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CENTRAL PROJECTIONS OF A PAIR OF ACCOMPANYING SPACES TO A LINEAR TWO-DIMENSIONAL SPACE OF FUNCTIONS WITH A CONTINUOUS FIRST DERIVATIVE

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M. Laitoch defined a central projection of bundles of integrals relative to the differential equation (q): $y'' = q(t)y$ with a given basis. There is considered a mapping among linear combinations $\alpha y + \beta y'$ and $\gamma y + \delta y'$ of the integral y relative to (q) and its derivative y', where the numbers of bases $[\alpha, \beta]$ and $[\gamma, \delta]$ are satisfying the condition $\alpha\delta - \beta\gamma \neq 0$.

The present paper deals with properties of a central projection of functions of a pair of accompanying spaces $P\varrho[\alpha, \beta]$ and $P\sigma[\gamma, \delta]$ to a linear two-dimensional space of functions with a continuous first derivative. The definitions and properties regarding these accompanying space have been discussed in [8] and [9]. We investigate the course of the central projection in dependence on the extreme points of the spaces $P\varrho[\alpha, \beta]$ and $P\sigma[\gamma, \delta]$ and their connection with transformations of these spaces. In conclusion we are showing assumptions under which the central dispersion of bundles of integrals of the differential equation (q) in [3].

Throughout this paper we assume $S \subset C_1(i)$ to be a regular two-dimensional space of a certain type and the set $S' \subset C_0(i)$ of derivatives of all functions relative to S to be a regular two-dimensional space of a certain type as well. Next we assume every function $y \in S$ and its derivative y' to be independent on the interval i and shall be concerned with two accompanying spaces $P\varrho[\alpha, \beta]$ and $P\sigma[\gamma, \delta]$ to the space S. The accompanying spaces $P\varrho[\alpha, \beta]$ and $P\sigma[\gamma, \delta]$ are, respectively, the sets of all functions $\varrho(\alpha y + \beta y')$ and $\sigma(\gamma y + \delta y')$, where $\alpha, \beta, \gamma, \delta$ are real constants different from zero, satisfying the condition $\alpha\delta - \beta\gamma \neq 0$ and $\varrho > 0$,

$\sigma > 0$ are functions continuous on the interval i. We assume the spaces $P\varrho[\alpha, \beta]$ and $P\sigma[\gamma, \delta]$ to be regular and of a certain type on the interval i. Let (u, v) be a basis of the space S. Then the characteristic or the phase of the basis $(\varrho(\alpha u + \beta u'), \varrho(\alpha v + \beta v'))$ relative to the space $P\varrho[\alpha, \beta]$ will be written as $f(t)$ or $\varphi(t)$, $t \in i$; the characteristic or the phase of the basis $(\sigma(\gamma u + \delta u'), \sigma(\gamma v + \delta v'))$ relative to the space $P\sigma[\gamma, \delta]$ will be written as $p(t)$ or $\psi(t)$, $t \in i$. The function $w = uv' - u'v$ is the Wronskian of functions of the basis (u, v) relative to S.

In [8] there are studied the zeros of functions of the space $P\varrho[\alpha, \beta]$. If it holds for the function $y \in S$ and for the point $t_0 \in i$ that $y(t_0) = 0$ and $y'(t_0) = 0$, then t_0 is a zero of type 1. If $y(t_0) = 0$ and $\dfrac{y'(t_0)}{y(t_0)} = -\dfrac{\alpha}{\beta}$, then t_0 is a zero of type 2. From our considerations will be excluded such zeros of type 1 which are the limit points of extremes of the function relative to the space S' having its zeros value at these points. In other words, we assume that there exist

$$\lim_{t \to t_0-} \frac{y'(t)}{y(t)} \quad \text{and} \quad \lim_{t \to t_0+} \frac{y'(t)}{y(t)},$$

with $y \in S$, for every $t_0 \in i$.

Definition 1. *Let $t_1, t_2 \in i$, $t_1 < t_2$. If there exists a function $y \in S$ such that $\varrho(t_1)(\alpha y(t_1) + \beta y'(t_1)) = 0$ and $\sigma(t_2)(\gamma y(t_2) + \delta y'(t_2)) = 0$, we say that the orderer pair of spaces $\{P\varrho[\alpha, \beta], P\sigma[\gamma, \delta]\}$ has a central projection ζ. The function $\zeta(t)$ assigning a first zero $t_2 \in i$ (if any), $t_2 > t_1$, of the function $\sigma(\gamma y + \delta y')$ to every zero $t_1 \in i$ of the function $\varrho(\alpha y + \beta y')$, will be called the central projection of an orderer pair of spaces $\{P\varrho[\alpha, \beta], P\sigma[\gamma, \delta]\}$.*

Convection 1. *For the sake of brevity we shall speak, hereafter, of the central projection ζ of the orderer pair of spaces $\{P\varrho[\alpha, \beta], P\sigma[\gamma, \delta]\}$ from Definition* 1 *as the projection ζ.*

Lemma 1. *Let the projection ζ be defined at the point $t_0 \in i$. Then $\zeta(t_0) > t_0$.*
The statement is evident.

Theorem 1. *Let $t_1, t_2 \in i$. The projection ζ is defined at the point t_1 assuming there the value t_2 exactly if $t_1 < t_2$ and a basis (u, v) of the space S exists such that the functions u, v and the points t_1, t_2 satisfy the following equation*

$$\begin{vmatrix} \alpha u(t_1) + \beta u'(t_1) & \alpha v(t_1) + \beta v'(t_1) \\ \gamma u(t_2) + \delta u'(t_2) & \gamma v(t_2) + \delta v'(t_2) \end{vmatrix} = 0 \tag{1}$$

and the function u, v and the points t_1, $t \in i$ do not satisfy equation (1) *for any point $t \in (t_1, t_2)$.*

The statement follows from Theorem 6 (see [9]).

Theorem 2. *Let $t_1, t_2 \in i$. The projection ζ is defined at the points t_1 assuming there the value t_2 exactly if $t_1 < t_2$ and* (*i*) *the function f and p are defined at the points t_1 and t_2 and $f(t_1) = p(t_2)$, respectively, whereby $f(t_1) \neq p(t)$ for every $t \in (t_1, t_2)$ for which p is defined.*

(*ii*) *the functions f and p are not defined at the points t_1 and t_2, respectively, whereby p is defined on the interval (t_1, t_2).*

The statement follows from Theorem 8 (see [9]).

Theorem 3. *Let $t_1, t_2 \in i$. The projection ζ is defined at the point t_1 assuming there the value t_2 exactly if $t_1 < t_2$ and*

$$\varphi(t_1) = \psi(t_2) + k\pi,$$

for k being an integer, holds, whereby $\varphi(t_1) \neq \psi(t) + k\pi$ for every $t \in (t_1, t_2)$.

The statement follows from Theorem 9 (see [9]).

Lemma 2. *Let the projection ζ be defined at the point $t_0 \in i$. Then it holds for the function $y \in S$ satisfying the equation $\varrho(t_0)(\alpha y(t_0) + \beta y'(t_0)) = 0$ that:*

1. *if $-\frac{\alpha}{\beta} > -\frac{\gamma}{\delta}$, then $y \neq 0$ on the interval $(t_0, \zeta(t_0)\rangle$,*

2. *if $-\frac{\alpha}{\beta} < -\frac{\gamma}{\delta}$, then y has at most one zero in the interval $(t_0, \zeta(t_0)\rangle$.*

Proof: By Lemma 1.2 [8] and by Theorem 2.2 [8] there is either $y(t_0) \neq 0$ and $\frac{y'(t_0)}{y(t_0)} = -\frac{\alpha}{\beta}$ or $y(t_0) = 0$, $y'(t_0) = 0$ and $\lim_{t \to t_0+} \frac{y'(t)}{y(t)} = +\infty$. In view of the definition of ζ we have $\sigma(t)(\gamma y(t) + \delta y'(t)) \neq 0$ for $t \in (t_0, \zeta(t_0))$.

1. Let $-\frac{\alpha}{\beta} > -\frac{\gamma}{\delta}$. If there were the zero point $T \in (t_0, \zeta(t_0)\rangle$ of the function y, then, by Theorem 2.1 [8] there would be $\lim_{t \to T-} \frac{y'(t)}{y(t)} = -\infty$ and the function $\frac{y'}{y}$ would assume the value $-\frac{\gamma}{\delta}$ on the interval (t_0, T) contradicting our assumption $\gamma y(t) + \delta y'(t) \neq 0$ on $(t_0, \zeta(t_0))$.

2. Let $-\frac{\alpha}{\beta} < -\frac{\gamma}{\delta}$. If there were two zeros $T_1, T_2 \in (t_0, \zeta(t_0)\rangle$, $T_1 < T_2$, of the function y, then, by Theorem 2.1 [8], the function $\frac{y'}{y}$ would assume the value $-\frac{\gamma}{\delta}$ within the interval (T_1, T_2), which, however, would conflict with the assumption $\gamma y(t) + \delta y'(t) \neq 0$ for $t \in (t_0, \zeta(t_0))$.

Theorem 4. *Let $t_1, t_2 \in i$. The projection ζ is defined at the point t_1 assuming there the value t_2 exactly if $t_1 < t_2$ and there exists a $y \in S$ such that either*

(*i*) $\frac{y'(t_1)}{y(t_1)} = -\frac{\alpha}{\beta}$ and $\frac{y'(t_2)}{y(t_2)} = -\frac{\gamma}{\delta}$, *whereby it holds if $y(t_0) \neq 0$, then $\frac{y'(t_0)}{y(t_0)} \neq -\frac{\gamma}{\delta}$, if $y(t_0) = 0$, then $y'(t_0) \neq 0$, for every $t_0 \in (t_1, t_2)$;*

or

(*ii*) $y(t_1) = 0$, $y'(t_1) = 0$ and $\frac{y'(t_2)}{y(t_2)} = -\frac{\gamma}{\delta}$, *whereby* $y(t_0) \neq 0$ *and* $\frac{y'(t_0)}{y(t_0)} \neq -\frac{\gamma}{\delta}$ *for every* $t_0 \in (t_1, t_2)$;

or

(*iii*) $\frac{y'(t_1)}{y(t_1)} = -\frac{\alpha}{\beta}$ *and* $y(t_2) = 0$, $y'(t_2) = 0$, *whereby* $y(t_0) \neq 0$ *and* $\frac{y'(t_0)}{y(t_0)} \neq -\frac{\gamma}{\delta}$ *for every* $t_0 \in (t_1, t_2)$.

Proof: I. Let $\zeta(t_1) = t_2$. Then it follows from Lemma 1.2 [8] and Theorem 2.2 [8] for the function $y \in S$ satisfying the equation $\varrho(t_1)(\alpha y(t_1) + \beta y'(t_1)) = 0$ that either $y(t_1) \neq 0$ and $\frac{y'(t_1)}{y(t_1)} = -\frac{\alpha}{\beta}$, or $y(t_1) = 0$, $y'(t_1) = 0$ and $\lim_{t \to t_1+} \frac{y'(t)}{y(t)} = +\infty$.

a) Let $-\frac{\alpha}{\beta} > -\frac{\gamma}{\delta}$. Then, by Lemma 2, $y \neq 0$ on the interval $(t_1, t_2\rangle$, thus $\frac{y'}{y} > -\frac{\gamma}{\delta}$ on the interval (t_1, t_2) which implies that either $\frac{y'(t_1)}{y(t_1)} = -\frac{\alpha}{\beta}$ and $\frac{y'(t_2)}{y(t_2)} = -\frac{\gamma}{\delta}$, or $y'(t_1) = 0$, $y(t_1) = 0$ and $\frac{y'(t_2)}{y(t_2)} = -\frac{\gamma}{\delta}$.

b) Let $-\frac{\alpha}{\beta} < -\frac{\gamma}{\delta}$. Then, by Lemma 2, there exists at most one point $T \in (t_1, t_2\rangle$ such that $y(T) = 0$. Let $y \neq 0$ on $(t_1, t_2\rangle$. Then either $\frac{y'}{y} > -\frac{\gamma}{\delta}$ on (t_1, t_2) which leads to $y(t_1) = 0$, $y'(t_1) = 0$ and $\frac{y'(t_2)}{y(t_2)} = -\frac{\gamma}{\delta}$, or $\frac{y'}{y} < -\frac{\gamma}{\delta}$ on (t_1, t_2) which leads to $\frac{y'(t_1)}{y(t_1)} = -\frac{\alpha}{\beta}$ and $\frac{y'(t_2)}{y(t_2)} = -\frac{\gamma}{\delta}$. Now, let $y(T) = 0$ for $T \in (t_1, t_2\rangle$. Then, if $T \neq t_2$, we get $\frac{y'}{y} < -\frac{\gamma}{\delta}$ on (t_1, T) and $\frac{y'}{y} > -\frac{\gamma}{\delta}$ on (T, t_2) which yields $\frac{y'(t_1)}{y(t_1)} = -\frac{\alpha}{\beta}$ and $\frac{y'(t_2)}{y(t_2)} = -\frac{\gamma}{\delta}$. If $T = t_2$, then $\frac{y'}{y} < -\frac{\gamma}{\delta}$ on (t_1, t_2) which yields $\frac{y'(t_1)}{y(t_1)} = -\frac{\alpha}{\beta}$ and $y'(t_2) = 0$, $y(t_2) = 0$.

II. Let one of the relations (i), (ii), (iii) hold. It is then obvious (from Definition 1) that $\zeta(t_1) = t_2$.

Corollary 1. *Let the projection* ζ *be defined at the point* $t_0 \in i$ *and let* $y \in S$ *be that function which satisfied the equation* $\varrho(t_0)(\alpha y(t_0) + \beta y'(t_0)) = 0$. *Let* $w \neq 0$ *on the interval* $(t_0, \zeta(t_0))$. Now,

1. *if* $-\frac{\alpha}{\beta} > -\frac{\gamma}{\delta}$, *then every function* $x \in S$ *has at most one zero on the interval* $\langle t_0, \zeta(t_0))$,

2. *if* $-\frac{\alpha}{\beta} < -\frac{\gamma}{\delta}$, *then every function* $x \in S$ *has at most two zeros on the interval* $(t_0, \zeta(t_0))$; *specially: if* $y \neq 0$ *on* $(t_0, \zeta(t_0))$, *then every function* $x \in S$ *has one zero at most.*

Corollary 2. *Let the projection* ζ *be defined at the point* $t_0 \in i$ *and let* $y \in S$ *be the function satisfying the equation* $\varrho(t_0)(\alpha y(t_0) + \beta y'(t_0)) = 0$. *Let* $\tau_i \in (t_0, \zeta(t_0))$, $i = 1, 2, \ldots, k$, *be zeros of the Wronskian* w. *Now*

1. *if* $-\frac{\alpha}{\beta} > -\frac{\gamma}{\delta}$ *then* $\frac{y'(\tau_i)}{y(\tau_i)} > -\frac{\gamma}{\delta}$,

2. *if* $-\frac{\alpha}{\beta} < -\frac{\gamma}{\delta}$ *then in case of* $y \neq 0$ *on* $(t_0, \zeta(t_0))$ *we have* $\frac{y'(\tau_i)}{y(\tau_i)} < -\frac{\gamma}{\delta}$ *and in case of* $T \in (t_0, \zeta(t_0))$ *being zero of the function* y, *we have* $\frac{y'(\tau_i)}{y(\tau_i)} < -\frac{\gamma}{\delta}$ *for all* $\tau_i < T$ *and* $\frac{y'(\tau_j)}{y(\tau_j)} > -\frac{\gamma}{\delta}$ *for all* $\tau_j > T$.

Theorem 5. *Let one of the following assumptions hold:*

(*i*) $-\frac{\alpha}{\beta} > -\frac{\gamma}{\delta}$ *and the space* S *be of type* $m \geqq 2$ *on* i,

(*ii*) $-\frac{\alpha}{\beta} < -\frac{\gamma}{\delta}$ *and the space* S *be of type* $m \geqq 3$ *on* i.

Then the projection ζ *is defined at least at one point of the interval* i.

Proof: (i) Let $-\frac{\alpha}{\beta} > -\frac{\gamma}{\delta}$ and $t_1, t_2 \in i$, $t_1 < t_2$, be the neighbouring zeros of the function $y \in S$. By Theorem 2.3 [8], the function $\frac{y'}{y}$ assumes then all values from $(-\infty, +\infty)$, i.e. also the values $-\frac{\alpha}{\beta}$ and $-\frac{\gamma}{\delta}$ on the interval (t_1, t_2). Let $\frac{y'(t_0)}{y(t_0)} = -\frac{\alpha}{\beta}$ hold for $t_0 \in (t_1, t_2)$. Since $-\frac{\alpha}{\beta} > -\frac{\gamma}{\delta}$ the function $\frac{y'}{y}$ assumes, with respect to Theorem 2.1 or Theorem 2.2 [8], the value $-\frac{\gamma}{\delta}$ on the interval (t_0, t_2). Thus ζ is defined at t_0.

(ii) Let $-\frac{\alpha}{\beta} < -\frac{\gamma}{\delta}$ and $t_1, t_2, t_3 \in i$, $t_1 < t_2 < t_3$, be the zeros of the function $y \in S$ with $y \neq 0$ on the intervals (t_1, t_2) and (t_2, t_3). By Theorem 2.3 [8], the function $\frac{y'}{y}$ assumes on every interval (t_1, t_2) and (t_2, t_3) all values from the interval $(-\infty, +\infty)$, hence the values $-\frac{\alpha}{\beta}$ and $-\frac{\gamma}{\delta}$ as well. Let $\frac{y'(t_0)}{y(t_0)} = -\frac{\alpha}{\beta}$ hold for $t_0 \in (t_1, t_2)$. If the function $\frac{y'}{y}$ assumes the value $-\frac{\gamma}{\delta}$ on (t_0, t_2), ζ is obviously defined at t_0. If $\frac{y'}{y} < -\frac{\gamma}{\delta}$ on (t_0, t_2) and $y'(t_2) = 0$, then ζ is defined at t_0 assuming there value t_2; if $\frac{y'}{y} < -\frac{\gamma}{\delta}$ on (t_0, t_2) and $y'(t_2) \neq 0$ then $\frac{y'}{y}$ assumes the value $-\frac{\gamma}{\delta}$ on (t_2, t_3) – hence, it is defined at t_0.

Lemma 3. *Let the projection ζ be defined on the interval $\langle a, b\rangle \subset i$. It then holds for every interval $j \subset \langle a, b\rangle$ that $\zeta \not\equiv$ constant on j.*

Proof: If $\zeta \equiv k$ were on an interval $j \subset \langle a, b\rangle$, k = constant, then $k \in i$ would we a sinfular point of the space $P\sigma[\gamma, \delta]$, which conflicts with the hypothesis about its regularity.

Theorem 6. *Let the following assumptions be satisfied: The projection ζ is defined on the interval $\langle a, b\rangle \subset i$ and assumes the values from the interval $(c, d) \subset i$; $w(t) \neq \neq 0$ for all $t \in \langle a, d)$; there lies no extreme point of the space $P\varrho[\alpha, \beta]$ in (a, b), and there lies no extreme point of the space $P\sigma[\gamma, \delta]$ in (c, d).*

Then the projection ζ is continuous and strictly monotonic on $\langle a, b\rangle$.

Proof: In view of the hypothesis $w \neq 0$ on $\langle a, d)$, every point on $\langle a, d)$ is a zero of type 2. We shall break up the proof into two parts: 1. if $-\frac{\alpha}{\beta} > -\frac{\gamma}{\delta}$ and 2. $-\frac{\alpha}{\beta} < -\frac{\gamma}{\delta}$.

1. Given $-\frac{\alpha}{\beta} > -\frac{\gamma}{\delta}$. By Lemma 2 and respecting the hypothesis $w \neq 0$ on $\langle a, d)$, it holds for every function $x \in S$ such that $\varrho(t_0)(\alpha x(t_0) + \beta x'(t_0)) = 0$, where $t_0 \in \langle a, b\rangle$ that $x \neq 0$ on $\langle t_0, \zeta(t_0)\rangle$, whereby $\zeta(t_0) \in (c, d)$. The function $\frac{x'}{x}$ is continuous on $\langle t_0, \zeta(t_0)\rangle$, $\frac{x'(t_0)}{x(t_0)} = -\frac{\alpha}{\beta}$, $\frac{x'(\zeta(t_0))}{x(\zeta(t_0))} = -\frac{\gamma}{\delta}$ and $\frac{x'(t)}{x(t)} > -\frac{\gamma}{\delta}$ for $t \in \langle t_0, \zeta(t_0))$. Let $y \in S$ be the function for which $\frac{y'(a)}{y(a)} = -\frac{\alpha}{\beta}$. There may now arise two alternatives for the function $\frac{y'}{y}$: either $\frac{y'}{y} > -\frac{\alpha}{\beta}$ on $(a, b\rangle$ or $\frac{y'}{y} < -\frac{\alpha}{\beta}$ on $(a, b\rangle$. The equality $\frac{y'(t)}{y(t)} = -\frac{\alpha}{\beta}$ for $t \in (a, b\rangle$ cannot arise with respect to Theorem 2.10 [8].

Given $\frac{y'}{y} < -\frac{\alpha}{\beta}$ $\left(\frac{y'}{y} > -\frac{\alpha}{\beta}\right)$ on $(a, b\rangle$, then it holds for every function $\frac{x'}{x}$, $x \in S$, assuming the value $-\frac{\alpha}{\beta}$ on (a, b) – let it be at the point $t_0 \in (a, b)$ – that $\frac{x'}{x} < -\frac{\alpha}{\beta}$ $\left(\frac{x'}{x} > -\frac{\alpha}{\beta}\right)$ on the interval $(t_0, b\rangle$.

Let first $\frac{y'}{y} < -\frac{\alpha}{\beta}$ on $(a, b\rangle$. If the function x were such that $\frac{x'(t)}{x(t)} > -\frac{\alpha}{\beta}$ for any $t \in (t_0, b\rangle$, where t_0 denotes a zero of the function $\varrho(\alpha x + \beta x')$, then, by Theorem 2.8 [8] and respecting the assumption that no extreme point of the space $P\varrho[\alpha, \beta]$ is lying in $\langle a, b\rangle$, there exists $\delta_0 > 0$ such that $\frac{x'(t)}{x(t)} < -\frac{\alpha}{\beta}$ holds for $t \in (t_0 - \delta_0, t_0)$. Since, however $\frac{x'}{x} \neq \frac{y'}{y}$ must be valid on $\langle a, t_0\rangle$, then, if $x \neq 0$ on $\langle a, t_0\rangle$ or if $T \in \langle a, t_0)$ so that $x(T) = 0$, i.e. $\lim_{t \to T+} \frac{x'(t)}{x(t)} = +\infty$, a point must exist in the interval (a, t_0) or (T, t_0) wherein the function $\frac{x'}{x}$ assumes the value $-\frac{\alpha}{\beta}$, and by Theorem 2.10 [8] an extreme point of the space $P\varrho[\alpha, \beta]$ must lie in the interval (a, t_0) or (T, t_0), which contradicts our assumption.

Completely analogous we can show if $\frac{y'}{y} > -\frac{\alpha}{\beta}$ on $(a, b\rangle$, then $\frac{x'}{x} > -\frac{\alpha}{\beta}$ on $(t_0, b\rangle$ if $\frac{x'(t_0)}{x(t_0)} = -\frac{\alpha}{\beta}$, $t_0 \in (a, b)$.

Let us now select the points $t_i \in (a, b)$, $i = 1, \ldots, k$,

$$a < t_1 < t_2 < \ldots < t_{k-1} < t_k < b, \tag{2}$$

taking k sufficiently great for

$$\zeta(a) > t_1, \zeta(t_1) > t_2, \ldots, \zeta(t_{k-1}) > t_k, \zeta(t_k) > b, \tag{3}$$

(which is possible with respect to Lemma 1) and let us denote by $x_i \in S$ the functions for which $\frac{x_i'(t_i)}{x_i(t_i)} = -\frac{\alpha}{\beta}$.

A. Let $\frac{y'}{y} < -\frac{\alpha}{\beta}$ on $(a, b\rangle$. Then $\frac{x_i'}{x_i} < -\frac{\alpha}{\beta}$ on $(t_i, b\rangle$ for every function $\frac{x_i'}{x_i}$ and it holds with respect to $w \neq 0$ on $\langle a, d)$:

$$\begin{aligned}
&\frac{y'}{y} < \frac{x_1'}{x_1} && \text{on } \langle t_1, (\zeta a)\rangle,\\
&\frac{x_1'}{x_1} < \frac{x_2'}{x_2} && \text{on } \langle t_2, \zeta(t_1)\rangle,\\
&\vdots\\
&\frac{x_{k-1}'}{x_{k-1}} < \frac{x_k'}{x_k} && \text{on } \langle t_k, \zeta(t_{k-1})\rangle,\\
&\frac{x_k'}{x_k} < \frac{y_1'}{y_1} && \text{on } \langle b, \zeta(t_k)\rangle,
\end{aligned} \tag{4}$$

where $y_1 \in S$ is the function for which $\frac{y_1'(b)}{y_1(b)} = -\frac{\alpha}{\beta}$. From the above relations we get the following inequalities

$$\zeta(a) < \zeta(t_1) < \zeta(t_2) < \ldots < \zeta(t_{k-1}) < \zeta(t_k) < \zeta(b). \qquad (5)$$

Evidently, the greater is k, the more so will hold the relations (3), (4) and thus also (5). The projection ζ is therefore increasing on $\langle a, b\rangle$ with respect to Lemma 3.

Let us now the continuity. With respect to the fact that ζ is defined as increasing on $\langle a, b\rangle$, it could have only points of discontinuity of the 1st kind on $\langle a, b\rangle$. So, it suffices to show that every point $t_0^* \in (\zeta(a), \zeta(b))$ is a functional value of ζ at a point of (a, b). Let $t_0^* \in (\zeta(t_{i-1}), \zeta(t_i))$. Following Lemma 1 [6] there exists an $x_0 \in S$ so that $\frac{x_0'(t_0^*)}{x_0(t_0^*)} = -\frac{\gamma}{\delta}$. In analogy with the method used in the paragraph before part A. concerning the proof of this Theorem, it can be shown that if $\frac{x_0'(t)}{x_0(t)} < -\frac{\gamma}{\delta}$ for a $t \in \langle\zeta(t_{i-1}), t_0^*)$, then, to satisfy the relation $\frac{x_{i-1}'}{x_{i-1}} \neq \frac{x_0'}{x_0}$ on the interval $\langle\zeta(t_{i-1}), t_0^*\rangle$, there would have to exist a point of $(\zeta(t_{i-1}), t_0^*)$ at which the function $\frac{x_0'}{x_0}$ assumes the value $-\frac{\gamma}{\delta}$, whereby if there exist a $T \in (\zeta(t_{i-1}), t_0^*)$ so that $x_0(T) = 0$, this point would lie on the interval (T, t_0^*). This would yield with respect to Theorem 2.10 [8] a contradiction with the assumption that the space $P\sigma[\gamma, \delta]$ has no extreme points on the interval (c, d).

Thus $\frac{x_0'}{x_0} > -\frac{\gamma}{\delta}$ on the interval $\langle\zeta(t_{i-1}), t_o^*)$ and since $\zeta(t_{i-1}) < t_0^* < \zeta(t_i)$, it must hold $\frac{x_{i-1}'}{x_{i-1}} < \frac{x_0'}{x_0} < \frac{x_i'}{x_i}$ on the interval $\langle t_i, \zeta(t_{i-1})\rangle$. If $x_0 \neq 0$ on (t_{i-1}, t_i), then we have the inequality $\frac{x_{i-1}'}{x_{i-1}} < \frac{x_0'}{x_0}$ also on (t_{i-1}, t_i), whence it follows that there exists a $t_0 \in (t_{i-1}, t_i)$ so that $\frac{x_0'(t_0)}{x_0(t_0)} = -\frac{\alpha}{\beta}$. If there exists a $T \in (t_{i-1}, t_i)$ so that $x_0(T) = 0$, then $\lim_{t\to T+} \frac{x_0'(t)}{x_0(t)} = +\infty$ and there exists again a $t_0 \in (T, t_i)$ so that $\frac{x_0'(t_0)}{x_0(t_0)} = -\frac{\alpha}{\beta}$. Since the point t_0^* was chosen arbitrarily, there obviously exists a point $t \in (a, b)$ to any point $t^* \in (\zeta(a), \zeta(b))$ so that $\zeta(t) = t^*$, which is the result we wished to proove.

B. Let $\frac{y'}{y} > -\frac{\alpha}{\beta}$ on $(a, b\rangle$. Then we get in (4) and (5) the reverse inequality, whence it follows that the projection ζ is descreaing on $\langle a, b\rangle$ and its continuity could be proved analogous to that carried out in part A.

2. Let $-\frac{\alpha}{\beta} < -\frac{\gamma}{\delta}$. Then, by Lemma 2, it holds for every function $x \in S$ such that $\varrho(t_0)(\alpha x(t_0) + \beta x'(t_0)) = 0$ with $t_0 \in \langle a, b\rangle$ that x has at most one zero in the interval $\langle t_0, \zeta(t_0)\rangle$, $\zeta(t_0) \in (c, d)$. The notation of functions and points from section 1. conserning the proof is preserved.

C. Let us first assume $y_1 \neq 0$ on the interval $\langle b, \zeta(b)\rangle$. Then $\frac{y_1'}{y_1}$ is continuous on $\langle b, \zeta(b)\rangle$ and $\frac{y_1'}{y_1} < -\frac{\gamma}{\delta}$ for $t \in \langle b, \zeta(b))$. Let us show that $x \neq 0$ on $\langle t_0, \zeta(t_0)\rangle$ for every function $x \in S$ such that $\varrho(t_0)(\alpha x(t_0) + \beta x'(t_0)) = 0$, where $t_0 \in \langle a, b)$. Then two possible cases for $\frac{x'}{x}$ arise from the continuity of the function $\frac{x'}{x}$ at t_0: either $\delta_0 > 0$ so that $\frac{x'}{x} > -\frac{\alpha}{\beta}$ or $\frac{x'}{x} < -\frac{\alpha}{\beta}$ on $(t_0, t_0 + \delta_0)$. It is readily seen that $x \neq 0$ on $\langle t_0, \zeta(t_0)\rangle$ is evident in case of $\frac{x'}{x} > -\frac{\alpha}{\beta}$ on $(t_0, t_0 + \delta_0)$. In case of $\frac{x'}{x} < -\frac{\alpha}{\beta}$ on $(t_0, t_0 + \delta_0)$ in assuming the existence of a zero of the function x in the interval $(t_0, \zeta(t_0))$, we are led to contradiction to the assumption of our Theorem saying that $P\sigma[\gamma, \delta]$ has no extreme points in the interval (c, d). The proof was carried out analogous to that in section 1. It turns out that ζ is continuous and increasing on $\langle a, b\rangle$ if $\frac{x_i'}{x_i} > -\frac{\alpha}{\beta}$ on the intervals $(t_i, b\rangle$ and $\frac{y'}{y} > -\frac{\alpha}{\beta}$ on the interval $(a, b\rangle$; is continuous and decreasing on $\langle a, b\rangle$ if $\frac{x_i'}{x_i} < -\frac{\alpha_i}{\beta}$ on the intervals $(t_i, b\rangle$ and $\frac{y'}{y} < -\frac{\alpha}{\beta}$ on the interval $(a, b\rangle$.

D. Let us now assume that $T_0 \in (b, \zeta(b))$ so that $y_1(T_0) = 0$ and let us show that $T_x \in (t_0, \zeta(t_0))$ so that $x(T_x) = 0$ for every function $x \in S$ such that $\varrho(t_0)(\alpha x(t_0) + \beta x'(t_0)) = 0$, where $t_0 \in \langle a, b)$. If such a point did not exist, then $\frac{x'}{x}$ would be continuous on $(t_0, \zeta(t_0))$ and with respect to the assumptions of our Theorem either $\frac{x'}{x} > -\frac{\alpha}{\beta}$ or $\frac{x'}{x} < -\frac{\alpha}{\beta}$ on $(t_0, b\rangle$ and it would hold $\frac{x'}{x} < -\frac{\gamma}{\delta}$ on $\langle t_0, \zeta(t_0))$. In case of $\frac{x'}{x} > -\frac{\alpha}{\beta}$, respecting the assumption $w \neq 0$ on $\langle a, d)$, i.e. $\frac{x'}{x} \neq \frac{y_1'}{y_1}$, we should be led in analogy with part 1. to the existence of a point in $(\zeta(t_0), \zeta(b))$, wherein $\frac{x'}{x}$ takes on the value $-\frac{\gamma}{\delta}$. This however would conflict with the assumption of Theorem 2.10 [8] saying that $P\sigma[\gamma, \delta]$ has no extreme points on (c, d). In case of $\frac{x'}{x} < -\frac{\alpha}{\beta}$ the contradiction is clear.

Thus every function x_i in $(t_i, \zeta(t_i))$ has a zero. Let us denote it by T_i and let T be a zero of the function y in $(a, \zeta(a))$. Assuming $w \neq 0$ on $\langle a, d)$ yields

$$T > T_1 > \dots > T_k > T_0,$$

if $\frac{y'}{y} > -\frac{\alpha}{\beta}$ on $(a, b\rangle$ and $\frac{x_i'}{x_i} > -\frac{\alpha}{\beta}$ on $(t_i, b\rangle$, i.e. the projection ζ is decreasing on $\langle a, b\rangle$, and

$$T < T_1 < \ldots < T_k < T_0,$$

if $\frac{y'}{y} < -\frac{\alpha}{\beta}$ on (a, T) and $\frac{x_i'}{x_i} < -\frac{\alpha}{\beta}$ on (t_i, T_i), i.e. the projection ζ is increasing on $\langle a, b\rangle$.

The continuity could be proved in analogy with section 1.

Theorem 7. *Let the following assumptions be satisfied: The projection ζ be defined on $\langle a, b\rangle \subset i$ taking on the values from $(c, d) \subset i$, for all $t \in \langle a, d)$ be $w(t) \neq 0$. There lies no extreme point of the space $P\varrho[\alpha, \beta]$ in the interval (a, d), and no extreme point of the space $P\sigma[\gamma, \delta]$ lies in (c, d).*

Then the projection ζ is continuous and increasing on $\langle a, b\rangle$.

Proof: We apply the results of the proof of Theorem 6 adopting also the notation therefrom.

1. Let $-\frac{\alpha}{\beta} > -\frac{\gamma}{\delta}$. There cannot occur case 1B. as in the proof of Theorem 6, because the assumption $\frac{y'}{y} > -\frac{\alpha}{\beta}$ on $(a, b\rangle$ gives the fact that at least one point must lie on the interval $(b, \zeta(a))$, wherein $\frac{y'}{y}$ takes on the value $-\frac{\alpha}{\beta}$. Following Theorem 2.10 [8] then there exists an extreme point of the space $P\varrho[\alpha, \beta]$ in the interval $(b, \zeta(a)) \subset (a, d)$, which is a contradiction. Thus the statement follows from section 1A. in the proof of Theorem 6, i.e. the projection ζ is increasing.

2. Let $-\frac{\alpha}{\beta} < -\frac{\gamma}{\delta}$. In analogy with section 1. concerning the proof of this Theorem, the inequalities $\frac{y'}{y} < -\frac{\alpha}{\beta}$ on $(a, b\rangle$ and $\frac{x_i'}{x_i} < -\frac{\alpha}{\beta}$ on $(t_i, b\rangle$ from case 2C. yields a contradiction to our assumptions. It follows from the inequalities $\frac{y'}{y} > -\frac{\alpha}{\beta}$ on $(a, b\rangle$ and $\frac{x_i'}{x_i} > -\frac{\alpha}{\beta}$ on $(t_i, b\rangle$ that the projection ζ is increasing. In case 2D. the inequalities $\frac{y'}{y} > -\frac{\alpha}{\beta}$ on $(a, b\rangle$ and $\frac{x_i'}{x_i} > -\frac{\alpha}{\beta}$ on $(t_i, b\rangle$ i.e. $T > T_1 > \ldots > T_k > T_0$ yield repeatedly to a contradiction to the assumptions of this Theorem. Namely, there would exist again at least one point of the interval (a, T), wherein $\frac{y'}{y} = -\frac{\alpha}{\beta}$. Thus, there may occur just the case $\frac{y'}{y} < -\frac{\alpha}{\beta}$ on (a, T) and $\frac{x_i'}{x_i} < -\frac{\alpha}{\beta}$ on (t_i, T_i), i.e. $T < T_1 < \ldots < T_k < T_0$, and it repeatedly holds that the projection ζ is increasing.

The continuity of the projection ζ was proved in the proof of Theorem 6.

Theorem 8. *Let the following assumptions be satisfied: The projection ζ be defined on the interval $\langle a, b\rangle \subset i$ taking on the values from $(c, d) \subset i$, $b < c$ and there lies* *an infinite number of zeros of the function w on the interval* $(b, c\rangle$, *for all* $t \in \langle a, b\rangle \cup$ $\cup\ (c, d)$ *be* $w(t) \neq 0$, *there lie no extreme points of the spaces* $P\varrho[\alpha, \beta]$ *and* $P\sigma[\gamma, \delta]$ *in* (a, b) *and* (c, d), *respectively.*

Then the projection ζ *is continuous and strictly monotonic on* $\langle a, b\rangle$.

Proof: Consider first exactly one zero of the Wronskian w lying an interval $(b, c\rangle$ written as τ. We continue to employ the notation introduced in the proof of Theorem 6.

1. Let $-\frac{\alpha}{\beta} > -\frac{\gamma}{\delta}$. All function $\frac{y'}{y}$, $\frac{y_1'}{y_1}$, $\frac{x_i'}{x_i}$ have the same value at the point τ and following Corollary 2 $\frac{y'(\tau)}{y(\tau)} > -\frac{\gamma}{\delta}$ holds.

A. Let $\frac{y'}{y} < -\frac{\alpha}{\beta}$ on $(a, b\rangle$. This yields the following inequalities:

$$
\begin{aligned}
&\frac{y'}{y} < \frac{x_1'}{x_1} && \text{on } \langle t_1, b\rangle\\
&\frac{x_1'}{x_1} < \frac{x_2'}{x_2} && \text{on } \langle t_2, b\rangle\\
&\vdots\\
&\frac{x_{k-1}'}{x_{k-1}} < \frac{x_k'}{x_k} && \text{on } \langle t_k, b\rangle\\
&\frac{x_k'(b)}{x_k(b)} < \frac{y_1'(b)}{y_1(b)},
\end{aligned}
\tag{6}
$$

$$
\frac{y'}{y} < \frac{x_1'}{x_1} < \frac{x_2'}{x_2} < \ldots \frac{x_{k-1}'}{x_{k-1}} < \frac{x_k'}{x_k} < \frac{y_1'}{y_1} \qquad \text{on } \langle b, \tau). \tag{7}
$$

If w changes its sign at τ, then the inequalities

$$
\frac{y'}{y} > \frac{x_1'}{x_1} > \frac{x_2'}{x_2} > \ldots > \frac{x_{k-1}'}{x_{k-1}} > \frac{x_k'}{x_k} > \frac{y_1'}{y_1} \tag{8}
$$

hold on $(\tau, c\rangle$ and

$$
\begin{aligned}
&\frac{y'}{y} > \frac{x_1'}{x_1} && \text{on } \langle c, \zeta(t_1)\rangle\\
&\frac{x_1}{x_1} > \frac{x_2}{x_2} && \text{on } \langle c, \zeta(t_2)\rangle\\
&\vdots\\
&\frac{x_{k-1}'}{x_{k-1}} > \frac{x_k'}{x_k} && \text{on } \langle c, \zeta(t_k)\rangle\\
&\frac{x_k'}{x_k} > \frac{y_1'}{y_1} && \text{on } \langle c, \zeta(b)\rangle.
\end{aligned}
\tag{9}
$$

Hence, the projection ζ is decreasing on $\langle a, b\rangle$. If w does not change its sign at τ, then (8) with a reverse inequality holds and moreover

$$\begin{aligned} &\frac{y'}{y} < \frac{x_1'}{x_1} && \text{on } \langle c, \zeta(a)\rangle \\ &\frac{x_1'}{x_1} < \frac{x_2'}{x_2} && \text{on } \langle c, \zeta(t_1)\rangle \\ &\vdots \\ &\frac{x_{k-1}'}{x_{k-1}} < \frac{x_k'}{x_k} && \langle c, \zeta(t_{k-1})\rangle \\ &\frac{x_k'}{x_k} < \frac{y_1'}{y_1} && \text{on } \langle c, \zeta(t_k)\rangle. \end{aligned} \tag{10}$$

Hence, the projection ζ is increasing on $\langle a, b\rangle$.

B. Let $\frac{y'}{y} > -\frac{\alpha}{\beta}$ on $(a, b\rangle$. Then in analogy with part A. of the proof of this Theorem, we obtain the following results: if w changes its sign at τ, then the projection ζ is increasing on $\langle a, b\rangle$, if w does not change its sign at τ, then the projection ζ is decreasing on $\langle a, b\rangle$.

2. Let $-\frac{\alpha}{\beta} < -\frac{\gamma}{\delta}$.

C. Letting $y_1 \neq 0$ on $\langle b, \zeta(b)\rangle$, then in analogy with part 1A. of this Theorem, we obtain the following results: Let $\frac{y'}{y} > -\frac{\alpha}{\beta}$ on $(a, b\rangle$. If w changes its sign at τ, then the projection ζ is decreasing on $\langle a, b\rangle$, if w does not change its sign at τ, then the projection ζ is increasing on $\langle a, b\rangle$. Let $\frac{y'}{y} < -\frac{\alpha}{\beta}$ on $(a, b\rangle$. If w changes its sign at τ, then the projection ζ is increasing on $\langle a, b\rangle$, if w does not change its sign at τ, then the projection ζ is decreasing on $\langle a, b\rangle$.

D. If the functions y_1, y, x_i have a zero in $(b, \zeta(b))$, $(a, \zeta(a))$, $(t_i, \zeta(t_i))$, respectively, then with respect to Corollary 2, these zeros are either all smaller or all greater than τ. In a manner analogous to that used in part 1A. in the proof of this Theorem on taking account of part 2D. in the proof of Theorem 6, we obtain the following results for the projection ζ increasing or decreasing on $\langle a, b\rangle$: If w changes its sign at τ, then under the assumption of $w \neq 0$ on $\langle a, d)$, the projection ζ being increasing and decreasing on $\langle a, b\rangle$ is decreasing and increasing on $\langle a, b\rangle$, respectively. If w does not change its sign at τ, then the same statements remain valid as under the assumption of $w \neq 0$ on $\langle a, d)$.

The continuity of the projection ζ could be proved analogous to that in the proof of Theorem 6.

If w has a finite number of zeros τ_i, $i = 1, 2, \ldots, k$, on $(b, c\rangle$, then we may proced for every τ_i analogous as in parts 1. and 2. regarding the proof of this Theorem, whence the statement follows.

Theorem 9. *Let the assumptions below be fulfilled: The projection ζ be defined on $\langle a, b\rangle \subset i$ taking on the values from $(c, d) \subset i$; for all $t \in \langle a, d)$ be $w(t) \neq 0$; $t_0 \in (a, b)$ be exactly one extreme point of the space $P\varrho[\alpha, \beta]$ in (a, b); there does not lie any extreme point of the space $P\sigma[\gamma, \delta]$ in (c, d). The projection ζ is then continuous on $\langle a, b\rangle$ and has an extreme at t_0.*

Proof: Following Theorem 2.8 [8] there exists an $x \in S$ such that $\varrho(t_0) \times \times (\alpha x(t_0) + \beta x'(t_0)) = 0$ and $\frac{x'}{x}$ has an extreme at t_0. Let $\frac{x'}{x}$ has a maximum at t_0. If $x \neq 0$ on $\langle t_0, \zeta(t_0)\rangle$ or there exists a $T_x \in (t_0, \zeta(t_0))$ such that $x(T_x) = 0$, then obviously $\frac{x'}{x} < -\frac{\alpha}{\beta}$ on $\langle a, t_0) \cup (t_0, b\rangle$ or $\langle a, t_0) \cup (t_0, T_x)$. We continue to use the notation from the proof of Theorem 6.

It then holds $\frac{y'}{y} > -\frac{\alpha}{\beta}$ on $(a, t_0\rangle$, for $t_i < t_0$ we have $\frac{x_i'}{x_i} > -\frac{\alpha}{\beta}$ on $(t_i, t_0\rangle$, for $t_j > t_0$ we have $\frac{x_j'}{x_j} < -\frac{\alpha}{\beta}$ on $(t_j, b\rangle$, if $x_j \neq 0$ on $\langle t_j, \zeta(t_j)\rangle$ or $\frac{x_j'}{x_j} < -\frac{\alpha}{\beta}$ on (t_j, T_j), where $T_j \in (t_j, \zeta(t_j))$ is a zero of the function x_j.

With reference to the proof of Theorem 6 the projection ζ is either increasing on $\langle a, t_0\rangle$ and decreasing on $\langle t_0, b\rangle$, or it is decreasing on $\langle a, t_0\rangle$ and increasing on $\langle t_0, b\rangle$. Next, the projection ζ is by Theorem 6 continuous on $\langle a, t_0\rangle$ and continuous on $\langle t_0, b\rangle$, thus it is continuous on $\langle a, b\rangle$ and has an extreme at t_0.

If $\frac{x'}{x}$ has a minimum at t_0, we may proced with the proof analogous as for the maximum.

Theorem 10. *Let the following assumptions be fulfilled: The projection ζ be defined on $\langle a, b\rangle \subset i$ assuming there the values from $(c, d) \subset i$; $w(t) \neq 0$ be valid for all $t \in \langle a, d)$; $t_0^* \in (c, d)$ be the isolated extreme point if the space $P\sigma[\gamma, \delta]$ on (c, d) with $\zeta(t_0) = t_0^*$, where $t_0 \in (a, b)$; there does not lie any extreme point of the space $P\varrho[\alpha, \beta]$ on (a, b). Then the projection ζ is discontinuous at t_0.*

Proof: Following Theorem 2.8 [8] there exists an $x \in S$ such that $\sigma(t_0^*) \times \times (\gamma x(t_0^*) + \delta x'(t_0^*)) = 0$ and $\frac{x'}{x}$ has an extreme at t_0^*. We continue to use the notation from the proof of Theorem 6.

1. Let $-\frac{\alpha}{\beta} > -\frac{\gamma}{\delta}$. Since $\frac{x'}{x} > -\frac{\gamma}{\delta}$ holds for $t \in \langle t_0, t_0^*)$ then $\frac{x'}{x}$ has a minimum at t_0^*. Thus, there exists $\varepsilon > 0$ such that $\frac{x'}{x} > -\frac{\gamma}{\delta}$ for $t \in (t_0^*, t_0^* + \varepsilon) \subset (c, d)$. Because of $t_0 \in (a, b)$ we consider the intervals $\langle a, t_0\rangle$ and $\langle t_0, b\rangle$. Since ζ is defined on the whole interval $\langle a, b\rangle$ and assumes there the values from the interval (c, d), then being assumed that $w \neq 0$ on $\langle a, d)$, it follows that the function $\frac{x'}{x}$

must take on the value $-\frac{\gamma}{\delta}$ on the interval $(t_0^* + \varepsilon, c)$. Let therefore $\frac{x'(t_1^*)}{x(t_1^*)} = -\frac{\gamma}{\delta}$, where $t_1^* \in (t_0^* + \varepsilon, c)$. It then holds – with respect to Theorem 6 and to the assumption, that t_0^* is an isolated extreme point of the space $P\sigma[\gamma, \delta]$ – that there exists an $\varepsilon_1 > 0$ such that the projection ζ is continuous for $t \in \langle t_0 - \varepsilon_1, t_0\rangle$ and it holds $\zeta(t) \leqq t_0^*$; the projection ζ is continuous for $t \in (t_0, t_0 + \varepsilon)$ and it holds $\zeta(t) > t_1^*$. t_0 is the point of discontinuity and more precisely, of the first kind.

2. Let $-\frac{\alpha}{\beta} < -\frac{\gamma}{\delta}$. Then $\frac{x'}{x}$ has a maximum at t_0^* if $x \neq 0$ on $\langle t_0, t_0^*\rangle$; $\frac{x'}{x}$ has a minimum at t_0 if x has a zero in (t_0, t_0^*). The proof proceeds in analogy with case 1.

Remark 1. *Let the projection ζ be defined on $\langle a, b\rangle$ assuming there the values from (c, d); let $w(t) \neq 0$ for $t \in \langle a, d)$ and $t_0 \in (a, b)$ be an isolated extreme point of the space $P\varrho[\alpha, \beta]$; $\zeta(t_0) \in (c, d)$ be an isolated extreme point of the space $P\sigma[\gamma, \delta]$. Let $x \in S$ be that function for which $\varrho(t_0)(\alpha x(t_0) + \beta x'(t_0)) = 0$. This enables us to prove in analogy with the proofs of Theorems 6, 9, 10 the validity of following relations:*

1. *Let $-\frac{\alpha}{\beta} > -\frac{\gamma}{\delta}$ with $\frac{x'}{x}$ having a maximum at t_0. Then the projection ζ has a minimum at t_0 and is discontinuous at t_0 with* $\lim_{t\to t_0} \zeta(t) > \zeta(t_0)$.

2. *Let $-\frac{\alpha}{\beta} > -\frac{\gamma}{\delta}$ with $\frac{x'}{x}$ having a minimum at t_0. Then the projection ζ is continuous at t_0 and has a maximum at t_0.*

3. Let $-\frac{\alpha}{\beta} < -\frac{\gamma}{\delta}$ with $\frac{x'}{x}$ *having a maximum at t_0 and $x \neq 0$ on $\langle t_0, \zeta(t_0)\rangle$. Then the projection ζ is continuous at t_0 and has a maximum at t_0.*

4. *Let $-\frac{\alpha}{\beta} < -\frac{\gamma}{\delta}$ with $\frac{x'}{x}$ having a minimum at t_0 and $x \neq 0$ on $\langle t_0, \zeta(t_0)\rangle$. Then the projection ζ has a minimum at t_0 and is discontinuous at t_0 with* $\lim_{t\to t_0} \zeta(t) > \zeta(t_0)$.

5. *Let $-\frac{\alpha}{\beta} < -\frac{\gamma}{\delta}$ with $\frac{x'}{x}$ having a maximum at t_0 and x having a zero in $(t_0, \zeta(t_0))$. Then the projection ζ has a minimum at t_0 and is discontinuous at t_0 with* $\lim_{t\to t_0} \zeta(t) > \zeta(t_0)$.

6. *Let $-\frac{\alpha}{\beta} < -\frac{\gamma}{\delta}$ with $\frac{x'}{x}$ having a minimum at t_0 and x having a zero in $(t_0, \zeta(t_0))$. Then the projection ζ is continuous at t_0 and has a maximum at t_0.*

In all the above cases the local extremes of the projection on $\langle a, b\rangle$ are in question.

The following Theorem 11 involves conditions sufficient for the existence of the projection ζ on the interval with an point $t_0 \in i$ at which the projection ζ is defined. The proof of this assertion proceeds completely analogous to those of Theorems 6, 8, 9, 10 and therefore it is left out.

Theorem 11. *Let the projection ζ be defined at the point $t_0 \in (a, b) \subset i$ assuming the value $\zeta(t_0) \in (c, d) \subset i$. Let next $w \neq 0$ on the interval (a, d). Then it holds:*

1. *Let $\zeta(t_0)$ not be an extreme point of the space $P\sigma[\gamma, \delta]$, then there exists a real number $h > 0$ such that the projection ζ is defined on the interval $(t_0 - h, t_0 + h)$.*
2. *Let t_0 not be an extreme point of the space $P\sigma[\alpha, \beta]$ and $\zeta(t_0)$ be an extreme point of the space $P\sigma[\gamma, \delta]$. Then there exists a real number $h > 0$ such that the projection ζ is defined on the interval $(t_0 - h, t_0\rangle$.*

Remark 2. *The first assertion of Theorem 11 remains valid if $\tau_i \in (t_0, \zeta(t_0))$ is a limited number of zeros of the Wronskian w. The latter assertion of Theorem 11 depends on the fact whether the Wronskian w changes the sign at its zeros or not. The projection ζ is defined either on the interval $(t_0 - h, t_0\rangle$ or on the interval $\langle t_0, t_0 + h)$.*

Theorem 12. *Let the projection ζ be defined on $j \subset i$. Then*

$$f(t) = p(\zeta(t))$$

holds for all $t \in j$ for which the characteristics f and p are defined. Next it holds

$$\varphi(t) = \psi(\zeta(t)) + k\pi$$

for k being an integer and $t \in j$.

The statement follows from Theorems 2 and 3.

Corollary 3. *Let the projection ζ be defined on $j \subset i$. Then*

$$[\alpha u(t) + \beta u'(t)]\,[\gamma v(\zeta(t)) + \delta v'(\zeta(t))] - \\ - [\alpha v(t) + \beta v'(t)]\,[\gamma u(\zeta(t)) + \delta u'(\zeta(t))] = 0 \tag{11}$$

holds for $t \in j$ and for the basis (u, v) of the space S.

Theorem 13. *Let the projection ζ be continuous and increasing or decreasing on $j_2 \subset i$ mapping this interval onto the interval $j_1 \subset i$. Then there exists exactly one transformation $T(z, \zeta, j_1, j_2)$ for which*

$$\varrho(\alpha y + \beta y') = T(\sigma(\gamma y + \delta y')), \quad \textit{where } y \in S.$$

Proof: The projection ζ satisfies properties 1, 2, 3 from Definition 4.1 [4] and also relation (11) for every $t \in j_2$, where (u, v) is the basis of the space S. Following Theorem 4.6 [4] there exists the transformation $T(z, \zeta, j_1, j_2)$ for which $T(\sigma(\gamma u + \delta u')) = \varrho(\alpha u + \beta u')$ and $T(\sigma(\gamma v + \delta v')) = \varrho(\alpha v + \beta v')$ holds. From this – with respect to Theorem 4.2 [4] – we have $\varrho(\alpha y + \beta y') = T(\sigma(\gamma y + \delta y'))$ for every function $y \in S$. By Theorem 4.4 [4] the modulus from T is a continuous functions on j_2 and

$$z(t) = \frac{\varrho(t)(\alpha y(t) + \beta y'(t))}{\sigma(\zeta(t))(\gamma y(\zeta(t)) + \delta y'(\zeta(t)))},$$

where $y \in S$ for all $t \in j_2$, for which $\sigma(\zeta(t))(\gamma y(\zeta(t)) + y'(\zeta(t))) \neq 0$. With respect to Definition 1 this yields the uniqueness of T.

Convection 2. *In all what follows we shall assume* $w \neq 0$, *and every function* $\frac{y'}{y}$ *with* $y \in S$ *being decreasing on every interval* $j \subset i$ *where it is defined. The space* S, S', $P\varrho[\alpha, \beta]$ *and* $P\sigma[\gamma, \delta]$ *are thus – with respect to Theorem* 2.13 [8] – *of the zeroth class on* i, *i.e. the phases of these spaces are monotone on the whole interval* i.

Theorem 14. *Between two neighbouring zeros of the function* $\varrho(\alpha y + \beta y') \in P\varrho[\alpha, \beta]$ *there lies exactly one zero of the function* $\sigma(\gamma y + \delta y') \in P\sigma[\gamma, \delta]$, *i.e. the zeros of the functions* $\varrho(\alpha y + \beta y')$ *and* $\sigma(\gamma y + \delta y')$ *separate themselves.*

The statement follows – with respect to Theorem 2.1 [8] and to Corollary 2.1 [8] – from the monotonicity of the function $\frac{x'}{x}$ on every interval on which it is defined.

Corollary 4. *If* $P\varrho[\alpha, \beta]$ *is a space of a finite type on* i, *then* $P\sigma[\gamma, \delta]$ *is a space of a finite type on* i; *specially: if* $P\varrho[\alpha, \beta]$ *is of type* m *then* $P\sigma[\gamma, \delta]]$ *is of type* $m + 1$ *at most. If* $P\varrho[\alpha, \beta]$ *is of an infinite type on* i, *then* $P\sigma[\gamma, \delta]$ *is of an infinite type on* i.

Theorem 15. *Let* S *be a space of an infinite type on* m $\left(m \geqq 2, \text{ if } -\frac{\alpha}{\beta} > -\frac{\gamma}{\delta};\ m \geqq 3, \text{ if } -\frac{\alpha}{\beta} < -\frac{\gamma}{\delta}\right)$ *on* $i = (a, b)$. *Let* $t_0 \in (a, b)$ *be the least point of* (a, b) *for which the following holds: There exists an* $y \in S$ *such that* $\frac{y'(t_0)}{y(t_0)} = -\frac{\alpha}{\beta}$ *and* $\frac{y'(t)}{y(t)} \neq -\frac{\gamma}{\delta}$ *for all* $t \in (t_0, b)$. *Let* $t_0^* \in (a, b)$ *be the greatest point of* (a, b), *for which the following holds: There exists an* $y_1 \in S$ *such that* $\frac{y_1'(t_0^*)}{y_1(t_0^*)} = -\frac{\gamma}{\delta}$ *and* $\frac{y_1'(t)}{y_1(t)} \neq -\frac{\alpha}{\beta}$ *for all* $t \in (a, t_0^*)$.

Then the projection ζ *is continuous and increasing on* (a, t_0) *mapping this interval onto the interval* (t_0^*, b).

Proof: Since the space S is defined on an open interval, the interval of definition of ζ is evidently an open interval. If a point $T \in (t_0, b)$ existed at which the projection ζ would be defined, then there would exist a function $x \in S$ such that $\frac{x'(\zeta(T))}{x(\zeta(T))} = -\frac{\gamma}{\delta}$ where $\zeta(T) \in (T, b)$. This would imply that the functions $\frac{y'}{y}$ and $\frac{x'}{x}$ would assume the same value at a point of $(T, \zeta(T))$, which with respect to the assumption $w \neq 0$ is impossible. Likewise may be shown that the projection ζ assumes the value $t < t_0^*$. By Theorem 7 the projection is increasing and continuous.

Theorem 16. *Let the assumptions of Theorem* 15 *be fulfilled and write* $j_2 = (a, t_0)$, $j_1 = (t_0^*, b)$. *Then there exists exactly one transformation* $T(z, \zeta, j_1, j_2)$ *for which*

$$\varrho(\alpha y + \beta y') = T(\sigma(\gamma y + \delta y')), \qquad \text{where } y \in S.$$

The statement follows from Theorems 13 and 15.

Theorem 17. *Let* $P\varrho[\alpha, \beta]$ *and* $P\sigma[\gamma, \delta]$ *be spaces of types* $+\infty$ *or* $-\infty$ *on the interval* $i = (a, b)$. *Let* $t_0, t_0^* \in (a, b)$ *be the points of Theorem* 15 *in so far as these points exist. Then:*

1. *if the spaces* $P\varrho[\alpha, \beta]$ *and* $P\sigma[\gamma, \delta]$ *are of type* $+\infty$ *on* (a, b), *then the projection* ζ *is defined, continuous and increasing on* (a, b) *mapping this interval onto* (t_0^*, b).

2. *if the spaces* $P\varrho[\alpha, \beta]$ *and* $P\sigma[\gamma, \delta]$ *are of type* $-\infty$ *on* (a, b), *then the projection* ζ *is defined, continuous and increasing on* (a, t_0) *mapping this interval onto* (a, b).

The statement follows from Theorems 14 and 15.

Theorem 18. *Let* $P\varrho[\alpha, \beta]$ *and* $P\sigma[\gamma, \delta]$ *be spaces of type* $\pm\infty$ *on* i. *Then the projection* ζ *is continuous and increasing on* i *mapping the interval* i *on itself.* The statement follows from Theorem 17.

Theorem 19. *Let* $P\varrho[\alpha, \beta]$ *and* $P\sigma[\gamma, \delta]$ *be spaces of type* $\pm\infty$ *on* i. *Then:*

1. $\zeta(t) = p^{-1}(f(t))$ *for all* $t \in i$, *for which the characteristics* p *and* f *are defined.*

2. $\zeta(t) = \psi^{-1}(\varphi(t))$, *where* $t \in i$ *and* φ *and* ψ *are phases satisfying the equation* $\varphi(t_0) = \psi(\zeta(t_0))$ *at a point* $t_0 \in i$.

With respect to Theorem 12, the statement follows from the monotonicity of the characteristics p, f and from phases φ, ψ on their intervals of definition.

Theorem 20. *Let* $P\varrho[\alpha, \beta]$ *and* $P\sigma[\gamma, \delta]$ *be spaces of type* $\pm\infty$ *on* i. *Then there exists exactly one total transformation* $T(z, \zeta)$ *of the space* $P\sigma[\gamma, \delta]$ *onto* $P\varrho[\alpha, \beta]$ *for which*

$$\varrho(\alpha y + \beta y') = T(\sigma(\gamma y + \delta y')), \qquad \text{where } y \in S.$$

The statement follows from Theorems 13 and 18.

Remark 3. *Following Theorem* 2.15 [8] *the set of integrals of the first accompanying equation* (q_1) *with bases* $[\alpha, \beta]$ *to the equation* (q) *(see* [2]) *is a two-dimensional accompanying space* $P\varrho[\alpha, \beta]$ *to the space* S *of the integrals of* (q), *where*

$$\varrho = \frac{1}{\sqrt{\alpha^2 - \beta^2 q}}.$$

The projection ζ *of bundles* $\alpha y + \beta y'$ *and* $\gamma y + \delta y'$, *where* y *is an integral of* (q), *is evidently a special case of the projection* ζ *of the pair of accompanying spaces*

$\{P\varrho[\alpha, \beta], P\sigma[\gamma, \delta]\}$ to the linear two-dimensional space S of functions with a continuous first derivative. Comparing the results of [3] we see that in case the spaces $S, S', P\varrho[\alpha, \beta]$ and $P\sigma[\gamma, \delta]$ are of the zeroth class, the projection ζ of the pair of spaces $\{P\varrho[\alpha, \beta], P\sigma[\gamma, \delta]\}$ has similar properties as the projection of bundles of integrals of (q) with bases $[\alpha, \beta]$ and $[\gamma, \delta]$.

ЦЕНТРАЛЬНАЯ ПРОЕКЦИЯ ПАРЫ СОПРОВОДИТЕЛЬНЫХ ПРОСТРАНСТВ К ЛИНЕЙНОМУ ДВУХРАЗМЕРНОМУ ПРОСТРАНСТВУ ФУНКЦИЙ С НЕПРЕРЫВНОЙ ПЕРВОЙ ПРОИЗВОДНОЙ

Резюме

Пусть $P\varrho[\alpha, \beta]$ и $P\sigma[\gamma, \delta]$ сопроводительные пространства к двухразмерному пространству $S \subset C_1(i)$, где $\alpha, \beta, \gamma, \delta$ не равные нулю вещественные постоянные, $\alpha\delta - \beta\gamma \neq 0$, и $\varrho > 0$, $\sigma > 0$ непрерывные функции на интервале i. Определяется центральная проекция упорядоченной пары пространств $\{P\varrho[\alpha, \beta], P\sigma[\gamma, \delta]\}$ и исследуются ее свойства. В работе определены необходимые и достаточные условия для существования проекции ζ в точке $t_0 \in i$ и показаны достаточные условия для существования проекции ζ на интервале. Исследуются свойства проекции ζ в зависимости от екстремальных точек пространств $P\varrho[\alpha, \beta]$ и $P\sigma[\gamma, \delta]$. Показывается, что проекция ζ не должна быть монотонной и не непрерывной на своем интервале определенности. В случае, когда проекция ζ непрерывна и монотонна на интервале $j_2 \subset i$, то она является амплитудой трансформации $T(z, \zeta, j_1, j_2)$ пространства $P\sigma[\gamma, \delta]$ в интервале $j_1 = \zeta(j_2)$ на пространство $P\varrho[\alpha, \beta]$ в интервале j_2.

Работа тоже касается пространств $P\varrho[\alpha, \beta]$ и $P\sigma[\gamma, \delta]$, которые нулевого класса — то есть, у которых ненаходятся екстремальные точки. В таком случае получим подобные результаты как в случае пространств интегралов первых сопроводительных уравнений к уравнению (q): $y'' = q(t)y$, где $q < 0$ непрерывная функция на интервале i, с базисами $[\alpha, \beta]$, $[\gamma, \delta]$. Проекция ζ непрерывна и возрастающая на своем интервале определения. В случае, что пространства $P\varrho[\alpha, \beta]$ и $P\sigma[\gamma, \delta]$ типа $\pm\infty$ на интервале i, то проекция ζ непрерывна и возрастающая на интервале i, отображает интервал i на себя и является амплитудой полной трансформации $T(z, \zeta)$ пространства $P\sigma[\gamma, \delta]$ на пространство $P\varrho[\alpha, \beta]$, причем $\varrho(\alpha y + \beta y') = T(\sigma(\gamma y + \delta y'))$ где $y \in S$.

CENTRÁLNÍ PROJEKCE DVOJICE PRŮVODNÍCH PROSTORŮ K LINEÁRNÍMU DVOJROZMĚRNÉMU PROSTORU FUNKCÍ SE SPOJITOU PRVNÍ DERIVACÍ

Souhrn

Nechť $P\varrho[\alpha, \beta]$ a $P\sigma[\gamma, \delta]$ jsou průvodní prostory k dvojrozměrnému prostoru $S \subset C_1(i)$, kde $\alpha, \beta, \gamma, \delta$ jsou reálné konstanty různé od nuly, $\alpha\delta - \beta\gamma \neq 0$, a $\varrho > 0$, $\sigma > 0$ jsou funkce spojité na intervalu i. Je definována centrální projekce uspořádané dvojice prostorů $\{P\varrho[\alpha, \beta], P\sigma[\gamma, \delta]\}$

a jsou zkoumány její vlastnosti. Jsou nalezeny nutné a postačující podmínky pro existenci projekce ζ v bodě $t_0 \in i$ a uvedeny postačující podmínky pro existenci projekce ζ na intervalu. Dále je vyšetřován průběh projekce ζ v souvislosti s extrémními body prostorů $P\varrho[\alpha, \beta]$ a $P\sigma[\gamma, \delta]$. Ukazuje se, že projekce ζ nemusí být monotonní ani spojitá na svém definičním intervalu. Z vlastností projekce ζ plyne za předpokladu její spojitosti a monotonnosti na intervalu $j_2 \subset i$, že je amplitudou transformace $T(z, \zeta, j_1, j_2)$ prostoru $P\sigma[\gamma, \delta]$ v intervalu $j_1 = \zeta(j_2)$ na prostor $P\varrho[\alpha, \beta]$ v intervalu j_2.

Dále jsou uvažovány prostory $P\varrho[\alpha, \beta]$ a $P\sigma[\gamma, \delta]$, které jsou nulté třídy, tj. nemají extrémní body. V tomto případě je situace podobná jako v prostorech řešení prvních průvodních rovnic k rovnici (q): $y'' = q(t)y$, kde $q < 0$ je funkce spojitá na intervalu i, při bázích $[\alpha, \beta]$ a $[\gamma, \delta]$. Projekce ζ je spojitá a rostoucí na svém definičním intervalu. V případě, že prostory $P\varrho[\alpha, \beta]$ a $P\sigma[\gamma, \delta]$ jsou navíc typu $\pm \infty$ na intervalu i, je projekce ζ spojitá a rostoucí na celém intervalu i, zobrazuje interval i na sebe a je amplitudou úplné transformace $T(z, \zeta)$ prostoru $P\sigma[\gamma, \delta]$ na prostor $P\varrho[\alpha, \beta]$, pro kterou platí $\varrho(\alpha y + \beta y') = T(\sigma(\gamma y + \delta y'))$, kde $y \in S$.

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