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SYMMETRY IN A CERTAIN SIMPLY PERTURBED HAMILTONIAN SYSTEM

JAN ANDRES

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Dedicated to Professor M.Laitoch on his 65th birthday

1. Recently an increasing attention has been paid to the study of the Lotka-Volterra system [1], namely

$$u' = \lambda u(1-v), \qquad v' = -\varrho v(1-u),$$

where λ, ϱ are control parameters. As is well-known such a system can be transformed by means of u=exp x, v=exp y to the form

$$x' = -g(y), \qquad y' = h(x)$$

with $g(y) = \lambda(\exp y - 1)$, $h(x) = \varrho(\exp x - 1)$. Therefore especially in case, when $h(x) = -f(x)$, $g(y) = -f(y)$ should be of interest in view of

$$\int_{x(0)}^{x(t)} f(s)ds + \int_{y(0)}^{y(t)} f(s)ds = \text{const.}$$

being the corresponding first integral. If the same hamiltonian system is considered under some small external forcing moreover, namely

$$x' = f(y) + \mu p(t), \qquad y' = - f(x) + \nu q(t), \tag{1}$$

where μ, ν are the positive small parameters, then the natural question appears with respect to the solutions $[x(t),y(t)] \in (1)$, having the property

$$\lim_{t\to\infty} [|x(t)| - |y(t)|] = 0, \tag{2}$$

called here as an asymptotic symmetry: "are these solutions also stable in the sense of Lagrange?", i.e.

$$\lim_{t\to\infty} \sup [|x(t)| + |y(t)|] < \infty. \tag{3}$$

In the following text we give an affirmative answer to this problem, assuming

$$f(u) \operatorname{sgn} u \geq \delta \qquad \text{for } |u| \geq R \tag{4}$$

or

$$f(u) \operatorname{sgn} u \leq -\delta \qquad \text{for } |u| \geq R, \tag{4'}$$

where δ, R are suitable positive constants and f(u) is an everywhere continuous function as well as those of p(t), q(t).

Replacing (4) or (4') by the restriction on the bounded and oscillatory function f(u) = -f(x), we will furthermore show the solvability of our problem for the equation

$$x' = f(x) + \nu q(t), \tag{5}$$

originated from (1) in the special case of (2), namely $x(t) \equiv -y(t)$.

Hence let the prescribed assumptions hold throughout all the following text.

2. Corollary 1. *Under (4) or (4') the following relation is satisfied for each solution* $[x(t),y(t)]$ *of* (1):

$$\limsup_{t \to \infty} \max [|x(t)|, |y(t)|] = \infty \Rightarrow \liminf_{t \to \infty} \min [|x(t)|, |y(t)|] \leq R,$$

provided μ, ν are sufficiently small.

P r o o f. We employ the Yoshizawa's technique [2, p.38] consisting of the construction of a Liapunov function $V(x,y)$ defined on R^2, which satisfies the following conditions:

(i) $\lim_{(|x|+|y|) \to \infty} V(x,y) = \infty$,

(ii) $\exists R_0 > 0,\ \delta_0 > 0 \ldots (|x|+|y|) > R_0 : \ldots V'_{(1)}(x,y) \leq -\delta_0$,

where

$$V'_{(1)}(x,y) = \limsup_{\Delta t \to 0} \left\{ [V(x(t+\Delta t), y(t+\Delta t)) - V(x(t), y(t))] (\Delta t)^{-1} \right\}_{(1)}$$

and therefore ensures the uniform-ultimate boundedness of all solutions of (1).

Hence let us define

$$V(x,y) := \int_0^x f(s)ds + \int_0^y f(s)ds + \varepsilon W(x,y),$$

where $W(x,y) := y\,\mathcal{X}_R(x) - x\,\mathcal{X}_R(y)$,

$$\mathcal{X}_R(z) := \begin{cases} z \ \ldots\ldots\ldots\ldots\ldots \text{for } |z| \leq R \\ R\,\mathrm{sgn}\, z \ \ldots\ldots\ldots \text{for } |z| \geq R \end{cases}$$

and ε be a suitable real, which magnitude will be specified later.

Since for $(x,y) \in R^2$ it is

$$|W(x,y)| \leq R\,[|x| + |y|] \tag{6}$$

and since (4) or (4') surely implies such $\varepsilon \neq 0$ that

$$\liminf_{|x| \to \infty} \left| \frac{F(x)}{\varepsilon R x} \right| > 1$$

or

$$\lim_{|x|\to\infty} \| F(x)| - |\varepsilon R x \| = \infty, \qquad (7)$$

where $$F(x) := \int_0^x f(s)ds,$$

the relation (i) follows immediately from (6), (7), provided $|\varepsilon| < |\frac{\delta}{R}|$.

Now let us fix ε in the foregoing way and estimate $V'_{(1)}(x,y)$ from (ii):

$$V'_{(1)}(x,y) = \mu p(t)f(x) + \nu q(t)f(y) + \varepsilon W'_{(1)}(x,y),$$

where

$$W'_{(1)} = \begin{cases} R[f(y)-f(x)] + R[\nu q(t)+\mu p(t)] \ldots\ldots\ldots\ldots x \geq R,\ y \leq -R \\ -Rf(x)+f(x)x-f(y)y-r+R\nu q(t) \ldots\ldots\ldots\ldots x \geq R,\ |y| \leq R \\ -R[f(y)+f(x)] + R[-\mu p(t)+\nu q(t)] \ldots\ldots\ldots\ldots x \geq R,\ y \geq R \\ -Rf(y)+f(y)y-f(x)x+r-R\mu p(t) \ldots\ldots\ldots\ldots |x| \leq R,\ y \geq R \\ R[f(x)-f(y)] - R[\mu p(t)+\nu q(t)] \ldots\ldots\ldots\ldots x \leq -R,\ y \geq R \\ Rf(x)-f(y)y+f(x)x-r-R\nu q(t) \ldots\ldots\ldots\ldots x \leq -R,\ y \geq R \\ R[f(x)+f(y)] + R[\mu p(t)-\nu q(t)] \ldots\ldots\ldots\ldots x \leq -R,\ y \leq -R \\ Rf(y)+f(y)y-f(x)x+r+R\mu p(t) \ldots\ldots\ldots\ldots |x| \leq R,\ y \leq -R \end{cases}$$

and $$r = \mu p(t)y + \nu q(t)x .$$

Letting

$$|p(t)| \leq : P, \qquad |q(t)| \leq : Q \qquad \text{for all } t, \qquad (8)$$

we obtain with respect to (4) (the case (4´) can be verified in an analogous way) for positive ε that

a) $|x| \geq R$, $|y| \geq R$:

$$\frac{V'_{(1)}(x,y)}{|\varepsilon|} \leq -R\left[|f(x)| + |f(y)|\right] + |\mu p(t)|\left(R + \left|\frac{f(x)}{\varepsilon}\right|\right) + |\nu q(t)| \times$$

$$\times \left(R + \left|\frac{f(y)}{\varepsilon}\right|\right) \leq -|f(x)|\left(R - \mu\frac{P}{|\varepsilon|}\right) - |f(y)|\left(R - \nu\frac{Q}{|\varepsilon|}\right) +$$

$$+ \mu PR + \nu QR,$$

while for negative ε that

b) $|x| \geq R$, $|y| \leq R$:

$$\frac{V'_{(1)}(x,y)}{|\varepsilon|} \leq -|f(x)x| + |f(x)|(R + |\mu \frac{p(t)}{\varepsilon}|) + |x \nu q(t)| + |f(y)| \times$$

$$\times (|y| + |\frac{\nu q(t)}{\varepsilon}|) + |\mu p(t) y| + |\nu q(t) R| \leq -|f(x)x| +$$

$$+ |f(x)|(R + |\mu \frac{P}{\varepsilon}|) + |x \nu Q| + |f(y)|(R + |\nu \frac{Q}{\varepsilon}|) + \nu QR + \mu PR,$$

c) $|x| \leq R$, $|y| \geq R$:

$$\frac{V'_{(1)}(x,y)}{|\varepsilon|} \leq -|f(y)y| + |f(y)|(R + |\nu \frac{q(t)}{\varepsilon}|) + |y \mu p(t)| + |f(x)| \times$$

$$\times (|x| + |\mu \frac{p(t)}{\varepsilon}|) + |\nu q(t) x| + |\mu p(t) R| \leq |f(y)y| + |f(y)| \times$$

$$\times (R + |\nu \frac{Q}{\varepsilon}|) + |y \mu P| + |f(x)|(R + |\mu \frac{P}{\varepsilon}|) + \nu QR + \mu PR.$$

It is evident that (ii) will be satisfied separately for a) or b) or c), provided $|\varepsilon| < \delta/R$ and μ, ν are small enough. Thus according to Yoshizawa's theorem the assertion of our corollary is implied immediately.

3. In this part we shall discuss the special case of (1) (x:=-y), namely (5) under the restriction that f(x) is bounded and oscillatory with isolated roots in the following sense: for each argument x they may be found such numbers $\beta_1 > \alpha_1 > x > \alpha_{-1} > \beta_{-1}$ that

$$f(\alpha_1) < 0, f(\beta_1) > 0, f(\alpha_{-1}) < 0, f(\beta_{-1}) > 0.$$

If the uniqueness condition is satisfied, then evidently, all the solutions of the equation resulting from (5) for $\nu = 0$, namely

$$x' + f(x) = 0$$

are bounded, which cannot be said about (5), if $\nu \neq 0$.

It will be very useful to be concerned still with the equation

$$x' + (-1)^k c_k(x-d) = (-1)^k k d_k + \nu q(t) - f_k(x) \quad (k\text{-integer}), \tag{9}$$

where $c_k > 0, d_k > 0, d$ are constants and $f_k(x)$ is such a continuous function for all x that

$$f(x) = f_L(x) + f_N(x), \tag{10}$$

$$f_L(x) = (-1)^k \; c_k(x-d) - kd_k \quad (f_N(x) = f_k(x)) \text{ in } x \in \langle x^*_{k-1}, x^*_k \rangle \tag{11}$$

that there always exists just one point $\bar{x}_k \in (x^*_{k-1}, x^*_k)$ with

$$f_L(\bar{x}_k) = f_N(\bar{x}_k) = 0. \tag{12}$$

Note. *Conditions (10) - (12) may be fulfilled e.g. for* $f(x) := \sin x$

$(f_L(x) := (-1)^k (x - k\pi), \; \max_{x \in R^1} |f_N(x)| = (\frac{\pi}{2} - 1), \; x \in \langle -\frac{\pi}{2} + k\pi, -\frac{\pi}{2} +$

$+ (k+1)\pi \rangle, \; \bar{x}_k = k\pi$; k-integer).

Corollary 2. *The condition*

$$\min\left[d(x^*_{k-1}, \bar{x}_k), \; d(x^*_k, \bar{x}_k)\right] > \frac{\nu Q + F_k}{c_k}, \tag{14}$$

where $F_k := \max_{x \in \langle x^*_{k-1}, x^*_k \rangle} f_N(x)$, *holding for* $k = 0, \pm 2, \pm 4$ (*cf. also* *(8), (10) - (12)), implies the boundedness of all solutions of (5).*

P r o o f. Let us assume on the contrary that $x(t)$ is such a divergent solution of (5) that

$$\limsup_{t \to \infty} x(t) = \infty$$

(the case, when $x(t)$ tends to "$-\infty$" may be performed just in the same way).

Then there must exist the last point $T_1 \geq 0$ with $x(T_1) = x^*_{k-1}$ and the first point $T_2 \geq T_1$ with $x(T_2) = x^*_k$, where k-is even. It is clear that $x(t) \in \langle x^*_{k-1}, x^*_k \rangle$ $(T_1 \leq t \leq T_2)$ can be represented by

$$x(t)=C\exp((-1)^{k+1}c_k t)+\int_0^t \exp((-1)^{k+1}c_k(t-\tau))[\nu q(t)-f_k(x(\tau))]d\tau + k\frac{d_k}{c_k}$$

with a constant C.

However, as for $t \geq 0$ and k-even the following estimation holds:

$$|x(t)| \leq |Ce^{-c_k t}| + \left|\frac{\nu Q + F_k}{c_k}\right| + \left|k\frac{d_k}{c_k}\right| ,$$

where $F_k := \max_{x \in \langle x^*_{k-1}, x^*_k \rangle} |f_N(x)|$, the existence of the point T_2 is impossible with respect to (14), a contradiction. This completes the proof.

<u>Note (continuation)</u>. *(14) is satisfied for* $f(x)=\sin x$, *if* $\pi/2 > \nu Q+F_k := (\pi/2-1)$, *i.e. if* $\nu < 1/Q$.

4. <u>Theorem</u>. *If the solution $[x(t),y(t)]$ of (1) is asymptotically symmetrical according to (2) under (4) or (4'), then (3) is necessarily implied, provided μ, ν are sufficiently small. If $x(t) = -y(t)$ moreover, then (3) follows even for the bounded and in the above way oscillatory function $f(x)$, provided (14).*

P r o o f. - follows immediately from the preceding two corollaries.

It can be seen from the proof of Corollary 1 that the first part of the assertion of our Theorem can be introduced in a more precise form, namely that *the possible asymptotically symmetrical solutions of (1) are even uniformly ultimately bounded.*

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SOUHRN

Symetrie v jistém jednoduše perturbovaném Hamiltonově systému

J a n   A n d r e s

V poznámce je zkoumána otázka, zda pro řešení systému (1) (těsně svázaného s rovnicemi Lotky-Volterry) s vlastností (2) vždy platí (3), a to i v případě dostatečně malých spojitých vnějších poruch. Jsou nalezeny postačující podmínky, zaručující pozitivní odpověď na daný problém.

РЕЗЮМЕ

Симметрия в одной просто возмущенной системе Гамильтона

Я н   А н д р е с

В заметке изучается вопрос, если свойство (2) принадлежащее решениям системы (1) (тесно связанной с уравнениями Лотки-Вольтерра) влечет всегда за собой тоже (3) , а даже в случае достаточно малых непрерывных внешних возмущений. Находятся достаточные условия, гарантирующие положительный ответ на данную проблему.

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