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ON A CERTAIN BOUNDARY VALUE PROBLEM FOR A FOURTH-ORDER ITERATED DIFFERENTIAL EQUATION

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Consider a fourth-order linear homogeneous differential equation of the form

$$Y^{IV}(t) + 10[q(t)\,Y'(t)]' + 3[3q^2(t) + q''(t)]\,Y(t) = 0 \tag{1}$$

with a function $q(t) \in \mathbf{C}^2(-\infty, +\infty)$, $q(t) > 0$ on the interval $\mathbf{I} = (-\infty, +\infty)$, arising by iteration of the differential equation of the second order

$$y''(t) + q(t)\,y(t) = 0 \tag{2}$$

(therefore the differential equation (1) will be also referred to as iterated equation). As we know the basis of this equation is formed by a quadruple of functions

$$[u^3(t), u^2(t)\,v(t), u(t)\,v^2(t), v^3(t)],$$

where $[u(t), v(t)]$ is the basis of (2). Thus the system of all solutions of (1) constitutes a four-parametric space of functions of the form

$$Y(t, C_1, \ldots, C_4) = \sum_{i=1}^{4} C_i u^{4-i}(t)\,v^{i-1}(t), \tag{3}$$

where $C_i \in \mathbf{R}$, $i = 1, \ldots, 4$, are arbitrary parameters and $\sum_{i=1}^{4} C_i^2 > 0$ (trivial solution not being considered).

Suppose the basis $[u(t), v(t)]$ of (2) to be oscillatory, which means that any (nontrivial) solution $y(t)$ of (2) on the interval $\mathrm{I} = (-\infty, +\infty)$ is oscillatory in the sense of [2]. Thus for brevity, we also speak of an oscillatory equation instead of the differential equation (2). It follows from the oscillatority of the basis $[u(t), v(t)]$ of (2) that any solution $Y(t)$ of (1) is oscillatory, i.e. equation (1) is oscillatory. We understand under the oscillatority of any nontrivial solution $y(t)$ [or $Y(t)$] of (2) [of (1)] a solution with infinitely many zeros both on the left and on the right from an arbitrary point $t \in (-\infty, +\infty)$.

In what follows we mean by "a solution of the differential equation" a nontrivial solution, only.

The homogeneous Sturm–Liouville boundary value problem has for the general (oscillatory) linear homogeneous differential equation of the fourth order

$$\sum_{i=0}^{4} p_i(t)\, Y^{(4-i)}(t) = 0,$$

where $p_i(t) \in C(-\infty, +\infty)$, $i = 1, \ldots, 4$, $p_0(t) \neq 0$, the following form

$$\begin{aligned} a_1 Y(t_0) + b_1 Y'(t_0) + c_1 Y''(t_0) + d_1 Y'''(t_0) &= 0, \\ a_2 Y(t_1) + b_2 Y'(t_1) + c_2 Y''(t_1) + d_2 Y'''(t_1) &= 0, \\ a_3 Y(t_2) + b_3 Y'(t_2) + c_3 Y''(t_2) + d_3 Y'''(t_2) &= 0, \\ a_4 Y(t_3) + b_4 Y'(t_3) + c_4 Y''(t_3) + d_4 Y'''(t_3) &= 0, \end{aligned} \qquad (\mathrm{SL}^0)$$

where $a_i, b_i, c_i, d_i \in \mathbf{R}$, $a_i^2 + b_i^2 + c_i^2 + d_i^2 > 0$, $i = 1, \ldots, 4$ and where $t_j \in \in (-\infty, +\infty)$, $j = 0, \ldots, 3$, $t_j \neq t_k$ for $j \neq k$; $j, k = 0, \ldots, 3$.

Thus we obtain the simpliest four-point boundary value problem for $b_i^2 + c_i^2 + + d_i^2 = 0$ (which $a_i \neq 0$, $i = 1, \ldots, 4$), which can be written in one and only one way as

$$Y(t_0) = 0,\ Y(t_1) = 0,\ Y(t_2) = 0,\ Y(t_3) = 0. \qquad (\mathrm{s})_4$$

The one-point up to the three-point boundary value problem of the above type are always solvable for (1). However, this assertion generally fails in case of the four-point boundary value problem.

And yet we sill now show that $(\mathrm{s})_4$ is always solvable for (1) if at least for one pair of mutually distinct points $t_j \in (-\infty, +\infty)$, $j = 0, \ldots, 3$, from the given quadruple (t_0, t_1, t_2, t_3) the relative boundary value problem

$$y(t_j) = 0, \qquad y(t_k) = 0, \qquad (\mathrm{s})_2$$

$j \neq k, j, k = 0, \ldots, 3$, is solvable for (2).

For completeness, let us first go through some more special cases, where with respect to the given quadruple of points $t_0, \ldots, t_3$, the boundary value problem for the corresponding n-tuples, $n \in \{2, 3, 4\}$, elected from the above quadruple, is solvable even for the differential equation (2) and all these cases will obviously be sufficient for the solution of $(\mathrm{s})_4$ relative to (1).

Let us recall only that in all what follows we need not keep to the ordering of points in such a quadruple (or n-tuples considered).

SOLVABILITY OF PROBLEM $(\mathrm{s})_4$

Statements I.–IV. presented at the end of this article immediately follow from our considerations in [3] or and [4] for finding out all possible bundles of solutions from (3) relative to the oscillatory differential equation (1), and their corresponding distribution of zeros (respecting the multiplicities). There is also shown that ayn solution $Y(t)$ from the corresponding bundles of solutions of such an equation may have either all triple zeros or the zeros will be alternately of multiplicity $\nu = 2$ and $\nu = 1$. Or all zeros of such a solution are simple without exception.

First and foremost we want utilize following theorems (or definitions):

Lemma.

Let $t_0 \in (-\infty, +\infty)$ be an arbitrary firmly chosen point. Then any solution $Y(t)$ of (1) vanishing at the point t_0 is the form

1) $Y(t) = C_1u^3(t) + C_2u^2(t)v(t) + C_3u(t)v^2(t)$, $C_3 \neq 0$, exactly if t_0 is a simple zero of the solution $Y(t)$,

2) $Y(t) = C_1u^3(t) + C_2u^2(t)v(t)$, $C_2 \neq 0$, exactly if t_0 is a double zero of the solution $Y(t)$,

3) $Y(t) = C_1u^3(t)$, $C_1 \neq 0$, exactly if t_0 is a triple zero of the solution $Y(t)$,
where $[u(t), v(t)]$ is such a basis of (2) that $u(t_0) = 0$.

Definition 1.

Let $t_0 \in (-\infty, +\infty)$ be an arbitrary firmly chosen point and let $Y(t)$ be an arbitrary solution of the oscillatory differential equation (1), vanishing at the point t_0 (we shall use the symbol ${}^{\nu}t_0$ to denote that the point t_0 is of multiplicity $\nu = 1, 2, 3$).

Then the n^{th} ($n = 1, 2, \dots$) zero of $Y(t)$ lying on the right [on the left] from the point ${}^{\nu}t_0$ ($\nu = 1, 2, 3$) will be called the n^{th} conjugate point from the right [from the left] to the point ${}^{\nu}t_0$. We indicate this by writing ${}^{\mu}t_n$ [or ${}^{\mu}t_{-n}$], where $\mu = 1, 2, 3$ denotes an appropriate multiplicity of this point.

Theorem 1.

Let ${}^{\nu}t_0 \in (-\infty, +\infty)$, $\nu = 1, 2, 3$, be an arbitrary firmly chosen point and let $Y(t)$ be such a solution of (1) where the point ${}^{\nu}t_0$ is its ν-multiple zero.

Then it holds:

1) Any $|k|$-*th* conjugate point ${}^{\mu}t_k$ ($k = \pm 1, \pm 2, \dots$) to the point ${}^{3}t_0$ is uniquely determined, whereby $\mu = 3$. At the same time there holds the inequality

$$ {}^{3}t_k < {}^{3}t_{k+1} . $$

2) Any $2|k|$-*th* conjugate point ${}^{\mu}t_{2k}$ ($k = \pm 1, \pm 2, \dots$) to the point ${}^{2}t_0$ is uniquely determined, whereby $\mu = 2$ and the set of all $|2k + 1|$-*th* points ${}^{\mu}t_{2k+1}$ ($k = \pm 1, \pm 2, \dots$), conjugate to the point ${}^{2}t_0$ forms an open interval $({}^{2}t_{2k}, {}^{2}t_{2k+2})$ where $\mu = 1$ and there hold the inequalities

$$ {}^{2}t_{2k} < {}^{1}t_{2k+1} < {}^{2}t_{2k+2} . $$

3a) If the first conjugate point ${}^{\mu}t_1$ to the point ${}^{1}t_0$ is uniquely determined, then any arbitrary $|k|$-*th* conjugate point ${}^{\mu}t_k$ ($k = \pm 1, \pm 2, \dots$) is uniquely determined as well, whereby $\mu = 1$. There holds the inequality

$$ {}^{1}t_k < {}^{1}t_{k+1} . $$

b) If a set of all first conjugate doints ${}^{\mu}t_1$ to the point ${}^{1}t_0$ forms an open interval, where $\mu = 2$, then any arbitrary $2 \mid k \mid$-*th* point ${}^{\varepsilon}t_{2k}$ ($k = \pm 1, \pm 2, \ldots$), conjugate to ${}^{1}t_0$ is uniquely determined, whereby $\varepsilon = 1$. The set of all $\mid 2k + 1 \mid$-*st* points ${}^{\varepsilon}t_{2k+1}$ ($k = \pm 1, \pm 2, \ldots$) conjugate to the point ${}^{1}t_0$ forms an open interval (${}^{1}t_{2k}$, ${}^{1}t_{2k+2}$), whereby $\varepsilon = 2$ and there hold the inequalities

$$ {}^{1}t_{2k} < {}^{2}t_{2k+1} < {}^{1}t_{2k+2}. $$

c) If a set of all first conjugate points ${}^{\mu}t_1$ to the point ${}^{1}t_0$ forms an open interval where $\mu = 1$, then any arbitrary $3 \mid k \mid$-*th* conjugate point ${}^{\varepsilon}t_{3k}$ ($k = \pm 1, \pm 2, \ldots$) to the point ${}^{1}t_0$ is uniquely determined, where $\varepsilon = 1$ and the set of all $\mid 3k + 1 \mid$-*st* conjugate points ${}^{\varepsilon}t_{3k+1}$ ($k = \pm 1, \pm 2, \ldots$) to the point ${}^{1}t_0$ forms an open interval (${}^{1}t_{3k}$, ${}^{1}t_{3k+2}$), where $\varepsilon = 1$, and the set of all $\mid 3k + 2 \mid$-*nd* conjugate points ${}^{\varepsilon}t_{3k+2}$ ($k = \pm 1, \pm 2, \ldots$) to the point ${}^{1}t_0$ forms an open interval (${}^{1}t_{3k+1}$, ${}^{1}t_{3k+3}$), where $\varepsilon = 1$ and there hold the inequalities

$$ {}^{1}t_{3k} < {}^{1}t_{3k+1} < {}^{1}t_{3k+2} < {}^{1}t_{3k+3}. $$

Definition 2.

Let the points ${}^{\nu}t_0$, ${}^{\mu}t_k \in (-\infty, +\infty)$, where ν, $\mu \in \{1, 2, 3\}$, $k = \pm 1, \pm 2, \ldots$, be conjugate points of a solution $Y_0(t)$ relative to (1).

We say that the point ${}^{\mu}t_k$ is a strongly conjugate point to the point ${}^{\nu}t_0$ exactiy if all solutions $Y(t)$ relative to (1) vanishing ν-times at the point ${}^{\mu}t_0$, are vanishing at the point ${}^{\mu}t_0$ as well.

Any conjugate point to the point ${}^{\nu}t_0$, being not a strongly conjugate point to ${}^{\nu}t_0$ will be called a weakly conjugate point to ${}^{\nu}t_0$.

Remark. It holds by the above definition: The point $t_k^* \in (-\infty, +\infty)$, $k = \pm 1, \pm 2, \ldots$, is a weakly conjugate point to ${}^{\nu}t_0 \in (-\infty, +\infty)$, where $\nu \in \{1, 2, 3\}$, exactly if among all solutions $Y(t)$ relative to (1) vanishing ν-times at ${}^{\nu}t_0$ there exist at least two solutions such that one of these vanishes at t_k^*, while the other does not.

Theorem 2.

Let ${}^{\nu}t_0$, ${}^{\mu}t_k \in (-\infty, +\infty)$, where $\nu, \mu \in \{1, 2, 3\}$, $k = \pm 1, \pm 2, \ldots$, be two conjugate points of the solution $Y(t)$ relative to (1).

Then the point ${}^{\mu}t_k$ is a strongly conjugate point to ${}^{\nu}t_0$ exactly if

1) either $\nu = \mu = 3$, $k = \pm 1, \pm 2, \ldots$,

2) or $\nu = \mu = 2$ and $k = 2m$, $m = \pm 1, \pm 2, \ldots$,

3) or $\nu = \mu = 1$ and

a) $k = 3m$, $m = \pm 1, \pm 2, \ldots$, if there exist simple weakly conjugate points to the points ${}^{\nu}t_0$, ${}^{\mu}t_k$,

b) $k = 2m$, $m = \pm 1, \pm 2, \ldots$, if there exist double weakly conjugate points to the points ${}^{\nu}t_0$, ${}^{\mu}t_k$,

c) $k = m$, $m \approx \pm 1, \pm 2, \ldots$, if there does not exist weakly conjugate points to the points ${}^{\nu}t_0$, ${}^{\mu}t_k$.

Theorem 3.

Let ${}^{\nu}t^*$, ${}^{\nu}t^{**} \in (-\infty, +\infty)$, where $\nu = 1, 2, 3$, be two arbitrary neighbouring strongly conjugate points of the solution $Y(t)$ relative to (1).

Then there may between ${}^{\nu}t^*$, ${}^{\nu}t^{**}$ lie at most two weakly conjugate points of the solution $Y(t)$, i.e. either none or exactly one or exactly two:

1) if $\nu = 3$, then there lies no weakly conjugate point of the solution $Y(t)$ between ${}^3t^*$, ${}^3t^{**}$,

2) if $\nu = 2$, then there always lies exactly one and namely a simple weakly conjugate point of the solution $Y(t)$ between ${}^2t^*$, ${}^2t^{**}$,

3) if $\nu = 1$, then there lies either no weakly conjugate point between ${}^1t^*$, ${}^1t^{**}$, or there lies exactly one and namely double weakly conjugate point, or there lie exactly two distinct points and namely simple weakly conjugate points of the solution $Y(t)$.

A fuller account of the distribution of all weakly conjugate points of an arbitrary solution $Y(t)$ (together with the corresponding bundle of such solutions) relative to (1) vanishing either at simple or at double points ${}^{\nu}t \in (-\infty, +\infty)$, $\nu = 1, 2$, is given by the following

Theorem 4.

Let $k = 0, \pm 1, \pm 2, \ldots$

1) Let $Y(t)$ be a solution relative to (1) vanishing at double strongly conjugate points. Then we can write for an arbitrary simple weakly conjugate point at which this solution vanishes

$$ {}^1t_{2k+1} \in ({}^2t_{2k}, {}^2t_{2k+2}), $$

where ${}^2t_{2k}$, ${}^2t_{2k+2}$ are two neighbouring mutually strongly conjugate points of this solution.

2) Let $Y(t)$ be a solution relative to (1) vanishing at simple strongly conjugate points. Then

a) it holds for an arbitrary double weakly conjugate point at which this solution vanishes that

$$ {}^2t_{2k+1} \in ({}^1t_{2k}, {}^1t_{2k+2}), $$

where ${}^1t_{2k}$, ${}^1t_{2k+2}$ are two neighbouring mutually strongly conjugate points of this solution,

b) it holds for a simple weakly conjugate point at which this solution vanishes either

$$ {}^1t_{3k+1} \in ({}^1t_{3k}, {}^1t_{3k+2}) \subset ({}^1t_{3k}, {}^1t_{3k+3}) $$

or

$$ {}^1t_{3k+2} \in ({}^1t_{3k+1}, {}^1t_{3k+3}) \subset ({}^1t_{3k}, {}^1t_{3k+3}), $$

where ${}^1t_{3k}$, ${}^1t_{3k+3}$ are two neighbouring mutually strongly conjugate points of this solution.

The general survey of zeros of the three-parametric bundle $Y(t, C_1, C_2, C_3)$ of solutions relative to (1) arising from the analysis of its algebraic structure is given by

Theorem 5.

Let $t_0 \in (-\infty, +\infty)$ be an arbitrary firmly chosen point. Consider the bundle

$$Y(t, C_1, C_2, C_3) = u(t)\,[C_1u^2(t) + C_2u(t)\,v(t) + C_3v^2(t)], \tag{S$_3$}$$

$C_i \in \mathbf{R}$, $i = 1, 2, 3$, $C_3 \neq 0$, of all solutions relative to (1), where $[u(t), v(t)]$ is a basis of the oscillatory differential equation (2) satisfying the condition

$$u(t_0) = 0, \qquad v'(t_0) = 0 \tag{P}$$

at the point t_0 [so that $u'(t_0) \neq 0$, $v(t_0) \neq 0$ and thus the point t_0 is a simple zero of the function $u(t)$]. Let T_1 denote the neighbouring zero of the function $u(t)$ lying on the right from t_0 [so that t_0, T_1 are neighbouring strongly conjugate points of an arbitrary solution $Y(t)$ relative to (1) from the bundle (S_3)].

Then

1) the sub-bundle of all solutions relative to (1) (up to an arbitrary nonzero multiplicative constant) exactly of the form

$$Y(t, C_1', C_2') = u(t)\,y_1^2(t, C_1', C_2'), \tag{S$_{31}$}$$

where $y_1(t, C_1', C_2') = C_1'u(t) + C_2'v(t)$, $C_i' \in \mathbf{R}$, $i = 1, 2$, $C_2' \neq 0$ stands for the double-parametric system of all solutions relative to (2) on the interval $(-\infty, +\infty)$ linearly independent of the function $u(t)$ corresponds to the condition

$$C_2^2 - 4C_1C_3 = 0.$$

Any solution from this system has in interval (t_0, T_1) exactly one zero, which is the double weakly point of the sub-bundle (S_{31}) of the solutions relative to (1),

2) the sub-bundle of all solutions relative to (1) (up to an arbitrary nonzero multiplicative constant) exactly of the form

$$Y(t, C_1', C_2', C_1'', C_2'') = u(t)\,y_1(t, C_1', C_2')\,y_2(t, C_1'', C_2''), \tag{S$_{32}$}$$

where $y_1(t, C_1', C_2') = C_1'u(t) + C_2'v(t)$, $y_2(t, C_1'', C_2'') = C_1''u(t) + C_2''v(t)$, $C_i', C_i'' \in \mathbf{R}$, $i = 1, 2$, $C_1'C_2'' \neq 0$, $C_1'C_2'' - C_2'C_1'' \neq 0$, stand for two double-parametric systems of all solutions relative to (2) such that any two functions from the three functions $u(t)$, $y_1(t)$, $y_2(t)$ are on the interval $(-\infty, +\infty)$ linearly independent corresponds to the condition

$$C_2^2 - 4C_1C_3 > 0.$$

Each of the two solutions $y_1(t)$, $y_2(t)$ [from the systems $y_1(t, C_1', C_2')$, $y_2(t, C_1'', C_2'')$ respectively, in an arbitrary choice of the constants C_i', $C_i'' \in \mathbf{R}$, $i = 1, 2$, satisfying the given conditions] has exactly in the interval (t_0, T_1) one zero, each of which is a simple weakly conjugate point of the sub-bundle (S_{32}) of the solutions relative to (1),

3) the sub-bundle of all solutions relative to (1) (up to an arbitrary nonzero multiplicative constant) exactly of the form

$$Y(t, C_1', C_2', C_1'', C_2'') = u(t)\, y^*(t, C_1', C_2', C_1'', C_2''), \qquad (S_{33})$$

where the four-parametric system of functions $y^*(t, C_1', C_2', C_1'', C_2'')$ stands for the sum of squares of the two linearly independent double-parametric systems of solutions $y_1(t, C_1', C_2') = C_1'u(t) + C_2'v(t)$, $y_2(t, C_1'', C_2'') = C_1''u(t) + C_2''v(t)$, $C_i', C_i'' \in \in \mathbf{R}$, $i = 1, 2$, $C_1'C_2'' - C_1''C_2' \neq 0$ (so that $C_i'^2 + C_i''^2 > 0$, $i = 1, 2$), relative to (2) having no zero on the interval $(-\infty, +\infty)$ corresponds to the condition

$$C_2^2 - 4C_1C_3 < 0.$$

In such a case the sub-bundle (S_{33}) of the solutions relative to (1), whose single and namely simple strongly conjugate points are the zeros of the function $u(t)$ has no weakly conjugate points.

STATEMENTS ON SOLVABILITY OF THE PROBLEM (s_4)

Statement I.

Let there be a solution $y_0(t)$ of the differential equation (2) for which

$$y_0(t_0) = y_0(t_1) = y_0(t_2) = y_0(t_3) = 0$$

holds [so that the four-point problem (s_4) is solvable even for (2)].

Then the problem (s_4) is solvable for (1) by means of any arbitrary function $Y(t)$ of the form

1) $Y_1(t) = y_0^3(t)$, whose all zeros $t_0, \ldots, t_3$ are always triple,

2) $Y_2(t) = y_0^2(t)\, y_1(t)$, where $y_1(t)$ is an arbitrary solution of (2) linearly independent of the solution $y_0(t)$ on the interval $(-\infty, +\infty)$ [hereafter in short: $y_1(t)\,\mathrm{N}\,y_0(t)$]. All points $t_0, \ldots, t_3$ are here double zeros of the solution $Y_2(t)$ and (according to the Sturm's theorem) they are separating themselves with the simple zeros of the function $y_1(t)$.

3) $Y_3(t) = y_0(t)\, y_1(t)\, y_2(t)$, where $y_i(t)$, $i = 1, 2$, are two arbitrary mutually linearly independent solutions of (2), each of them being a linearly independent of the function $y_0(t)$ on the interval $(-\infty, +\infty)$, i.e. $y_0(t)\,\mathrm{N}\,y_1(t)$, $y_0(t)\,\mathrm{N}\,y_2(t)$, $y_1(t)\,\mathrm{N}\,y_2(t)$, so that all points $t_0, \ldots, t_3$ are simple zeros of the solution $Y_3(t)$. Between any arbitrary two of these there always lies at least per one zero of each from both functions $y_i(t)$, $i = 1, 2$, being also mutually separating,

b) $Y_3(t) = y_0(t)\, y_1^2(t)$ is an arbitrary solution of (2), linearly independent of the solution $y_0(t)$ on the interval $(-\infty, +\infty)$ [i.e. $y_1(t)\,\mathrm{N}\,y_0(t)$]; all points $t_0, \ldots, t_3$ are simple zeros of the solution $Y_3(t)$ and (according to the Sturm's theorem) they are mutually separating with the zeros of the function $y_1(t)$, which are the double zeros of the solution $Y_3(t)$ relative to (1). Let us observe that the case 3) b) is dual to the case 2).

c) $Y_3(t) = y_0(t)[y_1^2(t) + y_2^2(t)]$, where $y_i(t)$, $i = 1, 2$, are two arbitrary mutually independent solutions of (2) [i.e. $y_1(t) \mathrm{N} y_2(t)$], so that all points $t_0, \ldots, t_3$ are simple zeros of the solution $Y_3(t)$. All zeros of the solution $Y_3(t)$ relative to (1) coincide exactly with all—simple—zeros of the solution $y_0(t)$ relative to (2) since for all $t \in (-\infty, +\infty)$ there holds

$$y_1^2(t) + y_2^2(t) > 0.$$

Statement II.

Let there be a solution $y_0(t)$ relative to (2) for which

$$y_0(t_0) = y_0(t_1) = y_0(t_2) = 0, \qquad y_0(t_3) \neq 0$$

holds [so that there certainly exists the solution $y_1(t)$ relative to (2)—and even a whole bundle of such solutions mutually distinct by a multiplicative nonzero constant—such that there holds: $y_1(t) \mathrm{N} y_0(t)$ on $(-\infty, +\infty)$ with $y_1(t_3) = 0$].

Then the problem (s_4) is solvable for the differential equation (1) by means of an arbitrary function $Y(t)$ of the form

1) $Y_1(t) = y_0^2(t) y_1(t)$, for which all three points t_0, t_1, t_2 are double zeros, while the point t_3 is its simple zero. All zeros of both functions $y_0(t)$, $y_1(t)$ are mutually separating on $(-\infty, +\pm)$ [so that there always lies at least one simple zero of the function $y_1(t)$ between the double zeros as t_0, t_1 as t_1, t_2].

2) a) $Y_2(t) = y_0(t) y_1(t) y_2(t)$, where $y_2(t)$ is a further arbitrary solution of (2), in which besides $y_0(t) \mathrm{N} y_1(t)$ also $y_0(t) \mathrm{N} y_2(t)$ and $y_1(t) \mathrm{N} y_2(t)$ holds. All points t_0, t_1, t_2, t_3 are simple zeros of the solution $Y_2(t)$. All zeros of the functions $y_0(t)$, $y_1(t)$, $y_2(t)$ are mutually separating on the interval $(-\infty, +\infty)$, so that between arbitrary two neighbouring zeros of each of these there always lies exactly one zero of the remaining two functions. Hence, there always lies at least per one zero of each from both functions $y_1(t)$ and $y_2(t)$ as between the points t_0, t_1 as between t_1, t_2.

b) $Y_2(t) = y_0(t) y_1^2(t)$, for which all three points t_0, t_1, t_2 are simple zeros and the point t_3 is its double zero [so that the case 2) b) is dual to the case 1)]. All zeros of both functions $y_0(t)$, $y_1(t)$ are mutually separating on $(-\infty, +\infty)$. Thus there always lies at least one double zero of the solution $Y_2(t)$ relative to (1) between the simple zeros as t_0, t_1 as t_1, t_2. Likewise, there lies at least one simple zero between two arbitrary double zeros of the solution $Y_2(t)$.

Statement III.

Let there be a solution $y_0(t)$ relative to (2) for which

$$y_0(t_0) = 0, \quad y_0(t_1) = 0, \quad y_0(t_i) \neq 0,$$

$i = 2, 3$, holds and a solution $y_1(t)$ relative to (2), $y_1(t) \mathrm{N} y_0(t)$ on $(-\infty, +\infty)$ such that

$$y_1(t_2) = 0, \qquad y_1(t_3) = 0$$

(so that $y_1(t_j) \neq 0$, $j = 0, 1$).

Then the problem (s_4) is solvable for (1) by an arbitrary function $Y(t)$ as the solution of (1) of the form

1) $Y_1(t) = y_0^2(t)\, y_1(t)$, for which both points t_0, t_1 are double zeros, while both points t_2, t_3 are its neighbouring simple zeros. As all zeros of both functions $y_0(t)$, $y_1(t)$ are mutually separating on $(-\infty, +\infty)$, there always lies at least one simple zero of the function $y_1(t)$ between the double zeros t_0, t_1. At the same time there lies always one double zero of the solution $Y_1(t)$ relative to (1) between the simple zeros t_2, t_3,

2) a) $Y_2(t) = y_0(t)\, y_1(t)\, y_2(t)$, where $y_2(t)$ is a further arbitrary solution relative to (2) in which besides $y_0(t) \mathrm{N}\, y_1(t)$ also $y_0(t) \mathrm{N}\, y_2(t)$ and $y_1(t) \mathrm{N}\, y_2(t)$ holds. All four points t_0, t_1, t_2, t_3 are simple zeros of the solution $Y_2(t)$ relative to (1). All zeros of the functions $y_0(t)$, $y_1(t)$, $y_2(t)$ are mutually separating on $(-\infty, +\infty)$, so that there always lies at least one zero as of the function $y_1(t)$ as of $y_2(t)$ between the points t_0, t_1. Likewise, there always lies at least one zero as of the function $y_0(t)$ as of $y_2(t)$ between the points t_2, t_3,

b) $Y_2(t) = y_0(t)\, y_1^2(t)$, for which both points t_0, t_1 are simple zeros, while both points t_2, t_3 are its double zeros [both cases 1) and 2) b) are mutually dual again]. It follows from the mutual separation of all zeros of $y_0(t)$, $y_1(t)$ on $(-\infty, +\infty)$ that there always lies at least one double [or simple] zero between an arbitrary two simple [or double] zeros of the solution $Y_2(t)$ relative to (1).

Statement IV.

Let there be a solution $y_0(t)$ relative to (2) for which

$$y_0(t_0) = 0, \; y_0(t_1) = 0, \; y_0(t_i) \neq 0,$$

$i = 2, 3$, holds and a solution $y_1(t)$ relative to (2), $y_1(t) \mathrm{N}\, y_0(t)$ on $(-\infty, +\infty)$ such that

$$y_1(t_2) = 0, \qquad y_1(t_3) \neq 0$$

[so that also $y_1(t_j) \neq 0, j = 0, 1$].

Then there certainly exists a solution $y_2(t)$ relative to (2) such that $y_2(t) \mathrm{N}\, y_0(t)$ and at the same time $y_2(t) \mathrm{N}\, y_1(t)$ on $(-\infty, +\infty)$ for which

$$y_2(t_3) = 0 \qquad [\text{and of course } y_2(t_k) \neq 0, k = 0, 1, 2].$$

In the above case is the problem (s_4) relative to (1) solvable by the function (up to an arbitrary non-zero multiplicative constant) exactly of the form

$$Y(t) = y_0(t)\, y_1(t)\, y_2(t),$$

so that all four points t_0, t_1, t_2, t_3 are its simple zeros. It follows from the fact that all zeros of the three functions $y_0(t)$, $y_1(t)$, $y_2(t)$ [being in pairs linear independent solutions of (2)] are mutually separating on $(-\infty, +\infty)$ that there lies at least one zero of each from both functions $y_1(t)$, $y_2(t)$ between the points t_0, t_1. At the same time there lies at least one zero of the function $y_2(t)$ [or $y_1(t)$, or $y_0(t)$] between the points t_0, t_2 [or t_0, t_3, or t_2, t_3].

Remark. In case that each of the points of the quadruple $t_0, t_1, t_2, t_3 \in \in (-\infty, +\infty)$, $t_i \neq t_j$, $i, j = 0, \ldots, 3$, is a zero always of one out of the four in pairs mutually linearly independent solutions $y_i(t)$, $i = 0, \ldots, 3$, on $(-\infty, +\infty)$ relative to (2), i.e. if $y_i(t) \, \mathrm{N} \, y_j(t)$, $i \neq j$, $i, j = 0, \ldots, 3$, the problem (s_4) relative to (1) is unsolvable.

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SOUHRN

POZNÁMKA K JEDNÉ OKRAJOVÉ ÚLOZE PRO ITEROVANOU DIFERENCIÁLNÍ ROVNICI 4. ŘÁDU

VLADIMÍR VLČEK

Práce se týká řešení homogenního čtyřbodového okrajového problému Sturm–Liouvilleova typu pro iterovanou obyčejnou lineární diferenciální rovnici 4. řádu. Ve čtyřech tvrzeních jsou ukázány všechny možné tvary svazků řešení, které se (s ohledem na násobnosti) anulují v dané čtveřici bodů za předpokladu, že se v těchto bodech anulují příslušná lineárně nezávislá řešení obyčejné lin. homogenní diferenciální rovnice 2. řádu, z níž uvažovaná rovnice vznikla iterací.

РЕЗЮМЕ

ЗАМЕЧАНИЕ ОБ ОДНОЙ КРАЕВОЙ ЗАДАЧЕ ДЛЯ ИТЕРИРОВАННОГО ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ 4-го ПОРЯДКА

ВЛАДИМИР ВЛЧЕК

Работа занимается решением однородной четырехточечной краевой задачи типа Штурма-Лиувилля для итерированного обыкновенного линейного дифференциального уравнения 4-го порядка.

В четырех утверждениях показаны всевозможные пучки решений, которые (взглядом к насобностьям) аннулируются в заданной четверке точек только в том предположении, что в этих точках аннулируются надлежащие линейно независимые решения обыкновенного линейного дифференциального уравнения 2-го порядка, из которого выше приведенное уравнение возникло после итерации.