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ON THE RADICAL IN HERMITIAN LMC ALGEBRAS

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Abstract.

V.Pták, s.f. Pták V. (1972) proved that the radical (in the sense of Jacobson) of the Hermitian Banach star algebra is equal to the kernel of the spectral norm. The present paper will show that this is true also for every complete locally multiplicatively convex Hermitian star algebra.

1. Notations and preliminaries.

All linear spaces are over the complex field C. The reader is assumed to be familiar with the basic concepts concerning the topological algebras, namely the Banach algebras, star algebras, locally multiplicatively convex /lmc/ algebras, notion of representation and irreducible representation of the topological algebra, spectra, Gelfand representation theory for the commutative case and so on. See Bonsall F., Duncan J. (1973), Michael E. (1952), Najmark A. (1968), Żelazko W. (1971). Let us now recall some notations and preliminary facts. Let A be a complete lmc-algebra. Let $\{q_\alpha\}_{\alpha\in\Sigma}$ be the directed set of submultiplicative seminorms on A separating points at A such that $A = \varprojlim A_\alpha$, where A_α denotes the completion of the normed algebra $\{A/\mathrm{Ker}\, q_\alpha, q_\alpha\}$. By π_α we denote the natural homomorphism mapping A onto A_α. Throughout, the spectrum of an element $x \in A$ will be denoted by $\sigma(x,A)$ pointing out that it is taken with respect to A. The corresponding spectral radius will be denoted by $|x|_\alpha^A$. Obviously, an element $x \in A$ is regular if and only if for each $\alpha\in\Sigma$ $\pi_\alpha(x)$ is regular in the algebra A_α yielding the equality $\sigma(x,A) = \bigcup_{\alpha\in\Sigma}\sigma(\pi_\alpha(x), A_\alpha)$. Then, for the spectral radius $|x|_\sigma^A = \sup|\pi_\alpha(x)|_\sigma^{A_\alpha}$. If no confusion is possible we use the following notations: for every $\alpha\in\Sigma$ $x_\alpha = \pi_\alpha(x)$, $|\pi_\alpha(x)|_\sigma^{A_\alpha} = |x|_\sigma^\alpha$, $p(\pi_\alpha(x)) = \sqrt{|x^* x|_\sigma^{A_\alpha}} = p_\alpha(x)$, $p(x) = \sqrt{|x^* x|_\sigma}$, where p and p_α mean the spectral norms in the corresponding algebras.

Definition 1.1. The element x from a star algebra A is said to be Hermitian, if $x^* = x$. The set of all Hermitian elements of A will be denoted by H(A). The element $x \in A$ is said to be normal if $x^*x = xx^*$. The set of all normal elements will be denoted by N(A).

Definition 1.2. The star algebra A is said to be Hermitian if the spectrum $\sigma(x)$ is real for any $x \in H(A)$.

Definition 1.3. The radical in the sense of Jacobson of the algebra A is the set

$$\mathrm{Rad}\, A = \{x \in A : f(x) = 0 \text{ for each strictly irreducible representation } f \text{ of } A\}$$

Recall now three well known propositions concerning the radical. Proofs may be found in Żelazko W. (1971).

Proposition 1.4. Let A be a complete lmc-star algebra with the unit element e, $\{q_\alpha\}_{\alpha\in\Sigma}$ the corresponding directed set of submultiplicative seminorms defining the topology on A. Then

(i) $|x|_\sigma = \sup_{\alpha\in\Sigma} \lim_n (q_\alpha(x^n))^{1/n} = \sup_{\alpha\in\Sigma} |x|_\sigma^\alpha$

(ii) If A is commutative, then

$|x|_\sigma = \sup \{|f(x)| :$ f-multiplicative continuous functional$\}$ on A

(iii) For each $\alpha\in\Sigma$ and for arbitrary $x \in A$ $|x|_\sigma^\alpha \leqq q_\alpha(x)$.

Proposition 1.5. Let A be an arbitrary algebra. Then

(i) Rad A $= \bigcap_{I_l\in\mathcal{J}_l} I_l = \bigcap_{I_p\in\mathcal{J}_p} I_p$, where $\mathcal{J}_l$, $\mathcal{J}_p$ is the set of all maximal modular left (right) ideals of A respectively.

(ii) If A has besides an unit element e, then Rad A $= \bigcap_{I_l\in\mathcal{J}_l} I_l = \bigcap_{I_p\in\mathcal{J}_p} I_p = \{x\in A : |ax|_\sigma = 0$ for each $a \in A\} = \{x\in A : |xa|_\sigma = 0$ for each $a\in A\}$. Here $\mathcal{J}_l$, $\mathcal{J}_p$ denotes the set of all maximal left (right) ideals of A respectively.

(iii) If A is besides commutative and has the unit element e, then Rad A $= \bigcap_{I\in\mathcal{J}} I$, where $\mathcal{J}$ denotes the set of all maximal ideals of A.

Proposition 1.6. Let A be a commutative Banach algebra with the unit element e. Then Rad A $= \{x\in A : |x|_\sigma = 0\} =$ Ker G $= \bigcap_{f\in\mathfrak{M}(A)}$ Kerf, where G denotes the Gelfand representation of A and $\mathfrak{M}(A)$ the set of all multiplicative functionals of A.

Corollary 1.7. Let A be a complete lmc-star algebra, $\{q_\alpha\}_{\alpha\in\Sigma}$

the corresponding directed set of seminorms. Then

(i) Rad A $= \{ x \in A : |ax|_{\sigma}^{\alpha} = 0$ for each $\alpha \in \xi$, $a \in A \} =$

$= \{ x \in A : |xa|_{\sigma}^{\alpha} = 0$ for each $\alpha \in \xi$, $a \in A \}$

(ii) If A is besides commutative, Rad A $= \bigcap_{f \in \mathfrak{M}^c(A)}$

Ker f $= \bigcap_{\alpha \in \xi} \bigcap_{f \in \mathfrak{M}^c(A)}$ Kerf = Ker G $= \bigcap_{\alpha \in \xi}$ Ker G_{α} =

$= \bigcap_{\alpha \in \xi} \{ x \in A : |x|_{\sigma}^{\alpha} = 0 \} = \{ x \in A : |x|_{\sigma} = 0 \}$

where G and G_{α} denote the corresponding Gelfand representations and $\mathfrak{M}^c(A)$, ($\mathfrak{M}^c(A_{\alpha})$) the sets of all continuous multiplicative functionals on A (A_{α}), respectively.

Proof: follows immediately from the precending propositions.

Q.E.D.

Theorem 1.8. Let A be a complete lmc-star algebra with the unit element e and the corresponding system of seminorms $\{q_{\alpha}\}_{\alpha \in \xi}$. Then the following conditions are equivalent:

(i) The algebra A is Hermitian.

(ii) $|\pi_{\alpha}(x)|_{\sigma}^{A_{\alpha}} = p(\pi_{\alpha}(x)) = p_{\alpha}(x)$ for each normal element $x \in A$ and for all $\alpha \in \xi$.

Proof: See Štěrbová AUPO (1983) Q.E.D.

Theorem 1.9. Let A be Hermitian lmc-star algebra which is complete. Let arbitrary $x \in A$ be given. Then $\sigma(x^* x) \geqq 0$.

Proof: See Štěrbová AUPO (1985) Q.E.D.

Lemma 1.10. Let A be same as in Theorem 1.9. For arbitrary a, $b \in H(A)$ and for each $\alpha \in \xi$ we have

$$|a^2b^2|_{\sigma}^{\alpha} \leqq |a^2|_{\sigma}^{\alpha} \, |b^2|_{\sigma}^{\alpha}$$

Proof: See Štěrbová AUPO (1985) Q.E.D.

2. The characterization of the radical by spectral norms.

In what follows all algebras are supposed to be Hermitian complete lmc-star algebras with the unit element e.

Proposition 2.1. Let $\alpha \in \Sigma$, $h, k \in H(A)$ be given such that $\sigma_\alpha(h) \geqq 0$, $\sigma_\alpha(k) \geqq 0$. Then $\sigma_\alpha(h+k) \geqq 0$.

Proof:

The simple spectral property together with Theorem 1.9. yields for arbitrary $\varepsilon > 0$ $\sigma((h+\varepsilon)^2) \geqq 0$ and $\sigma((k+\varepsilon)^2) \geqq 0$. For each integer $n = 1,2,\dots$ it follows $\sigma((h+\varepsilon)^2 + 1/n) > 0$ and $\sigma((k+\varepsilon)^2 + 1/n) > 0$. Using the results of Štěrbová, AUPO (1980, 1984) we can easily show the existence of the positive square roots $o_n \in H(A)$ of $((h+\varepsilon)^2 + 1/n)$ and $o_n' \in H(A)$ of $((k+\varepsilon)^2 + 1/n)$. This immediately shows that the sequence $\{o_n\}_{n=1}^{\infty}$ and $\{o_n'\}_{n=1}^{\infty}$ converge to the positive square root o_ε, o_ε' of $(h+\varepsilon)^2$ and $(k+\varepsilon)^2$, respectively.

Now the projection $\pi_\alpha(o_\varepsilon)$ is the positive square root of $(h_\alpha + \varepsilon)^2$ and by the unicity of the square root for Banach algebra, see Štěrbová AUPO (1980) we obtain $\pi_\alpha(o_\varepsilon) = (h_\alpha + \varepsilon)$. In the same way we obtain $\pi_\alpha(o_\varepsilon') = (k_\alpha + \varepsilon)$ where the last term denotes the unic positive square root of $(k_\alpha + \varepsilon)^2$ in A_α. This implies for any arbitrary $\varepsilon > 0$:

$$\sigma_\alpha(\varepsilon + h + \varepsilon + k) = \sigma(\pi_\alpha(\varepsilon + h) + \pi_\alpha(\varepsilon + k)) =$$

$$= \sigma(\pi_\alpha(o_\varepsilon) + \pi_\alpha(o_\varepsilon')) = \sigma(\pi_\alpha(o_\varepsilon + o_\varepsilon')) =$$

$$= \sigma_\alpha(o_\varepsilon + o_\varepsilon') \subset \sigma(o_\varepsilon + o_\varepsilon') \geqq 0$$

Thus

$$\sigma_\alpha(2\varepsilon + h + k) = 2\varepsilon + \sigma_\alpha(h+k) \geqq 0$$

The usual limitation procedure for $\varepsilon > 0$ yields.

$$\sigma_\alpha(h+k) \geqq 0 \qquad \text{Q.E.D.}$$

Corollary 2.2. Let $\{q_\alpha\}_{\alpha\in\xi}$ be the directed set of submultiplicative seminorms defining the topology on A. The spectral radius $|\ \ |_\sigma^\alpha$ is a subadditive function on H(A) for all $\alpha\in\xi$.

Proof:

Suppose conversely that there exist $\alpha\in\xi$ and $h,k\in H(A)$ such that $|h+k|_\sigma^\alpha > |h|_\sigma^\alpha + |k|_\sigma^\alpha$. Then $(h+k)\in H(A)$ and there obviously exists a real number $\lambda\in\sigma_\alpha(h+k)$ such that $\lambda>|h|_\sigma^\alpha+|k|_\sigma^\alpha$. Without any loss of generality we can suppose $\lambda > 0$. Now we can easily find positive numbers ε, λ_1, λ_2 such that $\lambda_1 > |h|_\sigma^\alpha$, $\lambda_2 > |k|_\sigma^\alpha$ and $\lambda = \lambda_1 + \lambda_2 + \varepsilon$. Elements (λ_1-h), (λ_2-k) are obviously hermitian and $\sigma_\alpha(\lambda_1-h) > 0$, $\sigma_\alpha(\lambda_2-k) > 0$, and the preceding theorem yields

$$\sigma_\alpha(\lambda_1 + \lambda_2 - h-k) \geqq 0$$

Hence $\sigma_\alpha(\lambda-h-k) = \sigma_\alpha(\lambda_1 + \lambda_2 + \varepsilon - h-k) > 0$ which contradicts to $\lambda\in\sigma_\alpha(h+k)$.

Q.E.D.

Proposition 2.3. Let $\{q_\alpha\}_{\alpha\in\xi}$ be the same as in Corollary 2.2. The spectral norm p_α is submultiplicative on A for all $\alpha\in\xi$.

Proof:

For all $\alpha\in\xi$ and for arbitrary $x,y\in A$

$$p_\alpha(xy) = (|y^*x^*x\,y|_\sigma^\alpha)^{1/2} = (|(x^*x)(y\,y^*)|_\sigma^\alpha)^{1/2}$$

By the assumption A is Hermitian, x^*x, $y\,y^*\in H(A)$ and by Theorem 1.9. the spectra of x^*x, $y\,y^*$ are nonnegative. This gives for all positive ε the existence of square roots $o_1, o_2\in H(A)$ of $(x^*x+\varepsilon)$, $(yy^*+\varepsilon)$. See Štěrbová AUPO (1980, 1984). Consequently $(\varepsilon + x^*x) = o_1^2$, $(\varepsilon + yy^*) = o_2^2$.

Lemma 1.10. gives for each $\varepsilon > 0$

$$|(x^*x+\varepsilon)(y\,y^*+\varepsilon)|_\sigma^\alpha = |c_1^2\,c_2^2|_\sigma^\alpha \leqq |c_1^2|_\sigma^\alpha\,|c_2^2|_\sigma^\alpha =$$

$$= |x^*x+\varepsilon|_\sigma^\alpha\,|y\,y^*+\varepsilon|_\sigma^\alpha = (|x^*x|_\sigma^\alpha+\varepsilon)(|y\,y^*|_\sigma^\alpha+\varepsilon) \qquad (1)$$

The last expression converges for $\varepsilon \to 0$ to $|x^*x|_\sigma^\alpha \cdot |y\,y^*|_\sigma^\alpha$.
Let us put $h = x^*x$, $k = y\,y^*$. By the subadditivity of the spectral radius $|\ |_\sigma^\alpha$ follows:

$$\big||(h+\varepsilon)(k+\varepsilon)|_\sigma^\alpha - |hk|_\sigma^\alpha\big| \leqq |(h+\varepsilon)(k+\varepsilon) - hk|_\sigma^\alpha \leqq$$

$$\leqq q_\alpha((h+\varepsilon)(k+\varepsilon) - hk) \leqq \varepsilon\, q_\alpha(h) + \varepsilon\, q_\alpha(k) + \varepsilon^2 \qquad (2)$$

Thus $|(x^*x+\varepsilon)(y\,y^*+\varepsilon)|_\sigma^\alpha$ converges for $\varepsilon \to 0$ to $|x^*x\,y\,y^*|_\sigma^\alpha$.
The usual limitation procedure for inequality (1) yields:

$$p_\alpha^2(xy) = |x^*x\,y\,y^*|_\sigma^\alpha \leqq |x^*x|_\sigma^\alpha\,|y\,y^*|_\sigma^\alpha = p_\alpha^2(x)\cdot p_\alpha^2(y)$$

Q.E.D.

Corollary 2.4. The spectral norm p is submultiplicative on A.
Proof:
It immediately follows from the precedinng proposition and by the welknown relation $p(x) = \sup_{\alpha\in\Sigma} p_\alpha(x)$.

Q.E.D.

Theorem 2.5. The following statements hold for all $\alpha \in \Sigma$:

(i) $\{x \in A : |ax|_\sigma^\alpha = 0 \text{ for all } a \in A\} \subset \operatorname{Ker} p_\alpha$

(ii) $\operatorname{Rad} A = \operatorname{Ker} p = \bigcap_{\alpha\in\Sigma} \operatorname{Ker} p_\alpha$

Proof:
(i) : Let $x \in \operatorname{Rad} A$. Then $x^*x \in \operatorname{Rad} A$ as the last is an ideal

in A. By Corollary 1.7. it follows $p(x) = (|x^*x|_\sigma^\alpha)^{1/2} = 0$. Suppose now that $p(x) = 0$ holds for $x \in A$. By Theorem 1.8., Theorem 1.9. then follows

$$|ax|_\sigma \leqq p(ax) \leqq p(x)\ p(a) = 0 \tag{3}$$

for all $a \in A$, where the last equality of (3) holds because of $p_\alpha(x) \cdot p_\alpha(a) = 0 \cdot p_\alpha(a) = 0$ for all $\alpha \in \Sigma$. The remainder of (i) is now evident.

(ii) :

Suppose for some $\alpha \in \Sigma$ and some $x \in A$ that $|ax|_\sigma^\alpha = 0$ for all $a \in A$. Especially for $a' = ax^*$ then

$$|(ax^*)x|_\sigma^\alpha = 0 \text{ for all } a \in A$$

and thus

$$|x^*x|_\sigma^\alpha = 0$$

hence

$$x \in \operatorname{Ker} p_\alpha$$

Q.E.D.

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SOUHRN

O RADIKÁLU V HERMITEOVSKÉ lmc ALGEBŘE

Dina Štěrbová

V práci je dokázáno, že radikál (ve smyslu Jacobsonově) úplné Hermiteovské lokálně m-konvexní algebry s involucí je roven jádru spektrální normy.

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РЕЗЮМЕ

ОБ РАДИКАЛАХ ВПОЛЬНЕ СИММЕТРИЧЕСКИХ ПОЛУНОРМИРОВАННЫХ КОЛЕЦ

ДИНА ЩЕРБОВА

В настоящей работе показывается, что радикал польного вполне симметрического полунормированного кольца с инволюцией является нуль-множеством спектральной нормы.