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LINEAR DIFFERENTIAL EQUATIONS OF THE SECOND ORDER WITH ELEMENTARY BASIC CENTRAL DISPERSIONS

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1. Introduction

This paper investigates differential equations of the type

$$y'' = q(t)\, y, \qquad q \in C^0(\mathbf{R}), \tag{q}$$

($\mathbf{R} := (-\infty, \infty)$) being oscillatory on $\mathbf{R}$ (i.e. $\pm\infty$ are the cluster points of zeros of any (nontrivial) solution of (q)). Equation (q) is often assumed such that its basic central dispersion is elementary. This property occurs e.g. in equation (q) whose coefficient q is a π-periodic function (on $\mathbf{R}$). Naturally, not every equation (q) with an elementary basic central dispersion has a π-periodic coefficient. The object of this paper is to investigate conditions under which (q) with an elementary basic central dispersion has a π-periodic coefficient q.

2. Basic concepts and properties

In all what follows the equations of type (q) are assumed to be oscillatory on $\mathbf{R}$. Trivial solutions of these equations are excluded from our considerations.

A function $\alpha \in C^0(\mathbf{R})$ is called the (first) phase of (q) if there exist independent solutions u, v of (q) such that

$$\operatorname{tg} \alpha(t) = u(t)/v(t) \qquad \text{for } t \in \mathbf{R} - \{t \in \mathbf{R};\, v(t) = 0\}.$$

The phase α of (q) is its coefficient q uniquely determined by the formula $q(t) =$

$= -\{\alpha, t\} - \alpha'^2(t)$, $t \in \mathbf{R}$, where $\{\alpha, t\} := \alpha'''(t)/(2\alpha'(t)) - (3/4)(\alpha''(t)/\alpha'(t))^2$ is the Schwarz derivative of α at the point t.

Every phase α of (q) possesses the following properties: (i) $\alpha \in C^3(\mathbf{R})$; (ii) $\alpha'(t) \neq 0$ for $t \in \mathbf{R}$; (iii) $\alpha(\mathbf{R}) = \mathbf{R}$.

The set of all functions α possessing the properties (i)–(iii) forms a group $\mathfrak{G}$ with respect to the composition of functions, called the group of phases. The group of phases $\mathfrak{G}$ has two important subgroups $\mathfrak{E}$ and $\mathfrak{H}$, $\mathfrak{E} \subset \mathfrak{H} \subset \mathfrak{G}$. The subgroup $\mathfrak{E}$ is constituted by phases of $y'' = -y$. The elements of the subgroup $\mathfrak{H}$ are elementary phases, i.e. those elements $\alpha \in \mathfrak{G}$ for which $\alpha(t + \pi) = \alpha(t) + \pi \operatorname{sign} \alpha'$, $t \in \mathbf{R}$.

If α is a phase of (q), then $\mathfrak{E}\alpha := \{\varepsilon\alpha;\ \varepsilon \in \mathfrak{E}\}$ is the set of phases of (q) and the coefficient q of (q) is a π-periodic function iff $\alpha(t + \pi) = \varepsilon\alpha(t)$, where $\varepsilon \in \mathfrak{E}$.

Let n be an integer, α be a phase of (q). Let α^{-1} denote the inverse function to the function α. The function $\varphi_n(t) := \alpha^{-1}(\alpha(t) + n\pi \operatorname{sign} \alpha')$, $t \in \mathbf{R}$, is called the central dispersion (of the first kind) of (q) with the index n. The function φ_1 will be written briefly φ and will be called the basic central dispersion of (q). We say that φ is an elementary basic central dispersion of (q) if $\varphi \in \mathfrak{H}$.

Let α be a phase of (q) and let φ be its basic central dispersion. Then $\mathfrak{H}\alpha := \{c\alpha;\ c \in \mathfrak{H}\}$ is the set of phases exactly of those equations having the basic central dispersion equal to φ. The above definitions and results may be found in [1], [2].

Lemma 1 ([3]). *The basic central dispersion φ of* (q) *is elementary iff φ is the basic central dispersion of*

$$y'' = q(t + \pi)\, y \tag{1}$$

as well.

It immediately follows from Lemma 1 that every equation (q) with a π-periodic coefficient q possesses an elementary basic central dispersion.

Let the coefficient q of (q) be a π-periodic function. Following the Floquet theory equation (q) may be associated with a certain quadratic equation, whose roots are called the characteristic multipliers of (q).

Lemma 2 ([2]–[6]). *Let φ be the basic central dispersion of* (q) *with a π-periodic coefficient q. Then:*

a) *all solutions of* (q) *are π-periodic* (*π-halfperiodic*) *iff*

$$\varphi_n(t) = t + \pi, \qquad t \in \mathbf{R},$$

where n is an even (*an odd*) *number;*

b) *equation* (q) *has real characteristic multipliers and all its solutions are not π-periodic or π-halfperiodic function iff there exist an $x \in \mathbf{R}$ and a positive integer n such that $\varphi_n(x) = x + \pi$ and $\varphi_n(t) \not\equiv t + \pi$ for $t \in \mathbf{R}$;*

c) *equation* (q) *has complex characteristic multipliers equal to $e^{\pm a\pi i}$, $a \in (0, 1)$, iff there exists a phase α of* (q) *and an integer n such that*

$$\alpha(t + \pi) = \alpha(t) + (2n + a)\,\pi \qquad \textit{for } t \in \mathbf{R}.$$

3. Main results

Theorem 1. *Let the oscillatory equation* (q) *have the elementary basic central dispersion. Then the function q is π-periodic iff the function* $p(t) := q(t + \pi) - q(t)$, $t \in \mathbf{R}$, *is π-periodic.*

Proof. Let the oscillatory equation (q) have the elementary basic central dispersion φ and let the function $p(t)$ be π-periodic. Then by Lemma 1 the equations (q) and (1) have the same basic central dispersion and there exists a $c \in \mathfrak{H}$ such that

$$\alpha(t + \pi) = c\alpha(t), \qquad t \in \mathbf{R}, \tag{2}$$

where α is a phase of (q). Let c be a phase of (g). Then we have from (2)

$$\begin{aligned} q(t + \pi) &= -\{\alpha, t + \pi\} - \alpha'^2(t + \pi) = -\{c\alpha, t\} - c'^2\alpha(t) \,.\, \alpha'^2(t) = \\ &= -\{c, \alpha(t)\}\, \alpha'^2(t) - \{\alpha, t\} - c'^2\alpha(t) \,.\, \alpha'^2(t) = \\ &= q(t) + (1 + g\alpha(t))\, \alpha'^2(t). \end{aligned}$$

Now, on our assumption, the function $(1 + g\alpha(t))\, \alpha'^2(t)$ is π-periodic on $\mathbf{R}$. Thus

$$\begin{aligned} (1 + g\alpha(t))\, \alpha'^2(t) &= (1 + g\alpha(t + \pi))\, \alpha'^2(t + \pi) = \\ &= (1 + gc\alpha(t))\, c'^2\alpha(t) \,.\, \alpha'^2(t), \end{aligned}$$

hence

$$(1 + gc(t))\, c'^2(t) = 1 + g(t), \qquad t \in \mathbf{R}. \tag{3}$$

Let c^{-1} be the inverse function to the function c. Then $c^{-1} \in \mathfrak{H}$. Let c^{-1} be a phase of (h). Then

$$\begin{aligned} -1 &= -\{c^{-1}c, t\} - c^{-1'^2}c(t) \,.\, c'^2(t) = -\{c^{-1}, c(t)\}\, c'^2(t) - \\ &- \{c, t\} - c^{-1'^2}c(t) \,.\, c'^2(t) = g(t) + (1 + hc(t))\, c'^2(t), \end{aligned}$$

hence

$$-1 = g(t) + (1 + hc(t))\, c'^2(t), \qquad t \in \mathbf{R}. \tag{4}$$

It follows from (3) and (4) that

$$(1 + gc(t))\, c'^2(t) = -(1 + hc(t))\, c'^2(t),$$

whence we get

$$g(t) = -2 - h(t), \qquad t \in \mathbf{R}. \tag{5}$$

It was shown in [7] that for the coefficient k of (k) having the basic central dispersion equal to $t + \pi$ the following holds: $\int_0^\pi k(t)\, \mathrm{d}t \geqq -\pi$ and $\int_0^\pi k(t)\, \mathrm{d}t = -\pi$ iff $k(t) = -1$ for $t \in \mathbf{R}$. From here and from (5) we get

$$-\pi \leqq \int_0^\pi g(t)\, \mathrm{d}t = -2\pi - \int_0^\pi h(t)\, \mathrm{d}t,$$

hence

$$\int_0^\pi h(t)\, \mathrm{d}t \leqq -\pi.$$

Thus, necessarily $h(t) = -1$ for $t \in \mathbf{R}$ and we get from (5) that $g(t) = -1$ for $t \in \mathbf{R}$. Consequently $c \in \mathfrak{E}$ and q is necessarily a π-periodic function as it follows from (2).

The proof of Theorem 1 is obvious in the opposite direction.

Theorem 2. *Let the equation* (q) *be oscillatory on* $\mathbf{R}$ *and let* q *be a* π*-periodic function. Let* φ *be the basic central dispersion of* (q). *It holds:*

(i) *if there exists a positive integer* n *such that* $\varphi_n(t) = t + \pi$ *for* $t \in \mathbf{R}$, *then the coefficients* p *of all equations* (p) *with the basic central dispersion equal to* φ *are* π*-periodic;*

(ii) *if the characteristic multipliers of* (q) *are real and for any positive integer* n *is* $\varphi_n(t) \not\equiv t + \pi$ *for* $t \in \mathbf{R}$, *then the coefficient* p $(\neq q)$ *of* (p) *having the basic central dispersion equal to* φ, *is not* π*-periodic;*

(iii) *if the equation* (q) *possesses complex characteristic multipliers equal to* $e^{\pm a\pi i}$, $0 < a < 1$, *and*

a) *a is an irrational number, then the coefficient* $p(\neq q)$ *of* (p) *having the basic central dispersion equal to* φ, *is not* π*-periodic,*

b) *a is a rational number, then there exist infinitely many various coefficients* p *of* (p) *having the basic central dispersion equal to* φ *and are* π*-periodic.*

Proof. Let the assumptions of Theorem 2 be satisfied.

(i) Let $\varphi_n(t) = t + \pi$, $t \in \mathbf{R}$, where n is a positive integer. Let α be a phase of (q), $c \in \mathfrak{H}$. Then $c\alpha(t + \pi) = c\alpha\varphi_n(t) = c(\alpha(t) + n\pi \operatorname{sign} \alpha') = c\alpha(t) + n\pi \operatorname{sign} \alpha' . \operatorname{sign} c'$. Let us put $\gamma(t) := c\alpha(t)$, $\varepsilon(t) := t + n\pi \operatorname{sign} \alpha' . \operatorname{sign} c'$, $t \in \mathbf{R}$. Then $\varepsilon \in \mathfrak{E}$ and $\gamma(t + \pi) = \varepsilon\gamma(t)$. Thus the coefficients p of equation (p) having the basic central dispersion equal to φ are π-periodic functions.

(ii) Let $\varphi_n(t) \not\equiv t + \pi$, $t \in \mathbf{R}$, for every positive integer n. Let the characteristic multipliers of (q) be real. Then, by Lemma 2 there exists an $x \in \mathbf{R}$ and a positive integer n: $\varphi_n(x) = x + \pi$. Let α be a phase of (q), sign $\alpha' = 1$. Then $\alpha\varphi(t) = \alpha(t) + \pi$ and $\alpha(t + \pi) = \varepsilon\alpha(t)$ for any $\varepsilon \in \mathfrak{E}$. Let $c \in \mathfrak{H}$ and $c\alpha$ be a phase of (p) and c be a phase of (g). Then

$$p(t) = -\{c\alpha, t\} - c'^2\alpha(t) . \alpha'^2(t) = -\{c, \alpha(t)\}\, \alpha'^2(t) - \\ - \{\alpha, t\} - c'^2\alpha(t) . \alpha'^2(t) = q(t) + (1 + g\alpha(t))\, \alpha'^2(t),$$

hence

$$p(t) = q(t) + (1 + g\alpha(t))\, \alpha'^2(t), \qquad t \in \mathbf{R}. \tag{6}$$

Let us assume that p is a π-periodic function. Then

$$q(t + \pi) + (1 + g\alpha(t + \pi))\, \alpha'^2(t + \pi) = q(t) + (1 + g\alpha(t))\, \alpha'^2(t),$$
$$(1 + g\varepsilon\alpha(t))\, \varepsilon'^2\alpha(t) . \alpha'^2(t) = (1 + g\alpha(t))\, \alpha'^2(t),$$
$$(1 + g\varepsilon(t))\, \varepsilon'^2(t) = 1 + g(t).$$

This implies

$$\varepsilon'(t)\sqrt{|1 + g\varepsilon(t)|} = \sqrt{|1 + g(t)|}, \qquad t \in R. \tag{7}$$

Let us put $x_1 := \alpha(x)$. Then $\alpha(x + \pi) = \varepsilon\alpha(x)$, hence $\varepsilon(x_1) = \alpha(x + \pi) = \alpha\varphi_n(x) = \alpha(x) + n\pi = x_1 + n\pi$. Integrating (7) from x_1 to t gives

$$\int_{\varepsilon(x_1)}^{\varepsilon(t)} \sqrt{|1 + g(s)|}\, ds = \int_{x_1}^{t} \sqrt{|1 + g(s)|}\, ds. \tag{8}$$

Upon substituting $s = u + n\pi$ in the integral on the left side of (8) we obtain

$$\int_{\varepsilon(x_1)}^{\varepsilon(t)} \sqrt{|1 + g(s)|}\, ds = \int_{x_1}^{\varepsilon(t)-n\pi} \sqrt{|1 + g(s)|}\, ds.$$

From this and from (8) we get

$$\int_{x_1}^{\varepsilon(t)-n\pi} \sqrt{|1 + g(s)|}\, ds = \int_{x_1}^{t} \sqrt{|1 + g(s)|}\, ds$$

hence

$$\int_{t}^{\varepsilon(t)-n\pi} |1 + g(s)|\, ds = 0 \qquad \text{for } t \in R. \tag{9}$$

We prove: $g = -1$. In the contrary case there exists a $t_0 \in \mathbf{R} : g(t_0) \neq -1$. Then, of course, we get from (9) that $\varepsilon(t) = t + n\pi$ in any neighbourhood of the point t_0 and from the properties of phases of $y'' = -y$ we get $\varepsilon(t) = t + n\pi$ for $t \in \mathbf{R}$. This implies $\alpha(t + \pi) = \alpha(t) + n\pi$ and $\varphi_n(t) = t + \pi$ for $t \in \mathbf{R}$ which is a contradiction. Thus $g = -1$ and we get from (6) that $p = q$.

(iii) Let (q) have complex characteristic multipliers equal to $e^{\pm a\pi i}$, $0 < a < 1$. Following Lemma 2 there exists a phase α of (q) and an integer n:

$$\alpha(t + \pi) = \alpha(t) + (2n + a)\pi, \qquad t \in \mathbf{R}. \tag{10}$$

Let $c \in \mathfrak{H}$, $c\alpha$ be a phase of (p) and c be a phase of (g). Then (6) is valid. Let us assume that p is a π-periodic function. Then (6) and (10) imply

$$(1 + g(\alpha(t) + a\pi))\, \alpha'^2(t) = (1 + g\alpha(t))\, \alpha'^2(t),$$

whence

$$g(t + a\pi) = g(t), \qquad t \in \mathbf{R}. \tag{11}$$

a) If a is an irrational number, then $g(t) = k$ (= a constant) for g is a continuous function and π and $a\pi$ are its period. Necessarily $k = -1$ and $p = g$ follows from (6);

b) Let a be a rational number and let $\mathfrak{H}_a$ be the set of all those $c \in \mathfrak{H}$ with the property: $c(t + a\pi) = c(t) + a\pi \operatorname{sign} c'$. H_a is clearly a subgroup of the group $\mathfrak{H}$. We see that $c\alpha(t + \pi) = c(\alpha(t) + (2n + a)\pi) = c\alpha(t) + (2n + a)\pi$ holds for every $c \in \mathfrak{H}_a$, sign $c' = 1$. Consequently $\mathfrak{H}_a\alpha := \{c\alpha;\, c \in \mathfrak{H}_a\}$ is the set of all phases of those equations with a π-periodic coefficient having the same basic central dispersion ($= \varphi$). Such equations are infinitely many.

ЛИНЕЙНЫЕ ДИФФЕРЕНЦИАЛЬНЫЕ УРАВНЕНИЯ ВТОРОГО ПОРЯДКА С ЭЛЕМЕНТАРНЫМИ ОСНОВНЫМИ ЦЕНТРАЛЬНЫМИ ДИСПЕРСИЯМИ

Резюме

Пусть

(q) $$y'' = q(t)\, y, \qquad q \in C^0(\mathbf{R}),$$

колеблющееся уравнение, $t_0 \in \mathbf{R}$ и $y(\neq 0)$ — решение уравнения (q), $y(t_0) = 0$. Пусть $\varphi(t_0)$ — первое справа сопряженное число с t_0. Тогда функция φ определена на $\mathbf{R}$ и называется основная центральная дисперсия уравнения (q). Если $\varphi(t + \pi) = \varphi(t) + \pi$, $t \in \mathbf{R}$, тогда φ называется элементарная основная центральная дисперсия уравнения (q).

Пусть q — π — периодическая функция. Уравнение (q) имеет элементарную основную центральную дисперсию φ только тогда, когда функция $p(t) := q(t + \pi) - q(t)$, $t \in \mathbf{R}$, π — периодическая (теорема 1).

Пусть q — π —периодическая функция и φ — основная центральная дисперсия уравнения (q). В теореме 2 исследуется структура всех уравнений типа $y'' = p(t)\, y$, $p \in C^0(\mathbf{R})$, которые имеют основную центральную дисперсию равну φ и p — π — периодическая функция. Эта структура исследуется с помощью свойств характеристических мультипликаторов уравнени (q).

LINEÁRNÍ DIFERENCIÁLNÍ ROVNICE 2. ŘÁDU S ELEMENTÁRNÍMI ZÁKLADNÍMI CENTRÁLNÍMI DISPERSEMI

Souhrn

Nechť

(q) $$y'' = q(t)\, y, \qquad q \in C^0(\mathbf{R}),$$

e oscilatorická rovnice, $t_0 \in \mathbf{R}$ a $y(\neq 0)$ je řešení rovnice (q), $y(t_0) = 0$. Nechť $\varphi(t_0)$ je první zprava od bodu t_0 ležící nulový bod řešení y. Pak funkce φ je definovaná na $\mathbf{R}$ a nazývá se základní centrální disperse rovnice (q). Jestliže $\varphi(t + \pi) = \varphi(t) + \pi$, $t \in \mathbf{R}$, pak φ se nazývá elementární základní centrální disperse rovnice (q).

Nechť q je π- periodická funkce. Rovnice (q) má elementární základní centrální dispersi φ právě když funkce $p(t) := q(t + \pi) - q(t)$, $t \in \mathbf{R}$, je π-periodická (věta 1).

Nechť q je π-periodická funkce a φ je základní centrální disperse rovnice (q). Ve větě 2 je vyšetřována struktura všech rovnic typu $y'' = p(t)\, y$, $p \in C^0(\mathbf{R})$, které mají základní centrální dispersi rovnu φ a mají π-periodický koeficient p. Tato struktura je vyšetřována prostřednictvím charakteristických multiplikátorů rovnice (q).

References

[1] Borůvka, O.: *Linear Differential Transformations of the Second Order*. The English Universities Press, London, 1971.

[2] Борувка, О.: *Теория глобальных свойств обыкновенных линейных дифференциальных уравнений второго порядка.* Дифференциальные уравнения, № 8, XII, 1976, 1347—1383.

[3] Borůvka, O.: *Lectures in the seminar of the Institute of Mathematics of the Czechoslovak Academy of Science in Brno.*
[4] Borůvka, O.: *Sur les blocs des équations différentielles $y'' = q(t)\,y$ aux coefficients périodiques.* Rend. Mat. (2), 8, S. VI, 1975, 519—532.
[5] Borůvka, O.: *Sur quelques compléments á la théorie de Floquet pour les équations différentielles du deuxième ordre.* Ann. mat. p. ed appl., S. IV, CII, 1975, 71—77.
[6] Neuman, F.: *Note on bounded non-periodic solutions of the second-order linear differential equations with periodic coefficients.* Mat. Nachrichten, 39, 1969, 217—222.
[7] Neuman, F.: *Linear differential equations of the second order and their applications.* Rend. Math. (3), Vol. 4, Serie VI, 1971, 559—617.

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