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A USEFUL PROPOSITION TO NONLINEAR DIFFERENTIAL SYSTEMS WITH A SOLUTION OF THE PRESCRIBED ASYMTOTIC PROPERTIES

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A study of differential equations and systems admitting some solutions of the prescribed properties is very important from the physical point of view /4/. The mathematical meaning of this problem consists of their finding to the given solutions. Thus, such a problem can be regarded as an inverse to the usual ones. Although nonconstructive approaches allowing to omit certain alternatives only are considered sometimes in this field, a "power" of sufficient conditions with respect to their necessity may be judged at that time at least.

Consider a system of first order differential equations

$$X' = F(t,X) \qquad /' \equiv \frac{d}{dt}/ , \tag{1}$$

*) The author gave a lecture on its applications at the Colloquium on the Qualitative Theory of Differential Equations (Szeged, August, 1984)

where $F \in C^0 : R^1 \times R^n \rightarrow R^n$ and assume that all solutions $X(t)$ are uniquely determined by the Cauchy's initial values

$$X(0) = X_0 \tag{2}$$

and continuously depend on them. Our aim is to secure the existence of a solution $X(t)$ of (1) with

$$\limsup_{|t| \to \infty} \| X(t) - \Omega(t) \| < \infty , \tag{3}$$

where $\Omega(t)$ is an everywhere continuous function of the prescribed asymptotic behaviour.

The following intuitively clear lemma which is only a slightly modified assertion from /5, pp.178-180/ allows us to realize it.

Lemma. Let $\{X_k(t)\}$ be the sequence of solutions of (1) such that every element $X_k(t)$ is defined on the interval $\langle -T_k, T_k \rangle$, where $\lim_{k \to \infty} T_k = \infty$ and

$$X_k(t) - \Omega(t) \in I \qquad t \in \langle -T_k, T_k \rangle , \tag{4}$$

where I denotes a bounded subset of R^n and $\Omega(t)$ is the above function. Then (1) possesses at least one solution $X(t)$ with (3).

Proposition. The system (1) admits a solution $X(t)$ with (3), when the following two conditions are satisfied:

$$\frac{F(0,X)}{|F(0,X)|} \neq \frac{F(0,-X)}{|F(0,-X)|} \qquad /F(0,X) \neq 0/ \tag{i}$$

for $\|X\| \geq R > 0$... great enough number.

(ii) a priori uniform boundedness of all solutions $X(t)$ of the problem (1) $\cap$ (1_μ), where

$$(1_\mu) \qquad X(\pm\mu T_k) = X(0) + \mu^i \Omega(\pm T_k) \qquad \mu \in (0,1),$$

$|T_k| \in (0,\infty)$ /see Lemma/ and $1 < i$...const.

P r o o f. Let us consider the sequence of the boundary value problems (1) $\cap$ (1_1) for $k = 1,2,\ldots$ and define in a corresponding way the modified translation operator T /see /5// as

$$T_\mu(X_0) := \begin{cases} \left[X(\pm\mu T_k, X_0) - X(0) - \mu^i \Omega(\pm T_k)\right] / (\pm\mu T_k) & \text{for } \mu \in (0,1] \\ F(0, X_0) & \text{for } \mu = 0, \end{cases}$$

where $X(t,X_0) = X(t,X(0))$ is the solution $X(t)$ of (1) with (2).

It is obvious that the problem (1) $\cap$ (1_1) is solvable for a fixed k if and only if

$$(5) \qquad T_1(X_0) = 0 .$$

But since we assume on a priori uniform boundedness of the expressions from (ii), which implies

$$T_\mu(X_0) \neq 0 \qquad \mu \in (0,1)$$

for $\|X_0\| \geq R$...great enough number, the satisfying

$$(6) \qquad T_0(X_0) \neq 0 \qquad \text{for } \|X_0\| = R$$

is enough instead of (5) (see /3, p. 20/, for more details see also /1/) and consequently, we may only assume

$$T_0(X_0) - (1 - \nu)T_0(-X_0) \neq 0 \quad \text{for} \quad \nu \in (0,1),$$

because the topological degree

$$d\left[T_0(X_0)-T_0(-X_0),\ \|X_0\| \leq R, 0\right] \neq 0 \quad \text{for} \quad \|X_0\| = R$$

with respect to the Borsuk theorem (see /3, p. 24/). That is, however, (i) for $F(0,X) \neq 0$.

This guarantees the existence of solutions $X_k(t)$ of (1) $\cap$ (1_1) with (4) on the interval $\langle -T_k, T_k \rangle$ for $k = 1,2,\ldots$ and that is why at least one solution $X(t)$ of (1) with (3) must exist with respect to Lemma, too.

R e m a r k. If there exist such a function $Y(t)$ and such a constant ω that

$$(7) \qquad F(t + \omega, Y(t + \omega)) \equiv F(t, Y(t)),$$

the existence problem of a solution $X(t)$ of (1), which is partially periodic (i.e. in one its component at least), is reasonable.

Since (7) yields for such solutions $X(t)$ that $X'(t+\omega) \equiv X'(t)$, it is necessary to replace condition (3) by

$$X(k\omega) - X(0) = \Omega(k\omega) \qquad /T_k = k\omega/$$

with one component of the vector $\Omega(\omega)$ not equal to zero at least and Lemma by the assumption of an "$\Omega(k\omega)$-periodicity" of the function $F(t,X)$ in X allowing to provide an ω-partial periodic prolongation of solutions $X(t)$ on the whole interval $(-\infty, \infty)$.

E x a m p l e. Consider the special system (1) for n = 2:

$$(1') \qquad X' = AX + F_0(t,X),$$

where $X = \begin{bmatrix} x \\ y \end{bmatrix}$, $A = \begin{bmatrix} 0 & 1 \\ c & 0 \end{bmatrix}$, $F_0(t,X) = \begin{bmatrix} 0 \\ f(x) + e(t) + c\frac{\Omega}{\omega}t \end{bmatrix}$

and $f(x + \Omega) \equiv f(x)$, $e(t + \omega) \equiv e(t)$.

In view of Proposition and Remark, it can be proved in an analogous way to /2/ that (1') admits for $0 \neq |c| < 1/\omega^2$ a solution $X(t)$ with $y(t + \omega) \equiv y(t)$ and $x(t + \omega) \not\equiv x(t)$.

Following the idea of R. R e i s s i g /6/, we can furthermore specify this solution $X(t)$ to be just one under the hypotheses

$$0 \leq c(x - y)^2 + [f(x) - f(y)](x - y) < (x - y)^2 ,$$

resp. $0 \geq [f(x) - f(y)](x - y) + c(x - y)^2$,

holding for all real x,y.

F u r t h e r i m p o r t a n t r e m a r k. Taking into account the boundedness of solutions of (1) $/\Omega(t) \equiv 0/$, it can be seen by stepwise critical reading the proofs in /7/ that the criteria, obtained there for the so called D'-divergent solutions via the Liapunov's direct method, guarantee simultaneously the uniform a priori boundedness of those to $(1) \cap (1_\mu)$ for $\mu \in (0,1\rangle$, and consequently a bounded solution is admitedd under (i).

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SOUHRN

JISTÁ UŽITEČNÁ PROPOZICE PRO NELINEÁRNÍ DIFERENCIÁLNÍ SYSTÉMY S ŘEŠENÍM PŘEDEPSANÝCH ASYMPTOTICKÝCH VLASTNOSTÍ

JAN ANDRES

V práci je vyslovena propozice, umožňující studium obecných diferenciálních systémů vzhledem k existenci jejich řešení, majících předem zadané vlastnosti. S využitím modifikovaného Krasnoselského lemmatu a výsledků teorie topologického stupně zobrazení je tento problém převeden na otázku a priorních odhadů řešení jisté posloupnosti okrajových úloh a lichosti normovaného operátoru pravých stran.

РЕЗЮМЕ

ОДНО ПОЛЕЗНОЕ ПРЕДПОЛОЖЕНИЕ ДЛЯ НЕЛИНЕЙНЫХ ДИФФЕРЕНЦИАЛЬНЫХ СИСТЕМ РЕШЕНИЕ КОТОРЫХ ОБЛАДАЕТ ЗАДАННЫМИ АСИМПТОТИЧЕСКИМИ СВОЙСТВАМИ

ЯН АНДРЕС

В работе сформулировано предложение, удобное для изучения общих дифференциальных систем по отношению к существованию их решений с заданными свойствами. При помощи обработанной леммы Красносельского и результатов теории топологической степени отображения приводят эту проблему к вопросу об априорных оценках решений одной серии краевых задач и нечетности нормированного оператора правых частей.

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