation

\[ y'' - p(t) y = f(t) \quad (11) \]

the function \( z(t) = \frac{y'(t)}{\sqrt{|p(t)|}}, t \in j \) represents a solution of the equation

\[ z'' - \left( p(t) + \sqrt{|p(t)|} \left( \frac{1}{\sqrt{|p(t)|}} \right)' \right) z = 2f(t) \left( \frac{1}{\sqrt{|p(t)|}} \right)' + \frac{f'(t)}{\sqrt{|p(t)|}} \quad (12) \]

and vicer versa, for every solution \( z \) of (12) the function \( z(t) \sqrt{|p(t)|} \) is the derivative of exactly one solution of (11).

**Proof.** Let \( y \) be a solution of (11) and \( z(t) = \frac{y'(t)}{\sqrt{|p(t)|}}, t \in j \). It is easily verified that the function \( z \) is a solution of (12).

Let \( z \) be a solution of (12) and \( v(t) = z(t) \sqrt{|p(t)|}, t \in j \). Assume that \( v \) is the derivative of a solution \( y \) of (11). Then the solution \( y \) and its derivative \( y' \) at \( x_0 \in j \) have necessarily the following values

\[ v_0 := \frac{1}{p(x_0)} (v'(x_0) - f(x_0)), \quad v'_0 := z(x_0) \sqrt{|p(x_0)|}. \]

Setting

\[ w(t) := v_0 + \int_{x_0}^{t} z(s) \sqrt{|p(s)|} \, ds, \quad t \in j, \]

then \( w'(t) = z(t) \sqrt{|p(t)|}, w(x_0) = v_0, w'(x_0) = v'_0 \) and further

\[ w''' = z'' \sqrt{|p|} + \frac{z'p'}{\sqrt{|p|}} + z(\sqrt{|p|})'' = \left[ \left( p - \frac{p''}{2p} + \frac{3}{4} \left( \frac{p'}{p} \right)^2 \right) z + 2f \left( \frac{1}{\sqrt{|p|}} \right)' + \frac{f'}{\sqrt{|p|}} \right] \sqrt{|p|} + \]

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Thus

$$w'' - \frac{p'}{p} w'' - pw' = -fp' + f'$$

From this we obtain

$$\left(\frac{w'' - pw}{p}\right)' = \left(\frac{f}{p}\right)'$$

$$w'' - pw = f + kp \quad (k = \text{a constant}).$$

The definition of the function $w$ and $w(x_0) = v_0$, $w'(x_0) = v'_0$ yields $k = 0$. This completes the proof of the Lemma.

**Remark 4.** Lemma 5 was proved in [1, p. 9] under the assumption $f(t) = 0$.

**Lemma 6.** Let $x_0 \in j$, $y'_0$ be arbitrary numbers, $l \in C^0(D)$, $l'(t, \mu) \in C^0(D)$, $k \in C^2(j)$, $k(t) < 0$ for $t \in j$. Let the second order fundamental dispersion $\psi$ of $(k)$: $y'' = k(t)y$ be defined $x_0 \in j$ and uniformly on $<x_0, \psi(x_0)>$

$$\lim_{\mu \to -\infty} \left\{2l(t, \mu)\left(\frac{1}{\sqrt{|k(t)|}}\right)' + \frac{l'(t, \mu)}{\sqrt{|k(t)|}}\right\} = -\infty,$$

$$\lim_{\mu \to +\infty} \left\{2l(t, \mu)\left(\frac{1}{\sqrt{|k(t)|}}\right)' + \frac{l'(t, \mu)}{\sqrt{|k(t)|}}\right\} = \infty.$$  \hspace{1cm} (13)

Let $\psi(t, \mu)$ be a solution of

$$y'' - k(t)y = i(t, \mu)$$  \hspace{1cm} (14)

satisfying the condition $\psi(x_0, \mu) = y'_0$. Setting

$$\mathcal{M}_1 := \{\psi'(x_0, \mu); \mu \in \mathbb{R}\},$$  \hspace{1cm} (15)

then

$$\mathcal{M}_1 = \mathbb{R}.$$  \hspace{1cm}

**Proof.** Let $\mathcal{M}_1$ be the set defined by (15). Let $\psi(t, \mu)$ be a solution of (14) satisfying the condition $\psi'(x_0, \mu) = y'_0$ and let $u(t, \mu) = \frac{\psi'(t, \mu)}{\sqrt{|k(t)|}}$, $(t, \mu) \in D$. Then $u(x_0, \mu) = \frac{y'_0}{\sqrt{|k(x_0)|}}$ and according to Lemma 5 $u(t, \mu)$ is a solution of
\[ y'' - \left( k(t) + \sqrt{|k(t)|} \left( \frac{1}{\sqrt{|k(t)|}} \right)'' \right) y = 0 \]

Putting

\[ p(t) := k(t) + \sqrt{|k(t)|} \left( \frac{1}{\sqrt{|k(t)|}} \right)'' , \quad t \in j , \]

\[ r(t, \mu) := 2l(t, \mu) \left( \frac{1}{\sqrt{|k(t)|}} \right)'' + \frac{l'(t, \mu)}{\sqrt{|k(t)|}} , \quad (t, \mu) \in D , \]

then \( u(t, \mu) \) is a solution of (7) where the functions \( p, r \) are defined by (16), \( u(x_0, \mu) = \frac{y'_0}{\sqrt{|k(x_0)|}} \). Since \( \psi \) is the 1st kind fundamental dispersion of (p), we have with respect to Lemma 2

\[ \mathcal{R} = \{ u(\psi(x_0), \mu); \mu \in \mathbb{R} \} = \left\{ v'(\psi(x_0), \mu) \frac{1}{\sqrt{|k(\psi(x_0))|}} ; \mu \in \mathbb{R} \right\} = \mathcal{M}_1 . \]

**Remark 5.** Let \( k \in C^2(j), \ k(t) < 0 \) and \( \left( \frac{1}{\sqrt{|k(t)|}} \right)' > 0 \) for \( t \in j, \ h \in C^1(j) \).

Setting \( l(t, \mu) := h(t) + \mu, (t, \mu) \in D \), then (13) applies uniformly on every compact subinterval of \( j \).

**Lemma 7.** Let \( x_0 \in j \), \( y'_0 \) be arbitrary numbers, \( l \in C^0(D), l'(t, \mu) \in C^0(D), k \in C^2(j), k(t) < 0 \) for \( t \in j \). Let the 2nd order fundamental dispersion \( \psi \) of (k) be defined at \( x_0 \), let the function \( l \) satisfy (13) for \( t \in \langle x_0, \psi(x_0) \rangle \) and

\[ 2l(t, \mu_1) \left( \frac{1}{\sqrt{|k(t)|}} \right)'' + \frac{l'(t, \mu_1)}{\sqrt{|k(t)|}} < 0 \]

\[ < 2l(t, \mu_2) \left( \frac{1}{\sqrt{|k(t)|}} \right)'' + \frac{l'(t, \mu_2)}{\sqrt{|k(t)|}} \]

for \( \mu_1 < \mu_2 \) and \( t \in \langle x_0, \psi(x_0) \rangle \). Let next \( v(t, \mu) \) be a solution of (14) satisfying the condition \( v'(x_0, \mu) = y'_0 \). Then the function

\[ \beta(\mu) := v'(\psi(x_0), \mu), \quad \mu \in \mathbb{R}, \]

is an increasing one on \( \mathbb{R} \) and \( \beta(\mathbb{R}) = \mathbb{R} \).

**Remark 6.** It follows from [5] that all solutions \( v(t, \mu) \) of (14) satisfying the condition \( v'(x_0, \mu) = y'_0 \) have the same values at the point \( (\psi(x_0), \mu) \). Therefore the function \( \beta \) is defined correctly by relation (17).
The proof of Lemma 7. Let \( v(t, \mu) \) be a solution of (14), \( v'(x_0, \mu) = y_0' \). We put \( u(t, \mu) := \frac{v'(t, \mu)}{\sqrt{|k(t)|}} \), \((t, \mu) \in D \). Then, with respect to Lemma 5, \( u \) is a solution of (7), where the functions \( p, r \) are defined by (16), \( u(x_0, \mu) = \frac{y_0}{\sqrt{|k(x_0)|}} \). Since \( \psi \) is the 1st kind fundamental dispersion of (p), the assumptions of Lemma 3 are satisfied and \( \beta(\mu) = v'(\psi(x_0), \mu) = \sqrt{|k(\psi(x_0))|} \cdot u(\psi(x_0), \mu) \) and the properties of the function \( \beta \) under proof, immediately follow from the properties of the function \( \alpha \) defined in Lemma 3.

3. We prove the following

**Theorem 1.** Let \( x_0, x_1 \in j, y_0, y_1 \) be arbitrary numbers, \( x_0 < x_1 \) and \( q \in C^0(D) \), \( r \in C^0(D) \). Let next (4) and (6) hold uniformly on every compact subinterval of \( j \). Then there exist numbers \( \lambda_0, \mu_0 \) such that the points \( (x_0, y_0), (x_1, y_1) \) are the 1st kind neighbouring knots relative to equation (1) with \( \lambda = \lambda_0, \mu = \mu_0 \), and to the initial condition \((x_0, y_0)\).

**Proof.** The function \( \varphi(x_0, \lambda) \) is continuous on its interval of definition with respect to Lemma 1 [3] and it follows from (4) holding by assumption uniformly on every compact subinterval of \( j \) that: \( \lim_{\lambda \to -\infty} \varphi(x_0, \lambda) = x_0 \) and there exists a number \( \lambda_1 \), where the function \( \varphi(x_0, \lambda) \) is mapping the interval \((-\infty, \lambda_1)\) onto the interval \((x_0, b)\). There exists therefore at least one number \( \lambda_0 \in (-\infty, \lambda_1) \): \( \varphi(x_0, \lambda_0) = x_1 \). Let \( u(t, \mu) \) be a solution of

\[
y'' - q(t, \lambda_0) y = r(t, \mu)
\]

\( u(x_0, \mu) = y_0 \). With respect to Lemma 2 then follows the existence of a number \( \mu_0 : u(x_1, \mu_0) = y_1 \).

**Corollary 2.** Let \( x_0, x_1 \in j, y_0, y_1 \) be arbitrary numbers, \( x_0 < x_1 \). Let \( q \) satisfy the assumption (i) and let \( r \) satisfy one of the following assumptions:

(i) \( r \in C^0(D) \) and (6) and (9) are satisfied for \( t \in j \),

(ii) \( r(t, \mu) \equiv \mu f(t) \), where \( f \in C^0(j) \) and \( \int_{x_0}^{x_1} f(t) y_1(t) \, dt = 0 \).

Here \( y_1 \) is a nontrivial solution of \( y'' = q(t, \lambda_0) y, y_1(x_0) = 0 \) and \( \lambda_0 \) is the number occurring in the statement of Lemma 1.

Then there exists exactly one value of the parameter \( \lambda \) which we write as \( \lambda_0 \) and exactly one value of the parameter \( \mu \) written as \( \mu_0 \) with the points \( (x_0, y_0), (x_1, y_1) \) being the 1st kind neighbouring knots relative to (1), where \( \lambda = \lambda_0 \) and \( \mu = \mu_0 \), and to the initial condition \((x_0, y_0)\).
Proof. With respect to Lemma 1 there exists exactly one number \( \lambda_0 : \varphi(x_0, \lambda_0) = x_1 \). We set \( p(t) := q(t, \lambda_0) \), \( t \in j \). If \( r \) satisfies the assumption (v) and (vi) then— with respect to Lemmas 3 and 4 respectively—there exists exactly one value of the parameter \( \mu \) written as \( \mu_0 \), where the equation \( y^{\prime\prime} - q(t, \lambda_0) y = r(t, \mu_0) \) has a solution \( u \) for which \( u(x_0) = y_0 \) and \( u(x_1) = y_1 \).

**Corollary 3.** Let \( x_0 \in j, x_1 \in j, y_0, y_1 \) be arbitrary numbers, \( x_0 < x_1 \). Let \( q \) satisfy the assumption (ii) and \( \inf \{ x; x \in j, x > x_0, p(x) < 0 \} < x_1 \). Let \( r \) satisfy either the assumption (v) or the assumption (vii)
\[
(vii) r(t, \mu) \equiv \mu f(t), \text{ where } f \in C^0(j) \text{ and } \int_{x_0}^{x_1} f(t) y_1(t) \, dt \neq 0 \text{ with } y_1 \text{ being a nontrivial solution of } y^{\prime\prime} = q(t, \lambda_1) y, y_1(x_0) = 0 \text{ and } \lambda_1 > 0 \text{ a number occurring in the statement of Lemma 1.}
\]

Then there exists exactly one positive value of the parameter \( \lambda \) written as \( \lambda_1 \) and exactly one value of the parameter \( \mu \) written as \( \mu_0 \) such that the points \( (x_0, y_0), (x_1, y_1) \) are the 1st kind neighbouring knots relative to (1), where \( \lambda = \lambda_1 \) and \( \mu = \mu_0 \), and to the initial condition \( (x_0, y_0) \).

Proof. With respect to Lemma 1 there exists exactly one number \( \lambda_1 > 0 \): \( \varphi(x_0, \lambda_1) = x_1 \). We set \( p(t) := q(t, \lambda_1), t \in j \). The rest of the proof proceeds completely analogous to the proof of Corollary 2.

**Corollary 4.** Let \( x_0 \in j, x_1 \in j, y_0, y_1 \) be arbitrary numbers, \( x_0 < x_1 \). Let \( q \) satisfy the assumption (ii) and \( \inf \{ x; x \in j, x > x_0, p(x) > 0 \} < x_1 \). Let \( r \) satisfy the assumption (v) or the assumption (vii), where instead of \( \lambda_1 \) we consider \( \lambda_2 < 0 \) occurring in the statement of Lemma 1. Then there exists one negative value of the parameter \( \lambda \) written as \( \lambda_2 \) and exactly one value of the parameter \( \mu \) written as \( \mu_0 \) such that the points \( (x_0, y_0), (x_1, y_1) \) are the 1st kind neighbouring knots relative to (1), where \( \lambda = \lambda_2 \) and \( \mu = \mu_0 \), and to the initial condition \( (x_0, y_0) \).

We refrain from proving these assertions since the proof is an exact repetition of the previous one.

**Theorem 2.** Let \( x_0 \in j, x_1 \in j, y_0', y_1' \) be arbitrary numbers, \( x_0 < x_1 \), \( q \in C^0(D) \), \( q'''(t, \lambda) \in C^0(D) \), \( r \in C^0(D) \), \( r'(t, \mu) \in C^0(D) \) and \( q(t, \lambda) < 0 \) for \( (t, \lambda) \in D \). Let (5) hold uniformly on every compact subinterval of \( j \) and let uniformly with respect to the variable \( t \) on every compact subinterval of \( j \):
\[
\lim_{\mu \to -\infty} \left\{ 2 r(t, \mu) \left( \frac{1}{\sqrt{|q(t, \lambda)|}} \right)' + \frac{r'(t, \mu)}{\sqrt{|q(t, \lambda)|}} \right\} = -\infty,
\]
\[
\lim_{\mu \to \infty} \left\{ 2 r(t, \mu) \left( \frac{1}{\sqrt{|q(t, \lambda)|}} \right)' + \frac{r'(t, \mu)}{\sqrt{|q(t, \lambda)|}} \right\} = \infty,
\]
(\( \lambda \in \mathbb{R} \)).
Then there exist numbers \( \lambda_0, \mu_0 \) where points \((x_0, y'_0), (x_1, y'_1)\) are the 2nd kind neighbouring knots relative to (1) (with \( \lambda = \lambda_0, \mu = \mu_0 \)) and to the initial condition \((x_0, y'_0)\).

**Proof.** We set \( q_1(t, \lambda) := q(t, \lambda) + \sqrt{|q(t, \lambda)|} \left( \frac{1}{\sqrt{|q(t, \lambda)|}} \right)^2, (t, \lambda) \in D. \) Then the 2nd kind fundamental dispersion \( \psi(t, \lambda) \) of (2) is equal to the 1st kind fundamental dispersion of the equation \( y'' = q_1(t, \lambda) y \). From the assumption (5) we can prove the existence of a number \( \lambda_0 : \psi(x_0, \lambda_0) = x_1 \) by a completely analogous method to that used in the first part of the proof of Theorem 1. We set \( r_1(t, \mu) := 2r(t, \mu) \left( \frac{1}{\sqrt{|g(t, \lambda_0)|}} \right)^2 + \frac{r'(t, \mu)}{\sqrt{|g(t, \lambda_0)|}} \), \((t, \mu) \in D. \) Let \( v(t, \mu) \) be a solution of \[ y'' - q_1(t, \lambda_0) y = r_1(t, \mu), v(x_0, \mu) = \frac{y'_0}{\sqrt{|g(t, \lambda_0)|}}. \] Then, with respect to Lemma 6, the function \( v(t, \mu) \sqrt{|g(t, \lambda_0)|} \) is the derivative of the exactly one solution of \( y'' - q_1(t, \lambda_0) y = r_1(t, \mu), \) written as \( u(t, \mu) \). Evidently \( u'(x_0, \mu) = y'_0. \) With respect to Lemma 6 there exists \( \mu_0, \) and (1) (with \( \lambda = \lambda_0, \mu = \mu_0 \)) has the solution \( u(t, \mu_0) \) satisfying \( u'(x_0, \mu_0) = y_0, u'(x_1, \mu_0) = u'(\psi(x_0, \mu_0)) = y'_1. \) Thus Theorem 2 is proved.

**Corollary 5.** Let \( x_0 \in j, x_1 \in j, y'_0, y'_1 \) be arbitrary numbers, \( x_0 < x_1. \) Let \( q \) satisfy the assumption (iii), \( r \in C^0(D), r'(t, \mu) \in C^0(D) \) and
\[
2r(t, \mu_1) \left( \frac{1}{\sqrt{|g(t, \lambda_0)|}} \right)^2 + \frac{r'(t, \mu_1)}{\sqrt{|g(t, \lambda_0)|}} < 2r(t, \mu_2) \left( \frac{1}{\sqrt{|g(t, \lambda_0)|}} \right)^2 + \frac{r'(t, \mu_2)}{\sqrt{|g(t, \lambda_0)|}} \quad \text{for } \mu_1 < \mu_2 \text{ and } (t, \lambda) \in D. \tag{19}
\]
Let next (18) be true for every \((t, \lambda) \in D.\)

Then there exists exactly one value of the parameter \( \lambda \) written as \( \lambda_0 \) and exactly one value of the parameter \( \mu \) written as \( \mu_0 \) such that the points \((x_0, y'_0), (x_1, y'_1)\) are the 2nd kind neighbouring knots relative to (1) (with \( \lambda = \lambda_0, \mu = \mu_0 \)) and to the initial condition \((x_0, y'_0)\).

**Proof.** Since \( q \) satisfies the assumption (iii), there exists exactly one number \( \lambda_0 : \psi(x_0, \lambda_0) = x_1 \). Then the statement of Corollary 5 follows from inequality (19) (where we put \( \lambda_0 \) in place of \( \lambda \)) and from Lemma 7.

**Corollary 6.** Let \( x_0 \in j, x_1 \in j, y'_0, y'_1 \) be arbitrary numbers, \( x_0 < x_1. \) Let \( q \) satisfy the assumption (iv), \( r \in C^0(D), r'(t, \mu) \in C^0(D) \) and let (18) for \((t, \lambda) \in D\) and (19) hold.

If \( \inf \{ x; x \in j, x > x_0, p(x) < 0 \} < x_1, \) then there exists exactly one positive value of the parameter \( \lambda \) written as \( \lambda_1 \) and exactly one value of the parameter \( \mu \) written as \( \mu_0 \) such that the points \((x_0, y'_0), (x_1, y'_1)\) are the 2nd kind neighbouring knots relative to (1) (with \( \lambda = \lambda_1, \mu = \mu_0 \)) and to the initial condition \((x_0, y'_0)\).
If \( \inf \{ x; x \in J, x > x_0, p(x) > 0 \} < x_1 \), then there exists exactly one negative value of the parameter \( \lambda \) written as \( \lambda_2 \) and exactly one value of the parameter \( \mu \) written as \( \mu_0 \) such that the points \((x_0, y'_0), (x_1, y'_1)\) are the 2nd order neighbouring knots relative to (1) (with \( \lambda = \lambda_2, \mu = \mu_0 \)) and to the initial condition \((x_0, y'_0)\).

The proof follows from Corollary 1 and from Lemma 7.

**BIBLIOGRAPHY**


**Souhrn**

**DVOUBODOVÝ OKRAJOVÝ PROBLÉM PRO NEHOMOGENNÍ LINEÁRNÍ DIFERENCIÁLNÍ ROVNICI 2. ŘÁDU**

SVATOSLAV STANĚK

V práci je vyšetřována lineární diferenciální rovnice

\[ y'' - q(t, \lambda) y = r(r, \mu), \]

kde \( q \in C^0(D), r \in C^0(D), D = J \times R, j = (a, b) (\infty \leq a < b \leq \infty) \), která závisí na dvou reálných parametrech \( \lambda, \mu \). Nechť \((x_0, y_0), (x_0, y'_0), (x_1, y_1), (x_1, y'_1)\) jsou libovolné body v \( D, x_0 < x_1 \). Jsou uvedeny postačující podmínky k tomu, aby:

(i) existovalo řešení \( y \) rovnice (1) pro něž \( y(x_0) = y_0, y(x_1) = y_1 \) a příslušná homogenní rovnice

\[ y'' = q(t, \lambda) y \]

(ii) existovalo řešení \( z \) rovnice (1) pro něž \( z'(x_0) = y'_0, z'(x_1) = y'_1 \) a příslušná...
homogenní rovnice (2) měla netriviální řešení u, kde u'(x_0) = u'(x_1) = 0 a u'(t) ≠ 0 pro t ∈ (x_0, x_1).
Rovněž je vyšetřována jednoznačnost řešení obou problémů.

Резюме

ДВУТОЧЕЧНАЯ ЗАДАЧА ДЛЯ НЕОДНОРОДНОГО ЛИНЕЙНОГО ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ ВТОРОГО ПОРЯДКА

СВАТОСЛАВ СТАНЕК

В работе исследуется неоднородное линейное дифференциальное уравнение

\[ y'' - q(t, \lambda) y = r(t, \mu), \quad (1) \]

где q ∈ C^0(D), r ∈ C^0(D), D = J × R, J = (a, b) (−∞ ≤ a − b ≤ ∞) которое зависит от двух действительных параметров λ, μ. Пусть (x_0, y_0), (x_0, y'_0), (x_1, y_1) и (x_1, y'_1) произвольные точки из D, x_0 < x_1. Приведены достаточные условия для того, чтобы

(i) существовало решение y уравнения (1), y(x_0) = y_0, y(x_1) = y_1 и одновременно соответствующее однородное уравнение

\[ y'' = q(t, \lambda) y \quad (2) \]

имело нетривиальное решение v, где v(x_0) = v(x_1) = 0 и v(t) ≠ 0 для t ∈ (x_0, x_1); (ii) существовало решение z уравнения (1), z'(x_0) = y'_0, z'(x_1) = y'_1 и одновременно соответствующее однородное уравнение (2) имело нетривиальное решение u, где u'(x_0) = u'(x_1) = 0 и u'(t) ≠ 0 для t ∈ (x_0, x_1).

Исследуется также однозначность решения обеих проблем.