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On the Floquet theory of differential equations \( y'' = Q(t)y \) with a complex coefficient of the real variable


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ON THE FLOQUET THEORY
OF DIFFERENTIAL EQUATIONS
y'' = Q(t)y WITH A COMPLEX COEFFICIENT
OF THE REAL VARIABLE

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1. Problem

A differential equation

\[ y'' = Q(t)y, \quad \text{Im} \ Q(t) \neq 0, \]  

is investigated, where $Q$ is a continuous and $\mathcal{F}$-periodic complex function on $\mathbb{R}$. From the Floquet theory (see for instance [7]) it then follows that there exist independent solutions $u, v$ of (Q) such that either

\[ u(t+\mathcal{T}) = \varphi u(t), \quad v(t+\mathcal{T}) = \varphi^{-1} v(t), \quad t \in \mathbb{R}, \quad 0 \neq \varphi \in \mathbb{C} \]  

(1)

or

\[ u(t+\mathcal{T}) = \varphi u(t) + v(t), \quad v(t+\mathcal{T}) = \varphi v(t), \quad t \in \mathbb{R}, \quad \varphi^2 = 1. \]  

(2)
Generally complex numbers \( \mathbb{C} \), \( \mathbb{C}^{-1} \) are called characteristic (or Floquet's) multipliers of \((Q)\).

In [2] - [6], [8], [9], [11], [12] the values of the characteristic multipliers of \((q): \ y'' = q(t) y\), \(q\) being a continuous \( \mathbb{T} \)-periodic real function on \( R \), where expressed by a phase and the \( (1st\ kind) \) central dispersion of \((q)\).

The present article offers a new look at the Floquet theory of \((Q)\) based on the phase theory point of view.

2. Basic notations, relations and preparatory lemmas

The symbol \( C^n(R) (C^n(R)) \), where \( n=0,1,2,\ldots, \) will refer to a set of real (complex) functions with continuous derivatives (on \( R \)) up to and including the order \( n \). Trivial solutions of linear equations will not be considered.

In analogy with [13] a function \( \alpha \in C^3(R) \) will be said to be a phase of an equation

\[ y'' = P(t) y, \quad P \in C^0(R), \quad Im P(t) \neq 0, \quad (P) \]

exactly if there exist independent solutions \( u, v \) of this equation such that,

a) \( u^2(t) + v^2(t) \neq 0 \) for \( t \in R \),

b) \( \alpha'(t) = -\frac{w}{u^2(t) + v^2(t)} \) for \( t \in R \), where \( w = uv' - u'v \).

If moreover \( \frac{\alpha(t_0)}{v(t_0)} = \frac{u(t_0)}{v(t_0)} \) at a point \( t_0 \in R \), where \( v(t_0) \neq 0 \), then \( \alpha \) is said to be a phase of the basis \((u,v)\) of \((P)\). In such a case \( u(t) = c \frac{\sin \alpha(t)}{\sqrt{\alpha'(t)}} \), \( v(t) = c \frac{\cos \alpha(t)}{\sqrt{\alpha'(t)}} \)

for \( t \in R \), where \( 0 \neq c \in C \).

A function \( \alpha \) is a phase of \((P)\) exactly if it is a solution (on \( R \)) of a nonlinear 3rd order differential equation

\[ -\{\alpha, t\} - \alpha^2(t) = P(t), \]

150
\[ \{\alpha, t\} := \frac{\alpha''''(t)}{2\alpha'(t)} - \frac{3}{4} \left( \frac{\alpha''(t)}{\alpha'(t)} \right)^2 \]

denotes the Schwarzian derivative of \( \alpha \) at the point \( t \).

If \( \alpha \) is a phase of \( (P) \), then every solution of \( (P) \) may be written either as
\[
\sin(\alpha(t) + c_2) \quad \frac{c_1}{\sqrt{\alpha''(t)}}
\]

or
\[
\frac{c_3}{\sqrt{\alpha'(t)}} \quad \frac{e^{i\alpha'(t)}}{\sqrt{\alpha'(t)}}
\]

where \( \gamma^2 = 1, c_1, c_2, c_3 \in \mathbb{C}, c_1 \neq 0 \neq c_3 \). The converse is valid, too: For arbitrary complex numbers \( c_1, c_2, c_3, c_1 \neq 0 \neq c_3 \), and a number \( \gamma, \gamma^2 = 1 \), the functions defined by (3) and (4) are solutions of \( (P) \). Hereby \( \sqrt{\alpha'(t)} \) means a continuous and single-valued branch of the square root of the function \( \alpha'(t) \).

If \( u \) is a solution of \( (P) \), \( u(t) \neq 0 \) for \( t \in \mathbb{R} \), then there exists a phase \( \alpha \) of \( (P) \) and a number \( c \in \mathbb{C}, c \neq 0 \), such that
\[ u(t) = c \frac{e^{i\alpha(t)}}{\sqrt{\alpha'(t)}}, \quad t \in \mathbb{R}. \]

All the above properties have been presented and proved in [13].

**Lemma 1.** Let \( \alpha \) be a phase of \( (P) \). Then
\[
(P(t)) = - \left\{ \alpha, t \right\} - \alpha'^2(t) = (i\alpha'(t) - \frac{\alpha''(t)}{2\alpha'(t)})' + (i\alpha'(t) - \frac{\alpha''(t)}{2\alpha'(t)})^2, \quad t \in \mathbb{R}.
\]

**Proof.** Setting \( u(t) := \frac{e^{i\alpha(t)}}{\sqrt{\alpha'(t)}} (\neq 0) \) for \( t \in \mathbb{R} \), then \( u \)

is a solution of \( (P) \). From the equalities \( \frac{u'}{u} = i\alpha' - \frac{\alpha''}{2\alpha'} \)
and \(-\{\alpha, t\} - \alpha^2(t)\) \(P(t) = \left(\frac{u'(t)}{u(t)}\right)^2 + \left(\frac{u''(t)}{u(t)}\right)^2\) then there follows the assertion of Lemma 1.

In analogy with [14] a function \(X\) will be said to be a (complete) transformator of \((P)\) if

(i) \(X \in \mathcal{C}^2(R), X'(t) \neq 0\) for \(t \in R, X(R) = R;\)

(ii) for every solution \(y\) of \((P)\) the function \(\frac{y[X(t)]}{X'(t)}\)

is again a solution of this equation.

The set of increasing transformators of \((P)\) constitutes a group \(\mathcal{L}_p^+\) relative to the composition of functions. We will say that \(\mathcal{L}_p^+\) is a planar group, if to every \((t_0, x_0) \in R \times R\) there exists exactly one function \(X \in \mathcal{L}_p^+\) such that \(X(t_0) = x_0\).

A transformator \(X\) of \((P)\), \(X'(t) > 0\) for \(t \in R\), will be called a central transformator of \((P)\) if

\[
\frac{y[X(t)]}{X'(t)} = y(t) \quad \text{for} \quad t \in R,
\]

where \(y^2 = 1\), for every solution \(y\) of \((P)\). The set of all central transformators of \((P)\) constitutes a group relative to the composition of functions, which we will write as \(\mathcal{L}_p^c\);

\(\mathcal{L}_p^c \subset \mathcal{L}_p^+\) (see [14]).

**Lemma 2.** Let \(\alpha\) be a phase of \((P)\). Then \(P\) is a \(\tau\)-periodic function exactly if the function \(\alpha(t+\tau)\) is a phase of \((P)\), too.

**Proof.** (\(\Rightarrow\)) Suppose \(P\) is a \(\tau\)-periodic function and set \(\beta(t) = \alpha(t+\tau), t \in R\). Then

\[
-\{\beta, t\} - \beta^2(t) = -\{\alpha, t+\tau\} - \alpha^2(t+\tau) = P(t+\tau) = P(t),
\]

so that

\[
-\{\beta, t\} - \beta^2(t) = P(t), \quad t \in R, \quad (5)
\]

152
whence it follows that $\beta$ is a phase of (P).

(\leftarrow\rightarrow) Suppose $\beta$ (defined analogous to the first part of the proof) is a phase of (P). Then (5) is true and consequently

$$ -\left\{\alpha(t+\alpha) \right\} - \alpha^2(t+\alpha) = P(t), \quad t \in \mathbb{R}. $$

It follows from this and from the equality $-\left\{\alpha(t) \right\} - \alpha^2(t) = P(t), \quad t \in \mathbb{R}$, that $P(t+\alpha) = P(t)$ for $t \in \mathbb{R}$.

Lemma 3. Let $a \in \mathbb{R}$, $\text{Re} P(t) + a \cdot \text{Im} P(t) \geq q(t)$ for $t \in \mathbb{R}$, where $q \in \mathcal{C}^0(\mathbb{R})$ and $(q): y'' = q(t)y$ be not oscillatory (i.e. any solution of (q) has at most a finite number of zeros on $\mathbb{R}$). Then any solution of (P) has at most a finite number of zeros on $\mathbb{R}$.

Proof. Suppose, there exists a solution $z$ of (P) with an infinite number of zeros, and $\alpha$ is their cluster point. Let $u$ be a solution of (q), $u(t) > 0$ for $t \neq b$ and $z(t_1) = z(t_2) = 0$ for $b \leq t_1 < t_2$, $z(t) \neq 0$ for $t \in (t_1, t_2)$. Since

$$ (z'(t)z(t))' = P(t)|z(t)|^2 + |z'(t)|^2, $$

then

$$ \int_{t_1}^{t_2} \left\{ |z'(s)|^2 + (\text{Re} P(s) + i \cdot \text{Im} P(s)) |z(s)|^2 \right\} ds = 0. $$

It then follows

$$ \int_{t_1}^{t_2} |z'(s)|^2 + \text{Re} P(s) |z(s)|^2 ds = 0, $$

$$ \int_{t_1}^{t_2} \text{Im} P(s) |z(s)|^2 ds = 0 $$

so that
\[ \int_{t_1}^{t_2} \left[ |z'(s)|^2 + q(s)|z(s)|^2 \right] ds \neq 0. \]

Since \( |z(t)|^2 = |z'(t)|^2 \) for \( t \in (t_1, t_2) \), we obtain
\[ \int_{t_1}^{t_2} \left[ r^2(s) + q(s)r^2(s) \right] ds \neq 0, \]

where \( r(t) := |z(t)|, t \in \mathbb{R} \). Then, by Lemma 1.3 ([15] p.3), the solution \( u \) has a zero on \((t_1, t_2)\), which is a contradiction.

3. Main results

In what follows we will investigate equations of the type
\[ y'' = Q(t)y, \quad Q \in C^0(\mathbb{R}), \quad \text{Im } Q(t) \neq 0, \quad Q(t+\pi) = Q(t) \]
for \( t \in \mathbb{R} \). (Q)

Lemma 4. There exists a phase \( \alpha \) of (Q) such that the function \( \text{id} - \frac{\alpha''}{2\alpha'} \) is \( \pi \)-periodic exactly if for a solution \( u \) of (Q)

\[ u(t+\pi) = \rho \cdot u(t), \quad u(t) \neq 0 \text{ for } t \in \mathbb{R} \] (6)

is valid, where \( 0 \neq \rho \in \mathbb{C} \).

Proof. (\( \Longrightarrow \)) Suppose there exists a phase \( \alpha \) of (Q) such that the function \( \text{id} - \frac{\alpha''}{2\alpha'} \) is \( \pi \)-periodic. If we set
\[ u(t) := \frac{e^{i\alpha(t)}}{\sqrt{\alpha'(t)}} (\neq 0), \quad t \in \mathbb{R}, \]
then \( u \) is a solution of (Q) and
\[ \frac{u'}{u} = \text{id} - \frac{\alpha'}{2\alpha} (:=p), \]
since \( u' \) is a \( 2\pi \)-periodic function. Further \( u(t) = u(0) \).
\(\exp(\int_0^t p(s)ds)\) which yields

\[u(t+\bar{T}) = \mathcal{G}_u(t), \] \text{where} \quad \mathcal{G}_u = \exp(\int_0^\alpha p(s)ds).

Let (6) hold for a solution \(u\) of (Q), where \(0 \neq \mathcal{G}_u \in \mathcal{C}\). Since \(u(t) \neq 0\) for \(t \in \mathbb{R}\), there exists a phase \(\alpha\) of (Q) and a \(c \in \mathcal{C}\) such that \(u(t) = c \frac{e^{i\alpha(t)}}{\sqrt{\alpha(t)}}\). On account of the fact that \(\frac{u'}{u}\) is a \(\mathcal{T}\)-periodic function and \(\frac{u'}{u} = i\alpha' - \frac{\alpha''}{2\alpha}\), it is clear that \(i\alpha' - \frac{\alpha''}{2\alpha}\) is also a function with a period \(\mathcal{T}\).

**Remark 1.** If (6) holds for a solution \(u\) of (Q), where \(0 \neq \mathcal{G}_u \in \mathcal{C}\), then \(\mathcal{G}_u\) is a characteristic multiplier of (Q).

**Remark 2.** If \(\beta\) is a phase of (P) and \(i\beta' - \frac{\beta''}{2\beta}\) is a \(\mathcal{T}\)-periodic function, then the coefficient \(P\) of (P) is also a \(\mathcal{T}\)-periodic function, as it readily follows from Remark 1.

**Remark 3.** In the terminology of transformators, equation (P) \(t + \mathcal{T} \in L_p^+(\mathbb{R})\) exactly if \(P\) is a \(\mathcal{T}\)-periodic function.

**Corollary 1.** Suppose there exists a phase \(\alpha\) of (Q) such that \(i\alpha' - \frac{\alpha''}{2\alpha}\) is a \(\mathcal{T}\)-periodic function. Then

\[
\frac{\sqrt{\alpha'}(\mathcal{T})}{\sqrt{\alpha'}(0)} \exp\left\{i(\alpha(\mathcal{T}) - \alpha(0))\right\}, \quad \frac{\sqrt{\alpha'}(2)}{\sqrt{\alpha'}(0)} \exp\left\{i(\alpha(0) - \alpha(2))\right\}
\]

are characteristic multipliers of (Q).

**Proof.** It follows from Remark 2 that the coefficient \(Q\) of (Q) is a \(\mathcal{T}\)-periodic function. Besides we obtain from the
proof (\(\implies\)) of Lemma 4 and Remark 1 that \(\exp(\int_0^x p(s)ds)\)

\(\exp(-\int_0^x p(s)ds)\), where \(p := i \alpha' - \frac{\alpha''}{\alpha'}\), are characteristic.

multipliers of \((Q)\). From this and from the equality

\[
\int_0^x p(s)ds = i(\alpha(x) - \alpha(0)) + \ln \frac{\alpha'(0)}{\alpha'(\theta)}
\]

immediately follows the assertion of Corollary 1.

**Lemma 5.** Suppose there exists a phase \(\alpha\) of \((Q)\) such that \(i \alpha'(t) - \frac{\alpha''(t)}{2 \alpha'(t)} = \rho(t)\), \(t \in \mathbb{R}\) is a \(\mathcal{F}\)-periodic function. Then for a phase \(\beta\) of \((Q)\)

\[
i \beta'(t) - \frac{\beta''(t)}{2 \beta'(t)} = \rho(t) \quad \text{for } t \in \mathbb{R}
\]

is fulfilled exactly if there exist \(k, k_1 \in \mathbb{C}\), \(k e^{2i\alpha(t)} \neq 1\) for \(t \in \mathbb{R}\) such that

\[
\beta(t) = \alpha(t) + \frac{i}{2} \ln(1 - ke^{2i\alpha(t)}) + k_1, \quad t \in \mathbb{R}. \tag{8}
\]

**Proof.** (\(\implies\)) Suppose \(\beta\) is such a phase of \((Q)\) that

\[
(p(t)) i \alpha'(t) - \frac{\alpha''(t)}{2 \alpha'(t)} = i \beta'(t) - \frac{\beta''(t)}{2 \beta'(t)}, \quad t \in \mathbb{R}. \tag{9}
\]

Then from Theorem 4 [13] there follows the equality \(\beta(t) = c[\alpha(t)] \frac{1}{(c_1 \cos z + c_2 \sin z)^2 + (c_3 \cos z + c_4 \sin z)^2}\)

\[
c^\prime(z) = \frac{1}{(c_1 \cos z + c_2 \sin z)^2 + (c_3 \cos z + c_4 \sin z)^2}
\]

for all \(z \in \mathbb{C}\), where \((c_1 \cos z + c_2 \sin z)^2 + (c_3 \cos z + c_4 \sin z)^2 \neq 0\) and \(c_1, c_2, c_3, c_4 \in \mathbb{C}\), \(c_2 c_3 - c_1 c_4 = 1\). Then

\[
\beta'(t) = c[\alpha(t)]. \alpha'(t), \quad \beta''(t) = c'[\alpha(t)]. \alpha''(t) + c'[\alpha(t)]. \alpha'^2(t)
\]

and on substituting in (9) we get
\[ i = i \cdot c \frac{\partial}{\partial t} [\alpha(t)] - \frac{c^n}{2c^e} [\alpha'(t)] \]

All solutions of the above equation are of the form \( c^* \alpha(t) = \frac{1}{1-ke^{2i\alpha(t)}} \), where \( k \in \mathbb{C} \) is an arbitrary number such that \( ke^{2i\alpha(t)} \neq 1 \) for \( t \in \mathbb{R} \). There is an infinite number of such \( k \) and if we proceed in the same manner as in [13] we may prove the Lebesgue measure (the complex number is taken as a point in Gauss plane) of the set of such numbers \( k \) equals to infinity. Here \( c^* \alpha \) has the form (10). In the case of \( k \neq 1 \) it suffices to put \( c_1 = 0, c_2 = \frac{i}{\sqrt{1-k}}, c_3 = \sqrt{1-k}, c_4 = \frac{-1}{\sqrt{1-k}} \) while in the case of \( k = 1 \) we put \( c_1 = -\frac{\sqrt{2}}{2}, c_2 = 0, c_3 = -i \frac{\sqrt{2}}{2}, c_4 = \sqrt{2} \). Hence \( \beta'(t) = \frac{\alpha'(t)}{1-k e^{2i\alpha(t)}} \) and integrating the latter equality from 0 to \( t \) gives

\[
\beta(t) = \beta(0) + \int_{0}^{t} \frac{\alpha'(s)ds}{1-k e^{2i\alpha(s)}} = \beta(0) + \alpha(t) + \frac{i}{2} \ln(1-ke^{2i\alpha(t)}) - \alpha(0) - \frac{i}{2} \ln(1-ke^{2i\alpha(0)}) = \alpha(t) + \frac{i}{2} \ln(1-ke^{2i\alpha(t)}) + k_1,
\]

where \( k_1 := \beta(0) - \alpha(0) - \frac{i}{2} \ln(1-e^{2i\alpha(0)}) \).

(\( \Leftrightarrow \)) Suppose \( \beta \) is the function defined by (8), where \( k, k_1 \in \mathbb{C}, ke^{2i\alpha(t)} \neq 1 \) for \( t \in \mathbb{R} \). By a direct computation it may be verified that \( \beta \) is a phase of (Q) and (7) is true.

Lemma 6. Let all solutions of (Q) not be \( \mathcal{P} \)-periodic or \( \mathcal{P} \)-halfperiodic and let there exist a phase \( \alpha \) of (Q) such that the function \( \iota \alpha' = \frac{\alpha''}{2\alpha'} \) (\( \iota : p_1 \)) is \( \mathcal{P} \)-periodic. Then
there exists at most one \( \mathcal{F} \)-periodic function \( p_2 \), \( p_1 \neq p_2 \), such that

\[
p_2 = i\beta' - \frac{\beta''}{2\beta'}
\]

for a phase \( \phi \) of \( (Q) \).

**Proof.** Following Remark 2, it suffices to prove that the Riccati equation

\[
\dot{u} + u^2 = Q(t)
\]

has at most two different \( \mathcal{F} \)-periodic solutions (defined on \( \mathbb{R} \)) under the assumption that all solutions of \( (Q) \) are not \( \mathcal{F} \)-periodic or \( \mathcal{F} \)-halfperiodic. First, the function \( p_1 \) is a \( \mathcal{F} \)-periodic solution of (11). We assume that there exist further two \( \mathcal{F} \)-periodic solutions \( p_2, p_3 \) of (11), \( p_1 \neq p_2 \), \( p_1 \neq p_3 \), \( p_2 \neq p_3 \). Integrating the equalities

\[
\frac{(p_3 - p_2)'}{p_3 - p_2} - \frac{(p_3 - p_1)'}{p_3 - p_1} = p_1 - p_2,
\]

\[
\frac{(p_2 - p_3)'}{p_2 - p_3} - \frac{(p_2 - p_1)'}{p_2 - p_1} = p_1 - p_3,
\]

\[
\frac{(p_3 - p_1)'}{p_2 - p_1} = -p_3 - p_1,
\]

from 0 to \( \mathcal{F} \) yields

\[
\int_0^\mathcal{F} (p_1(t) - p_2(t)) dt = 2im\mathcal{F}, \quad \int_0^\mathcal{F} (p_1(t) - p_3(t)) dt = 2in\mathcal{F},
\]

\[
\int_0^\mathcal{F} (p_1(t) + p_3(t)) dt = 2ir\mathcal{F},
\]

where \( m, n, s \) are integers, whence

\[
\int_0^\mathcal{F} p_1(t) dt = i(n+r)\mathcal{F}, \quad \int_0^\mathcal{F} p_2(t) dt = i(n+r-2m)\mathcal{F}, \quad \int_0^\mathcal{F} p_3(t) dt = i(r-n)\mathcal{F}.
\]
Since $p_1(t) = \frac{y_1'(t)}{y_1(t)}$, $p_2(t) = \frac{y_2'(t)}{y_2(t)}$, where $y_1$, $y_2$ are suitable independent solutions of (Q), $y_1(t) \neq 0$, $y_2(t) \neq 0$ for $t \in R$, there exist $k_1, k_2 \in C$ such that $y_i(t) = k_i \exp(\int_0^t p_i(s)ds)$, $i = 1, 2$, $t \in R$. Naturally, then

$$y_1(t+T) = k_1 \exp(\int_0^T p_1(s)ds) \exp(\int_0^T p_1(s)ds) = (-1)^{n+r} y_1(t)$$

(i=1, 2, $t \in R$), hence all solutions of (Q) are $\tilde{T}$-periodic or $\tilde{T}$-halfperiodic, which is a contradiction.

**Remark 4.** In assuming that all solutions of (Q) are $\tilde{T}$-periodic or $\tilde{T}$-halfperiodic, the Riccati equation (11) has infinitely many $\tilde{T}$-periodic solutions. All these solutions are of form $Y(t)$, where $y$ is a solution of (Q), $y(t) \neq 0$ for $t \in R$ (see Example 1). Here the main difference is in the number of periodic solutions of the Riccati equation in a real case, when even there the equation has at most two $\tilde{T}$-periodic solution (see [10]).

**Example 1.** The Riccati equation

$$u'' + u^2 = -4 + 16e^{8it}$$

has $\tilde{T}$-periodic solutions, say

$$u = -2i + 4ie^{4it} \cot(g^{4it} + c),$$

with $c \in C$ being an arbitrary number such that $\sin(e^{4it} + c) \neq 0$ for $t \in R$. This condition is fulfilled for $c = c_1 + ic_2$ such that $(c_1 + k\tilde{T})^2 + c_2^2 \neq 1$ for all integer $k$.

It becomes obvious that the investigation of $\tilde{T}$-periodicity of the function $i\alpha - \frac{\alpha^2}{2} \alpha$, where $\alpha$ is a phase of (Q), is essential. The remain part of this text is devided into three cases:
Case 1 - there exists a phase $\alpha$ of (Q) such that its derivative $\alpha'$ is a $\gamma$-periodic function (and then the function $i\alpha' - \frac{\alpha''}{2\alpha'}$, too, is $\gamma$-periodic);

Case 2 - there exists such a phase $\alpha$ of (Q) that its derivative $\alpha'$ is not a $\gamma$-periodic function and $i\alpha' - \frac{\alpha''}{2\alpha'}$ is a $\gamma$-periodic function;

Case 3 - there exists no such phase $\alpha$ of (Q) that $i\alpha' - \frac{\alpha''}{2\alpha'}$ is a $\gamma$-periodic function.

**Theorem 1.** Suppose $\bar{\phi}$ is a characteristic multiplier of (Q), $|\bar{\phi}| \leq 1$. Then, there exist independent solutions $u,v$ of (Q), $u(t)v(t) \neq 0$ for $t \in \mathbb{R}$ satisfying (1) exactly if there exists a phase $\alpha$ of (Q), $k_1, k_2 \in \mathbb{R}, 0 \leq k_1 \leq (1 + \text{sign } k_2)\gamma$, $k_1 \neq 2\gamma$, $k_2 \geq 0$ and an integer $n$ such that

$$
\alpha(t+\gamma) = \alpha(t) + (k_1 + 2n\gamma) + ik_2, \ t \in \mathbb{R},
$$

$$
\phi = e^{k_2-ik_1} \text{ and } \psi = \frac{\alpha'(t+\gamma)}{\sqrt{\alpha(t)}} (= \pm 1).
$$

**Proof.** (\(\rightarrow\)) Let $\bar{\phi}$ be a characteristic multiplier of (Q), $|\bar{\phi}| \leq 1$ and $u,v$ be independent solutions of (Q) satisfying (1), $u(t)v(t) \neq 0$ for $t \in \mathbb{R}$. Setting $U = \frac{1}{2}(u+v), V := \frac{1}{2}(v-u)$ yields that $U,V$ are independent solutions of (Q) and $U^2(t) + V^2(t) \neq 0$ for $t \in \mathbb{R}$. Let $\alpha$ be a phase of the basis $(U,V)$ of (Q). Then there exists a $c \in \mathbb{C}, c \neq 0$, such that

$$
U(t) = c \frac{\sin \alpha(t)}{\sqrt{\alpha(t)}}, \ V(t) = c \frac{\cos \alpha(t)}{\sqrt{\alpha(t)}}, \ t \in \mathbb{R},
$$

(see [13]). Since

$$
c \frac{\sin \alpha(t)}{\sqrt{\alpha(t)}} = \frac{1}{2}(u(t)+v(t)), \ c \frac{\cos \alpha(t)}{\sqrt{\alpha(t)}} = \frac{1}{2}(v(t)-u(t)),
$$

$$
t \in \mathbb{R} \ (14)
$$
then
\[ \frac{c^2}{\alpha'(t+\tau)} = \frac{1}{4}(\Phi . u(t) + \Phi^{-1} . v(t))^2 - \frac{1}{4}(\Phi^{-1} . v(t) - \Phi . u(t))^2 = u(t)v(t) = \frac{c^2}{\alpha'(t)}. \]

Naturally, then \( \alpha'(t+\tau) = \alpha'(t) \) and therefore for an \( a \in \mathbb{C} \) we get
\[ \alpha(t+\tau) = \alpha(t) + a, \quad t \in \mathbb{R}. \] (15)

Let \( \psi = \frac{\sqrt{\phi'(t+\tau)}}{\sqrt{\phi(t)}} \). Evidently, \( \psi \) is either equal to 1 or equal to -1. From the definition of \( U, V \) and from (1), (13) - (15) it follows from one side
\[ V(t+\tau) + iU(t+\tau) = \psi c \frac{\exp i(\chi(t)+\alpha)}{\sqrt{\alpha(t)}} \]
and from the other side
\[ V(t+\tau) + iU(t+\tau) = \frac{1}{2}(\Phi^{-1} . v(t) - \Phi . u(t)) + \]
\[ + \frac{1}{2}(\Phi . u(t) + \Phi^{-1} . v(t)) = i\Phi^{-1} . v(t) = \]
\[ = \Phi^{-1} . (V(t) + iU(t)) = c\Phi^{-1} \frac{\exp i\alpha(t)}{\sqrt{\alpha(t)}}. \]

Thus \( \Phi = \psi e^{-i\alpha} \) and if \( a = a_1 + ia_2 \) is \( |a| = e^{\pi} = 1 \), whence \( a \equiv 0 \). Next let \( a_1 = k_1 + 2n\pi \), where \( 0 \leq k_1 < 2\pi \) and \( n \) is an integer, Setting \( k_2 := a_2 (\equiv 0) \), we get from (15) formula (12) and \( \Phi = \psi \exp(-i(k_1 + 2n\pi)) + k_2 = \psi \exp(k_2 - k_1) \).

It remains to prove that in case of \( a_2 = 0 \), i.e. where \( |a| = 1 \), the number \( k_1 \) may be chosen to that \( 0 \leq k_1 < 2\pi \). In case of \( \pi < k_1 < 2\pi \) we consider the phase \( \beta := -\alpha \) in place of the phase \( \alpha \) of (Q). Then it follows from (15)
\[ \beta(t+\omega) = \beta(t) - a = \beta(t) - k_1 - 2n\omega = \]
\[ = \beta(t) + (2\omega - k_1) - 2(n+1)\omega \]

and in place of the integer \( n \) in (12) we put the integer \(-(n+1)\) and in place of the number \( k_1 \) we put \( 2\omega - k_1 \). Evidently \( 0 < 2\omega - k_1 < \omega \).

\( \langle \cdots \rangle \) Let \( \alpha \) be a phase of (Q), \( k_1, k_2 \in \mathbb{R} \), \( 0 \leq k_1 \leq 1 \) = \((1+\text{sign } k_2)\omega \), \( k_1 \neq 2\omega \), \( k_2 \equiv 0 \) and \( n \) be an integer such that (12) is true. Let \( \gamma = \frac{\sqrt{\alpha'(t+\omega)}}{\sqrt{\alpha(t)}} \) and set \( \mathcal{Q} := \gamma \cdot \exp(k_2 - i k_1) \), \( U(t) := \frac{\sin \alpha(t)}{\sqrt{\alpha'(t)}} \), \( V(t) := \frac{\cos \alpha(t)}{\sqrt{\alpha'(t)}} \), \( u(t) := -iU(t) + V(t) \), \( v(t) := iU(t) + V(t) \) for \( t \in \mathbb{R} \).

Then \( |\mathcal{Q}| = 1 \), \( u \), \( v \) are independent solutions of (Q), \( u(t)v(t) = U^2(t) + V^2(t) \neq 0 \),

\[ u(t+\omega) = \cos \alpha(t+\omega) - i \frac{\sin \alpha(t+\omega)}{\sqrt{\alpha'(t+\omega)}} = \frac{\exp(-i\alpha(t+\omega))}{\sqrt{\alpha(t+\omega)}} = \]
\[ = \mathcal{Q} \left[ \cos \alpha(t) - i \frac{\sin \alpha(t)}{\sqrt{\alpha'(t)}} \right] = \mathcal{Q} \cdot u(t), \]

\[ v(t+\omega) = \cos \alpha(t+\omega) + i \frac{\sin \alpha(t+\omega)}{\sqrt{\alpha'(t+\omega)}} = \frac{\exp(i\alpha(t+\omega))}{\sqrt{\alpha(t+\omega)}} = \]
\[ = \mathcal{Q}^{-1} \left[ \cos \alpha(t) + i \frac{\sin \alpha(t)}{\sqrt{\alpha'(t)}} \right] = \mathcal{Q}^{-1} \cdot v(t), \quad t \in \mathbb{R}, \]

and \( \mathcal{Q} \), \( \mathcal{Q}^{-1} \) are characteristic multipliers of (Q).

Corollary 2. Let \( \alpha \) be a phase of (Q). All solutions of (Q) are \( \omega \)-periodic or \( \omega \)-halfperiodic exactly if

\[ \alpha(t+\omega) = \alpha(t) + k\omega, \quad t \in \mathbb{R}, \quad (16) \]

where \( k = 2n + \frac{1}{2}(1-\varepsilon) \) or \( k = 2n + \frac{1}{2}(1+\varepsilon) \), \( n \in \mathbb{Z} \) and \( \varepsilon = \frac{1}{\sqrt{\alpha'(t)}} \) (= \pm 1 for \( t \in \mathbb{R} \)).
Proof. ($\Rightarrow$) Suppose all solutions of (Q) are $\mathcal{F}$-periodic or $\mathcal{F}$-halfperiodic. The functions $\frac{e^{i\alpha}(t+\gamma)}{\sqrt[\alpha(t)]{}}$, $\frac{e^{-i\alpha}(t)}{\sqrt[\alpha(t)]{}}$ are independent solutions of (Q) and

$$
\frac{e^{i\alpha}(t+\gamma)}{\sqrt[\alpha(t)]{}} = \psi \frac{e^{i\alpha}(t)}{\sqrt[\alpha(t)]{}} ,
$$

$$
\frac{e^{-i\alpha}(t)}{\sqrt[\alpha(t)]{}} = \psi \frac{e^{-i\alpha}(t)}{\sqrt[\alpha(t)]{}} , t \in \mathbb{R},
$$

where $\psi^2 = 1$. Here all solutions for $\psi = 1$ ( $\psi = -1$) are $\mathcal{F}$-periodic ( $\mathcal{F}$-halfperiodic). On multiplying out both sides of (17) we get $\alpha'(t+\gamma) = \alpha'(t)$, thus for any $a \in \mathbb{C}$ we have $\alpha(t+\gamma) = \alpha(t) + a$ for $t \in \mathbb{R}$. Then from (17) there follows $e^{ia} = \psi e$, $e^{-ia} = \psi e$, with $e = \frac{\sqrt[\alpha(t+\gamma)]{}}{\sqrt[\alpha(t)]{}}$. If $a = a_1 + ia_2$ we have $a_2 = 0$, for $\psi e = 1$ we get $\cos a_1 = 1$ and for $\psi e = -1$ we get $\cos a_1 = -1$. In this way $a_1 = (2n + \frac{1}{2}(1-\varepsilon))\gamma$ for $\psi = 1$ and $a_1 = (2n + \frac{1}{2}(1+\varepsilon))\gamma$ for $\psi = -1$, where $n$ is an appropriate integer.

($\Leftarrow$) Suppose $\alpha$ is a phase of (Q) satisfying (16), where $n$ is an integer, $e = \frac{\sqrt[\alpha(t+\gamma)]{}}{\sqrt[\alpha(t)]{}}$ and $k = 2n + \frac{1}{2}(1-\varepsilon)$($k = 2n + \frac{1}{2}(1+\varepsilon)$). Let us put $u(t) := \frac{e^{i\alpha}(t)}{\sqrt[\alpha(t)]{}}$, $v(t) := \frac{e^{-i\alpha}(t)}{\sqrt[\alpha(t)]{}}$ for $t \in \mathbb{R}$. Then $u$, $v$ are independent solutions of (Q), $u(t)v(t) \neq 0$, $u(t)' = u(t)$, $v(t)' = v(t)$ ($u(t+\gamma) = -u(t)$, $v(t+\gamma) = -v(t)$), $t \in \mathbb{R}$. It immediately follows from this that all solutions of (Q) are $\mathcal{F}$-periodic ($\mathcal{F}$-halfperiodic).

Remark 5. In the terminology of central transformators of (Q) all solutions of (Q) are $\mathcal{F}$-periodic or $\mathcal{F}$-halfperiodic exactly if $t+\gamma$ is a central transformator of (Q).
Remark 6. If all solutions of \((Q)\) are \(\mathcal{T}\)-periodic or \(\mathcal{T}\)-halfperiodic, then the value of the number \(k\) in Corollary 2 will generally depend on the choice of the phase of \((Q)\) — as it becomes apparent from Example 1 \([14]\).

In the following theorem we present certain sufficient conditions for the derivative \(\dot{\alpha}\) of a phase \(\alpha\) of \((Q)\) to be \(\mathcal{T}\)-periodic.

**Theorem 2.** Suppose there exists a number \(a \in \mathbb{R}\) such that \(\text{Re } Q(t) + a \cdot \text{Im } Q(t) \leq q(t)\) for \(t \in \mathbb{R}\), where \(q \in C^0(\mathbb{R})\) and \(y'' = q(t)y\) is a nonoscillatory equation. Then one of the following two mutually excluding situations arises:

(i) there exist independent solutions of \((Q)\) such that \(u(t)v(t) \neq 0\) for \(t \in \mathbb{R}\) and (1) holds, where \(\varphi^2 \neq 1\);

(ii) there exist independent solutions \(u, v\) of \((Q)\) such that \(v(t) \neq 0\) for \(t \in \mathbb{R}\) and (2) holds.

**Proof.** From Lemma 3 there follows that every solution of \((Q)\) has at most a finite number of zeros. Consequently, every solution \(u\) of \((Q)\) satisfying the equality \(u(t+\mathcal{T}) = \mathcal{Q} \cdot u(t)\) on \(\mathbb{R}\), where \(0 \neq \mathcal{Q} \in \mathbb{C}\), has no zeros on \(\mathbb{R}\), i.e. \(u(t) \neq 0\) for \(t \in \mathbb{R}\). Especially from this there follows that all solutions of \((Q)\) cannot be \(\mathcal{T}\)-periodic or \(\mathcal{T}\)-halfperiodic. The statement of the Theorem readily follows from the results of the Floquet theory.

**Lemma 7.** Suppose all solutions of \((Q)\) are not \(\mathcal{T}\)-periodic or \(\mathcal{T}\)-halfperiodic. Let \(\alpha, \beta\) be such phases of \((Q)\) that

\[
\alpha(t+\mathcal{T}) = \alpha(t) + a, \quad t \in \mathbb{R}, \quad (18)
\]

\[
\beta(t+\mathcal{T}) = \beta(t) + b, \quad t \in \mathbb{R}, \quad (19)
\]

where \(a, b \in \mathbb{C}\). Then either \(a = b\) (in this case \(\alpha(t) - \alpha(0) = \beta(t) - \beta(0)\) for \(t \in \mathbb{R}\)) or \(a = -b\) (in this case \(\alpha(t) - \alpha(0) = -[\beta(t) - \beta(0)]\) for \(t \in \mathbb{R}\)).
Proof. We may assume without loss of generality $\alpha(0) = \beta(0)$. In the contrary case we assume instead of phases $\chi(t)$ and $\beta(t)$, the phases $\alpha(t) - \alpha(0)$ and $\beta(t) - \beta(0)$, respectively. By Corollary 2 the numbers $a, b$ cannot be equal to an integral multiple of $\pi$. Next, from Theorem 4 [13] there follows the existence of $k_1, k_2, k_3, k_4 \in \mathbb{C}$, $k_1 k_4 - k_2 k_3 \neq 0$ such that

$$\beta'(t) = \frac{(k_2 k_3 - k_1 k_4) \alpha'(t)}{(k_1 \cos \alpha(t) + k_2 \sin \alpha(t))^2 + (k_3 \cos \alpha(t) + k_4 \sin \alpha(t))^2},$$

$t \in \mathbb{R}$. \hspace{1cm} (20)

Placing $t$ instead of $t + \pi$ in (20) then from (18) and (19) we obtain

$$\beta'(t) = \frac{(k_2 k_3 - k_1 k_4) \alpha'(t)}{(k_1 \cos \alpha(t) + k_2 \sin \alpha(t) + a)^2 + (k_3 \cos \alpha(t) + k_4 \sin \alpha(t) + a)^2},$$

$t \in \mathbb{R}$. \hspace{1cm} (21)

Since $a$ is not equal to an integral multiple of $\pi$, there follows from (20) and (21) that $(k_1 \cos \alpha(t) + k_2 \sin \alpha(t))^2 + (k_3 \cos \alpha(t) + k_4 \sin \alpha(t))^2$ is a constant function on $\mathbb{R}$, thus $\beta'(t) = c \alpha'(t)$, where $c \in \mathbb{C}$ is an appropriate number, $c \neq 0$. From the equality $Q(t) = -\{\alpha, t\} - \alpha^2(t) = -\{\beta, t\} - \beta^2(t)$ we obtain $c^2 = 1$. If $c = 1$, then $\beta'(t) = \alpha'(t)$ and therefore $\beta(t) = \alpha(t)$ for $t \in \mathbb{R}$ and $a = b$. If $c = -1$, then $\beta'(t) = -\alpha'(t)$ and therefore $\beta(t) = -\alpha(t)$ for $t \in \mathbb{R}$ and $a = -b$.

**Corollary 3.** Let all solutions of (Q) not be $\pi$-periodic or $\pi$-halfperiodic. If for any phase $\alpha$ of (Q) relation (12) holds, where $0 \leq k_1 \leq (1 + \text{sign } k_2)\pi$, $k_1 \neq 2\pi$, $k_2 \geq 0$, then the value of the integer $n$ in this formula does not depend on the choice of the phase $\alpha$ of (Q) and it is defined uniquely by (Q).
Proof. Suppose $\alpha$, $\beta$ are the phases of $(Q)$ such that

$$
\alpha(t+\tau) = \alpha(t) + (k_1 + 2n\tau) + ik_2, \quad t \in \mathbb{R},
$$

$$
\beta(t+\tau) = \beta(t) + (s_1 + 2m\tau) + is_2, \quad t \in \mathbb{R},
$$

where $0 \leq k_1 \leq (1 + \text{sign } k_2)\tau$, $0 \leq s_1 \leq (1 + \text{sign } s_2)\tau$, $k_1 \neq 2\tau \neq s_1$, $k_2 \not\equiv 0$, $s_2 \not\equiv 0$ and $m, n$ are integers. By Lemma 6 there is either $\alpha(t) - \alpha(0) = \beta(t) - \beta(0)$ or $\alpha(t) - \alpha(0) = -[\beta(t) - \beta(0)]$. If $\alpha(t) - \alpha(0) = \beta(t) - \beta(0)$, then $k_1 = s_1$, $k_2 = s_2$ and $m = n$. If $\alpha(t) - \alpha(0) = -[\beta(t) - \beta(0)]$, then $k_1 + 2n\tau + ik_2 = -(s_1 + 2m\tau + is_2)$, whence $k_2 = -s_2$ and from the assumptions to $k_2, s_2$ we obtain $k_2 = s_2 = 0$. Then by Theorem 1 $|Q| = 1$, where $Q$ is one of the characteristic multipliers of $(Q)$. Then, however, $0 \neq k_1 \neq 2\tau \neq s_1 \neq 0$, because in the contrary case (by Corollary 2) all solutions of $(Q)$ would be $\tau$-periodic or $\tau$-halfperiodic. Hence $k_1, s_1 \in (0, \tau)$, whence we get $0 < k_1 + s_1 < 2\tau$. From the other side it holds $k_1 + s_1 = -2(m + n)\tau$, which is a contradiction.

Theorem 3. Suppose $\alpha$ be a phase of $(Q)$ and

$$
\alpha(t+\tau) = \alpha(t) + a, \quad t \in \mathbb{R}, \quad (22)
$$

where $0 \neq a \in \mathbb{C}$. Then a function $\beta$ is a phase of $(Q_1)$,

$$
\beta(t+\tau) = \beta(t) + a, \quad t \in \mathbb{R}, \quad (23)
$$

and

$$
\frac{\sqrt{\alpha(t+\tau)}}{\sqrt{\alpha(t)}} = \frac{\sqrt{\beta(t+\tau)}}{\sqrt{\beta(t)}} \quad (\neq 1), \quad (24)
$$

exactly if

$$
\beta(t) = k + d \int_0^t e^{i\gamma(s)} \alpha'\gamma(s) ds, \quad t \in \mathbb{R}, \quad (25)
$$
where \( k \in \mathbb{C}, \gamma \in \mathbb{C}^2(\mathbb{R}), c \in \mathbb{C}^3(\mathbb{R}), \gamma(t+\tilde{\gamma}) = \gamma(t) + 4n\tilde{\gamma}(n \in \mathbb{Z}), c(t+\tilde{\gamma}) = c(t) + \tilde{\gamma}, c'(t) > 0 \) for \( t \in \mathbb{R} \) and

\[
(\gamma') d = a \left[ \int_0^\infty e^{i\gamma(s)} \alpha'(s) ds \right]^{-1}.
\]

**Proof.** (\( \Longleftrightarrow \)) Let \( k, d, c, \gamma \) satisfy the assumptions of Theorem 3 and the function \( \beta \) be defined by formula (25). Then

\[
\beta(t+\tilde{\gamma}) = k + d \int_0^\infty e^{i\gamma(s)} \alpha'(s) ds + d \int_0^\infty e^{i\gamma(s)} \alpha'(s) ds = c(t) + c(t+\tilde{\gamma}),
\]

since the function \( e^{i\gamma(t)}\alpha'(t) \) is \( \mathcal{F} \)-periodic and

\[
d \int_0^\infty e^{i\gamma(s)} \alpha'(s) ds = a. \]

Next \( \beta \in \mathbb{C}^3(\mathbb{R}) \) and \( \beta'(t) =

\[
de^{i\gamma}(c(t))(\alpha(c(t)))' \neq 0 \text{ for } t \in \mathbb{R}. \text{ Thus } \beta \text{ is a phase of any } (Q_1). \text{ We denote } f_{\alpha'}(t) (f_{\beta'}(t)) \text{ a continuous single-valued branch of the argument of the function } \alpha'(\beta') \text{ on } \mathbb{R}. \text{ Then for an integer } m \text{ there is } f_{\alpha'}(t+\tilde{\gamma}) = f_{\alpha'}(t) + 2m\tilde{\gamma}. \text{ From the equality } \beta' = de^{i\gamma}(c(t))(\alpha(c(t)))' \text{ there follows the existence of an integer } j \text{ such that}

\[
f_{\beta'}(t) = \gamma(c(t)) + f_{\alpha'}(c(t)) + 2j\tilde{\gamma} + \text{Arg } d,
\]

whence we get

\[
f_{\beta'}(t+\tilde{\gamma}) = \gamma(c(t)+\tilde{\gamma}) + f_{\alpha'}(c(t)+\tilde{\gamma}) + 2j\tilde{\gamma} + \text{Arg } d = \gamma(c(t)) + f_{\alpha'}(c(t)) + 2(j+m+2n)\tilde{\gamma} + \text{Arg } d = f_{\beta'}(t) + 2(m+2n)\tilde{\gamma},
\]

i.e. (24) is true, whereby

\[
\frac{\sqrt{\alpha'(t+\tilde{\gamma})}}{\sqrt{\alpha'(t)}} = \frac{\sqrt{\beta'(t+\tilde{\gamma})}}{\sqrt{\beta'(t)}} = (-1)^m.
\]
Let $\beta$ be a phase of $(Q_1)$ satisfying (23), where $0 \not\equiv a \in C$. We put

$$A(t) := \int_0^t |\alpha'(s)| ds, \quad B(t) := \int_0^t |\beta'(s)| ds, \quad t \in \mathbb{R}.$$ 

Then $A, B$ are increasing functions on $\mathbb{R}$, $A, B \in C_1(\mathbb{R})$. Because of $|\alpha'(t+\tau)| = |\alpha'(t)|$, $|\beta'(t+\tau)| = |\beta'(t)|$ we have

$$A(t+\tau) = A(t) + a, \quad B(t+\tau) = B(t) + b, \quad t \in \mathbb{R},$$

where $a = A(\tau) > 0$, $b = B(\tau) > 0$. Setting $C(t) := \frac{a}{b} B(t)$, $c(t) := A^{-1}(C(t))$, $t \in \mathbb{R}$, yields

$$C(t+\tau) = C(t) + a, \quad t \in \mathbb{R},$$

and $c(t+\tau) = A^{-1}(C(t) + a) = A^{-1}(C(t)) + \tau = c(t) + \tau$, sign $c' = 1$, $c(0) = 0$.

From the equality $C(t) = A(c(t))$ it follows that $\int_0^t |\beta'(s)| ds = \frac{b}{a} \int_0^t |\alpha'(s)| ds$ whence

$$|\beta'(t)| = \frac{b}{a} c'(t) |\alpha'(c(t))|, \quad t \in \mathbb{R}. \quad (26)$$

Let us put $\varphi(t) := \frac{\beta'(t)}{(\alpha'(c(t)))}$, $t \in \mathbb{R}$. Then $|\varphi(t)| = \frac{b}{a}$, $\varphi(t+\tau) = \varphi(t)$, $\varphi \in \mathcal{C}_0(\mathbb{R})$. Let $f_{\varphi}$ denote a continuous and single-valued branch of the argument of the function $\varphi$ and $f_{\alpha'}$, $f_{\beta'}$ be defined analogous to the proof (\cite{1}) above. Then for some integers $k, j$ there holds

$$f_{\varphi}(t) = f_{\alpha'}(t) - f_{\alpha'}(c(t)) + 2j\tau,$$

$$f_{\alpha'}(t+\tau) = f_{\alpha'}(t) + 2k\tau,$$

and from (24) there follows the existence of an integer $n$:

$$f_{\alpha'}(t+\tau) - f_{\alpha'}(t) = f_{\alpha'}(t+\tau) - f_{\alpha'}(t) + 4n\tau.$$
Furthermore

\[ f\varphi(t+\gamma) = f\varphi(t) = f\varphi(t) - \varphi(c(t)) + 2j\beta + 4n\gamma \]

Therefore there exist an integer n and a function \( \gamma, \gamma \in C^2(\mathbb{R}) \), \( \gamma(t+\gamma) = \gamma(t) + 4n\gamma \) such that the function \( \varphi \) may be written as \( \varphi(t) = de^{i\int c(t)} \), where \( d = \frac{b_1}{a_1} \). From the definition of functions \( \varphi, \gamma \) and from (26) we obtain \( \beta'(t) = de^{i\gamma(c(t))} \). Integrating the last equality from 0 to t we get

\[ \beta(t) = \beta(0) + d \int_0^t e^{i\gamma(c(s))} \alpha(c(s)) \alpha'(s) ds = \beta(0) + D \int_0^t e^{i\gamma(s)} \alpha'(s) ds. \]

From this and from (23) it follows

\[ \beta(0) + d \int_0^{c(t)} e^{i\gamma(s)} \alpha'(s) ds + a = \]

\[ \beta(0) + d \int_0^{c(t)+\gamma} e^{i\gamma(s)} \alpha'(s) ds \]

and consequently

\[ a = d \int_0^{c(t)+\gamma} e^{i\gamma(s)} \alpha'(s) ds = d \int_0^{\gamma} e^{i\gamma(s)} \alpha'(s) ds. \]

If we put \( d = a \left[ \int_0^{\gamma} e^{i\gamma(s)} \alpha'(s) ds \right]^{-1} \) and \( k := \]

169
\[ \beta(0) + d \int_0^\infty e^{i\varphi(s)}\alpha'(s)ds, \]
then the phase \( \beta \) may be written in the form of (25).

**Remark 7.** Let all solutions of (Q) not to be \( \mathcal{S} \)-periodic or \( \mathcal{S} \)-halfperiodic. It follows from Corollary 3 that a phase \( \alpha \) of (Q), for which (12) holds - where \( 0 \leq k_1 \leq (1 + \text{sign } k_2)\mathcal{S} \), \( k_1 \neq 2\mathcal{S} \), \( k_2 \leq 0 \) and \( n \) is an integer - is uniquely determined up to an additive constant.

Remark 7 justifies us to the following

**Definition 1.** Let all solutions of (Q) not be \( \mathcal{S} \)-periodic or \( \mathcal{S} \)-halfperiodic, \( n \) being an integer, and \( \nu^2 = 1 \). We say that the pair of numbers \((n, \nu)\) (it this order) is a significant pair of numbers of (Q) if there exists a phase \( \alpha \) of (Q) such that (12) holds, where \( 0 \leq k_1 \leq (1 + \text{sign } k_2)\mathcal{S} \), \( k_1 \neq 2\mathcal{S} \), \( k_2 \leq 0 \) and \( \nu = \frac{\sqrt{\alpha'(0)}}{\sqrt{\alpha'(t)}} \) for \( t \in \mathbb{R} \).

**Theorem 4.** Let \((n, \nu)\) be the significant pair of numbers of (Q) and \( \alpha \) be such a phase of (Q) that (12) holds, where \( 0 \leq k_1 \leq (1 + \text{sign } k_2)\mathcal{S} \), \( k_1 \neq 2\mathcal{S} \), \( k_2 \leq 0 \), \( \nu = \frac{\sqrt{\beta'(\mathcal{S})}}{\sqrt{\beta'(0)}} \).

Then \((n, \nu)\) is the significant pair of numbers of (Q₁) and the equations (Q) and (Q₁) have equal characteristic multipliers exactly if

\[
Q_1(t) = Q(c(t))c^{-2}(t) - \{c, t\} + (\alpha(c(t)))^{-2}(1 - d^2 e^{-2i\varphi}(c(t))) + \frac{c^{-2}(t)}{4} \left[ 2i\varphi(c(t))\frac{\alpha''(c(t))}{\alpha'(c(t))} - 2i\varphi'(c(t)) - \varphi''(c(t)) \right],
\]

for \( t \in \mathbb{R} \) (27)

where \( \varphi \in C^2(\mathbb{R}), c \in C^3(\mathbb{R}), \varphi(t + \mathcal{S}) = \varphi(t) + 4n\mathcal{S} (n \in \mathbb{Z}), c(t + \mathcal{S}) = c(t) + \mathcal{S}, c'(t) > 0 \) for \( t \in \mathbb{R} \) and

\[
d = \left( \int_0^\infty e^{-i\varphi'(s)}\alpha'(s)ds \right)^{-1}(k_1 + 2n\mathcal{S} + ik_2).
\]

170
Proof. \((\implies)\) Let \((n, \nu)\) be significant numbers of \((Q_1)\) and let the equations \((Q)\) and \((Q_1)\) have equal characteristic multipliers. From Theorem 1, Corollary 3 and from its proof then there follows the existence of such a phase \(\beta\) of \((Q_1)\) that

\[
\beta(t + \mathbf{x}) = \beta(t) + (k_1 + 2n \mathbf{x}) + ik_2, \quad t \in \mathbb{R},
\]

and \(\nu = \frac{\sqrt{\beta'(0)}}{\sqrt{\beta(0)}}\). By Theorem 3 naturally \(\beta(t) = h + \)

\[
c(t) + d \int_0^t e^{i\mathbf{z}(s)}\alpha'(s)ds,
\]

where \(h \in \mathbb{C}\) and \(d, c, \mathbf{z}\) satisfying the assumptions stated in the Theorem. From the equality \(Q_1(t) = -\{\beta, t\} - \beta^2(t)\) we get with some modification the form of (27) for the coefficient \(Q_1\) of \((Q_1)\).

\((\implies)\) Let the function \(Q_1\) be defined by (27), where \(d, c, \mathbf{z}\) satisfy the assumptions of the Theorem. A direct calculation shows that the function \(\beta(t) = d \int_0^t e^{i\mathbf{z}(s)}\alpha'(s)ds,\)

t \(\in \mathbb{R},\) is a phase of \((Q_1)\). By Theorem 3 there hold (23) and (24), thus from Theorem 1 it follows that \((n, \nu)\) is the significant pair of numbers of \((Q)\) and \((Q_1)\) and both equations have equal characteristic multipliers.

Corollary 4. Suppose the group of increasing transformers \(L_Q^*\) of \((Q)\) is planar. Then there exist independent solutions \(u, v\) of \((Q)\), \(u(t)v(t) \neq 0\) for \(t \in \mathbb{R}\) satisfying (1).

Proof. By Corollary 1 \([14]\) there exists a function \(Y \in C^2(\mathbb{R})\), \(Y(0) = 1, Y'(0) > 0\) for \(t \in \mathbb{R}\) and a number \(c \in \mathbb{C}\), \(c^2 \in \mathbb{C} - \mathbb{R}\) such that the function \(\alpha(t) = cY(t)\) for \(t \in \mathbb{R}\), is a phase of \((Q)\). Let \(c = c_1 + ic_2\). Then \(c_1c_2 \neq 0\) and it follows from the equality \(-\{\alpha, t\} - \alpha^2(t) = Q(t)\) that

\[
-2c_1c_2Y^{'2}(t) = \text{Im } Q(t), \quad \text{hence } Y'(t) = \sqrt{\frac{-\text{Im } Q(t)}{2c_1c_2}}\] for \(t \in \mathbb{R},\)

171
where \( Y^2 = 1 \). Naturally, then \( Y' \) and thus also \( \alpha' \) are \( \mathcal{R} \)-periodic functions and from Theorem 1 there immediately follows the assertion of the Corollary.

**Remark 8.** If there exist a function \( Y \in C^3(\mathbb{R}) \), \( Y(\mathbb{R}) = \mathbb{R}, \ Y'(t) > 0 \) for \( t \in \mathbb{R} \) and a number \( c \in \mathbb{C}, c^2 \in \mathbb{C} - \mathbb{R} \) such that the function \( \alpha(t):= c.Y \) is a phase of \( (Q) \), i.e. the group of increasing transformators of \( (Q) \) is planar (see Corollary 1 [14]), then it follows from Corollary 1 that

\[
\sqrt{\frac{Y'(-\pi)}{Y'(\pi)}} \exp\left[i\int_{-\pi}^{\pi} c(Y(t) - Y(0))\right], \sqrt{\frac{Y'(-\pi)}{Y'(\pi)}} \exp\left[i\int_{-\pi}^{\pi} c(Y(0) - Y(\pi))\right]
\]

are characteristic multipliers of \( (Q) \).

**Case 2**

**Theorem 5.** There exist independent solutions \( u, v \) of \( (Q) \) such that \( u(t) \neq 0 \) for \( t \in \mathbb{R} \), \( v \) has a zero at a point of \( R \) satisfying \( (1) \) exactly if there exist a phase \( \alpha \) of \( (Q) \), an integer \( n \), and \( x \in \mathbb{R} \) so that \( \alpha'(t) \) is not \( \mathcal{R} \)-periodic function, the function \( i\alpha'(t) - \frac{\alpha'(t)}{2\alpha(t)} \) is \( \mathcal{R} \)-periodic and \( \alpha(x+n\mathcal{R}) = \alpha(x) + n\mathcal{R} \).

**Proof.** (\( \Rightarrow \)) Suppose there exist independent solutions \( u, v \) of \( (Q) \) for which \( (1) \) holds, \( u(t) \neq 0 \) for \( t \in \mathbb{R} \), \( v \) having a zero on \( R \). Then, by Lemma 4, there exists a phase \( \alpha \) of \( (Q) \) such that function \( i\alpha' - \frac{\alpha'}{2\alpha} \) is \( \mathcal{R} \)-periodic and on account of the fact that \( v \) has a zero on \( R \), then by Theorem 1, the function \( \alpha' \) is not \( \mathcal{R} \)-periodic. It next follows from Theorem 8 and Theorem 5 [13] that there exist numbers \( c_1, c_2 \in \mathbb{C}, c_1 \neq 0: v(t) = c_1\frac{\sin(\alpha(t)+c_2)}{\sqrt{\alpha'(t)}} \) for \( t \in \mathbb{R} \). For an \( x \in \mathbb{R} \) let \( v(x) = 0. \) Then it follows from \( (1) \) that \( v(x+n\mathcal{R}) = v(x) = 0 \) i.e. there exist such integers \( n_1, n_2 \) that \( \alpha(x) = -c_2 + n_2\mathcal{R} \),
\( \alpha(x+T) = -c_2 + n_2 T \), whence \( \alpha(x+T) = \alpha(x) + nT \), \( n := n_2 - n_1 \) being an integer.

(\(-\)) Suppose there exists a phase \( \alpha \) of \( (Q) \) such that \( \alpha \) is not \( T \)-periodic, the function \( i\alpha' - \frac{\alpha''}{2\alpha} \) is \( \mathcal{F} \)-periodic and there exist an integer \( n \) and an \( x \in \mathbb{R} \): \( \alpha(x+T) = \alpha(x) + nT \).

From Lemma 4 there then follows the existence of a solution \( u \) of \( (Q) \), \( u(t) \not= 0 \) for \( t \in \mathbb{R} \), satisfying (6), where \( 0 \not= \varrho \in \mathbb{C} \). If we put \( v(t) := \frac{\sin(\alpha(t)-\alpha(x))}{\sqrt{\alpha'(t)}} \) for \( t \in \mathbb{R} \), then \( v \) is a solution of \( (Q) \), \( v(x) = 0 \). Thus \( u, v \) are independent solutions of \( (Q) \) and \( v(x+T) = \frac{\sin(\alpha(x+T)-\alpha(x))}{\sqrt{\alpha'(x+T)}} = \frac{\sin nT}{\sqrt{\alpha'(x+T)}} = 0 \). Therefore \( v(x) = v(x+T) = 0 \) and thus \( v(t+T) = \mathcal{T} \cdot v(t) \) for \( t \in \mathbb{R} \), where \( \mathcal{T} \in \mathbb{C} \) is a suitable number, \( \mathcal{T} \not= 0 \). From the Floquet theory we have \( \mathcal{T} = \mathcal{p}^{-1} \). We see that the solutions \( u, v \) of \( (Q) \) satisfy (1).

Remark 9. If there exist such independent solutions \( u, v \) of \( (Q) \) that \( u(t) \not= 0 \) for \( t \in \mathbb{R} \), \( v \) having a zero on \( \mathbb{R} \) and (1) is valid, then the Riccati equation (11) has exactly one \( \mathcal{F} \)-periodic solution.

Corollary 5. Let \( \alpha \) be such a phase of \( (Q) \) that \( \alpha' \) is not a \( \mathcal{F} \)-periodic function, \( i\alpha' - \frac{\alpha''}{2\alpha} \) is a \( \mathcal{F} \)-periodic function and for an \( x \in \mathbb{R} \) we have \( \alpha(x+T) = \alpha(x) + nT \), \( n \) being an integer. Then

\[
(-1)^n \frac{\sqrt{\alpha'(x+T)}}{\sqrt{\alpha'(x)}}, \quad (-1)^n \frac{\sqrt{\alpha'(x)}}{\sqrt{\alpha'(x+T)}},
\]

are the values of the characteristic multipliers of \( (Q) \).

Proof. From Corollary 1 and from its proof we find that

\[
\exp(\int_x^{x+T} p(s)ds), \exp(-\int_x^{x+T} p(s)ds), \text{where } p := i\alpha' - \frac{\alpha''}{2\alpha},
\]

are
the characteristic multipliers of \((Q)\). Since

\[
\int_{x}^{x+\hat{\gamma}} p(s)ds = i\left[\alpha(x+\hat{\gamma}) - \alpha(x)\right] - \frac{1}{2} \ln \alpha'(x+\hat{\gamma}) - \ln \alpha'(x) = i \hat{\gamma} + \ln \frac{\sqrt{\alpha'(x+\hat{\gamma})}}{\sqrt{\alpha'(x+\hat{\gamma})}}
\]

then

\[
\exp(\int_{x}^{x+\hat{\gamma}} p(s)ds) = (-1)^{n} \left(\frac{\sqrt{\alpha'(x+\hat{\gamma})}}{\sqrt{\alpha'(x+\hat{\gamma})}}\right)^{\gamma}
\]

where \(\gamma^{2} = 1\).

**Remark 10.** The result of Corollary 5 may be proved also in other way. If we put \(\beta(t) := \alpha(t) - \alpha(x), \nu(t) := \frac{\sin \beta(t)}{\nu(t)}\), \(t \in \mathbb{R}\), then \(\beta\) is a phase of \((Q)\) and \(\nu\) is a solution of this equation, \(\nu(x) = \nu(x+\hat{\gamma}) = 0\). Hence, the equality \(\nu(t+\hat{\gamma}) = \varphi^{-1} \cdot \nu(t)\) holds for \(t \in \mathbb{R}\), where \(\varphi^{-1}\) is one of the characteristic multipliers of \((Q)\). By differentiating the equality

\[
\frac{\sin \beta(t+\hat{\gamma})}{\sqrt{\beta'(t+\hat{\gamma})}} = \frac{\sin \beta(t)}{\sqrt{\beta'(t)}}
\]

and setting now in the resulting equality \(x\) instead of \(t\), we obtain (with some modification) the equality

\[
\frac{\beta'(x+\hat{\gamma})}{\sqrt{\beta'(x+\hat{\gamma})}} \cos \beta(x+\hat{\gamma}) = \varphi^{-1} \frac{\beta'(x)}{\sqrt{\beta'(x)}} \cos \beta(x),
\]

whence \(\varphi = (-1)^{n} \frac{\sqrt{\beta'(x+\hat{\gamma})}}{\beta'(x+\hat{\gamma})}\), thus \((-1)^{n} \frac{\sqrt{\alpha'(x+\hat{\gamma})}}{\sqrt{\alpha'(x+\hat{\gamma})}}\) are the characteristic multipliers of \((Q)\).

**Remark 11.** Let a phase \(\alpha\) of \((Q)\) satisfy the assumptions of Corollary 5. Let further \(p := i\alpha' - \frac{\alpha''}{2 \alpha}\). From Lemma 5 there follows that then for every phase \(\beta\) of \((Q)\) for which

\[
\frac{\beta'(x+\hat{\gamma})}{\sqrt{\beta'(x+\hat{\gamma})}} = p,
\]

we have \(\beta(x+\hat{\gamma}) = \delta(x) + n\hat{\gamma}\).
Theorem 6. There exist independent solutions \( u, v \) of (Q), \( v(t) \neq 0 \) for \( t \in \mathbb{R} \) for which (2) is valid exactly if the function

\[
\alpha(t) = \frac{i}{2} \ln \left[ P(t) - \frac{2it}{a^2} \right], \quad t \in \mathbb{R}
\]

is a phase of (Q), where \( 0 \neq a \in \mathbb{C} \), \( P \in \mathbb{C}^3(\mathbb{R}) \) is a \( \tau \)-periodic function, \( P(t) \neq \frac{2it}{a^2}, \quad iP'(t) + \frac{2it}{a^2} = \sqrt{iP'(t+\tau) + \frac{2it}{a^2}} \)

for \( t \in \mathbb{R} \).

Proof. \((\implies)\) Let (2) hold, where \( u, v \) are independent solutions of (Q), \( v(t) \neq 0 \) for \( t \in \mathbb{R} \). Then there exists such a phase \( \alpha \) of (Q) that

\[
v(t) = e^{i\alpha(t)}, \quad t \in \mathbb{R}.
\]

Every solution of (Q) may be written as \( y(t) = v(t) \int_a^t \frac{ds}{\sqrt{v^2(s)}} + b \), where \( a, b \in \mathbb{C} \). An easy calculation shows that the function \( u \) satisfies (2) exactly if

\[
u(t) = v(t) \int_a^t \frac{ds}{\sqrt{v^2(s)}} + b,
\]}

where \( b \in \mathbb{C} \) is an arbitrary constant and \( \int_a^t \frac{ds}{\sqrt{v^2(s)}} = 1 \) (with respect to the \( \tau \)-periodicity of \( v^2 \) we see that \( \int_t^{t+\tau} \frac{ds}{\sqrt{v^2(s)}} = a \) constant). Then
\[
\frac{1}{a} = \oint_{\eta} \frac{d\eta}{v^2(s)} = \oint_{\eta} \alpha'(s)e^{-2i\alpha(s)}ds = \\
= \frac{i}{2} \left[ e^{-2i\alpha(t+\tau)} - e^{-2i\alpha(t)} \right],
\]
hence
\[
e^{-2i\alpha(t+\tau)} = e^{-2i\alpha(t)} \frac{2ie^t}{a}.
\]
From the latter equality then follows the existence of a such a \(T\)-periodic function \(P \in C^3(\mathbb{R})\), \(P(t) \neq \frac{2iet}{aT}\), \(P'(t) \neq \frac{2ie}{aT}\) for \(t \in \mathbb{R}\) that the function \(e^{-2i\alpha(t)}\) may be written as
\[
e^{-2i\alpha(t)} = P(t) - \frac{2iet}{aT} \text{ for } t \in \mathbb{R},
\]
whence \(\alpha(t) = \frac{i}{2} \ln(P(t) - \frac{2iet}{aT})\). From the last relation and from the equality \(v(t) = \frac{\alpha(t)}{\sqrt{\alpha(t)}}\) (with some modification) we obtain
\[
v(t) = \mathcal{G} \frac{\sqrt{2}}{\sqrt{1P'(t)+\frac{2e}{aT}}} \text{ for } t \in \mathbb{R}, \text{ where } \mathcal{G}^2 = 1.
\]
It follows from the assumption \(v(t+T) = \mathcal{G} \cdot v(t)\) that
\[
\sqrt{1P'(t+T)+\frac{2e}{aT}} = \mathcal{G} \sqrt{1P'(t)+\frac{2e}{aT}} \text{ for } t \in \mathbb{R}.
\]

(\(\leftarrow\)) Let the function \(\alpha\) defined by (28) be a phase of (Q) where the function \(P\) and the number \(a\) satisfy the assumptions of the Theorem. Putting \(v(t):=\frac{e^{i\alpha(t)}}{\sqrt{\alpha(t)}}\) yields \(v(t) \neq 0\),
\[
v(t) = \mathcal{G} \frac{\sqrt{2}}{\sqrt{1P'(t)+\frac{2e}{aT}}}, \text{ } v(t+T) = \mathcal{G} \cdot v(t) \text{ for } t \in \mathbb{R}, \text{ where } \mathcal{G}^2 = 1. \text{ Let us put further } u(t):= v(t) \left[ a \int_{0}^{t} \frac{ds}{v^2(s)} + \right] 176
Then \( u \) is a solution of (Q) and
\[
\begin{aligned}
u(t) &= \frac{ia}{2} v(t) e^{-2i\pi(t)} = \frac{\alpha}{2} \sqrt{\frac{1}{2}} \left( \frac{iP(t) + \frac{2\pi t}{a^2}}{\sqrt{iP'(t) + \frac{2\pi}{a^2}}} \right).
\end{aligned}
\]
Consequently \( u(t+\pi) = \Phi \cdot u(t) + v(t) \). So, we have proved that there exist independent solutions \( u, v \) of (Q), \( v(t) \neq 0 \) for \( t \in \mathbb{R} \), satisfying (2).

**Corollary 6.** There exist independent solutions \( u, v \) of (Q), \( v(t) \neq 0 \) for \( t \in \mathbb{R} \), for which (2) is valid exactly if
\[
Q(t) = -\frac{1}{2} \frac{P''(t) - \frac{2\pi i}{a}}{P'(t) - \frac{2\pi i}{a}} + \frac{3}{4} \left( \frac{P''(t) - \frac{2\pi i}{a}}{P'(t) - \frac{2\pi i}{a}} \right)^2 \quad \text{for } t \in \mathbb{R}
\]
where \( 0 \neq a \in \mathbb{C}, P \in \mathcal{C}^3(\mathbb{R}) \) is a \( \pi \)-periodic function, \( P(t) \neq \frac{2\pi i}{a} \), \( P'(t) \neq \frac{2\pi i}{a} \), \( \sqrt{iP'(t) + \frac{2\pi}{a^2}} = \Phi \sqrt{iP'(t) + \frac{2\pi}{a^2}} \)
for \( t \in \mathbb{R} \).

**Proof.** This immediately follows from the preceding Theorem and from the fact that \( \alpha \) is a phase of (Q) exactly if it is a solution (on \( \mathbb{R} \)) the equation \( Q(t) = -\{\alpha, t\} - \alpha^2(t) \).

**Example 2.** Consider the equation
\[
y'' = \frac{4e^{2it}(1 - e^{2it})}{(1 + 2e^{2it})^2} y.
\]

The functions \( v(t) = \sqrt{\frac{1}{1 + 2e^{2it}}} \) and \( u(t) = -\frac{t - ie^{2it}}{\sqrt{\frac{1}{1 + 2e^{2it}}} \sqrt{1 + 2e^{2it}}} \)
are its independent solutions for which \( v(t+\pi) = v(t), u(t+\pi) = u(t) + v(t) \) for \( t \in \mathbb{R} \).
Case 3

Theorem 7. An equation (Q) has independent solutions u, v satisfying
\[ u(t+\mathcal{T}) = \varphi_u(t), \quad v(t+\mathcal{T}) = \varphi_v(t), \quad t \in \mathbb{R}, \quad \varphi^2 \neq 1, \quad (29) \]
where \( u, v \) have zeros on \( \mathbb{R} \) exactly if there exists such a phase \( \alpha \) of (Q) that \( \alpha' \) is not a \( \mathcal{T} \)-periodic function and
\[ \alpha(t_1) = n_1 \mathcal{T}, \quad \alpha(t_2) = \frac{\mathcal{T}}{2} + n_2 \mathcal{T}, \quad (30) \]
where \( t_1, t_2 \in [0, \mathcal{T}) \), \( t_1 \neq t_2 \) and \( k_1, k_2, n_1, n_2 \) are integers.

In this case \((-1)^{k_1-n_1} \frac{\sqrt{\alpha'(t_1+\mathcal{T})}}{\sqrt{\alpha'(t_1)}}\), \((-1)^{k_1-n_1} \frac{\sqrt{\alpha'(t_2+\mathcal{T})}}{\sqrt{\alpha'(t_2)}}\) (or also \((-1)^{k_2-n_2} \frac{\sqrt{\alpha'(t_2+\mathcal{T})}}{\sqrt{\alpha'(t_2)}}\), \((-1)^{k_2-n_2} \frac{\sqrt{\alpha'(t_2+\mathcal{T})}}{\sqrt{\alpha'(t_2)}}\)) are the characteristic multipliers of (Q).

Proof. (\( \Rightarrow \)) Let there exist independent solutions \( u, v \) of (Q) satisfying (29), both having zeros on \( \mathbb{R} \). Without loss of generality we may assume \( u^2(t) + \sqrt{2}(t) \neq 0 \) for \( t \in \mathbb{R} \). From (29) there follows that \( u, v \) have zeros on \([0, \mathcal{T})\). Suppose now \( u(t_1) = v(t_2) = 0 \), where \( t_1, t_2 \in [0, \mathcal{T}), \quad t_1 \neq t_2 \). Let \( \alpha \) be a phase of the basis \((u, v)\) of (Q). Then \( u(t) = c \frac{\sin \alpha(t)}{\sqrt{\alpha'(t)}}, \quad v(t) = c \frac{\cos \alpha(t)}{\sqrt{\alpha'(t)}} \) for \( t \in \mathbb{R}, \quad c \in \mathbb{C}, \quad c \neq 0 \). Since \( u(t_1+\mathcal{T}) = u(t_1) = 0, \quad v(t_2+\mathcal{T}) = v(t_2) = 0 \), we have
\[ \alpha(t_1+\mathcal{T}) = k_1 \mathcal{T}, \quad \alpha(t_1) = n_1 \mathcal{T}, \quad \alpha(t_2+\mathcal{T}) = \frac{\mathcal{T}}{2} + k_2 \mathcal{T}, \quad \alpha(t_2) = \frac{\mathcal{T}}{2} + n_2 \mathcal{T} \], where \( n_1, n_2, k_1, k_2 \) are integers. With
respect to \( q^2 \neq 1 \), it follows from Corollary 2 that \( \alpha' \) is not a \( \mathcal{A} \)-periodic function.

\( \Leftarrow \) Let there exist such a phase \( \alpha \) of \((Q)\) that \( \alpha' \) is not a \( \mathcal{A} \)-periodic function and \((30)\) is valid, where \( t_1, t_2 \in [0, \mathcal{A}) \), \( t_1 \neq t_2 \) with \( n_1, n_2, k_1, k_2 \) being integers.

Setting \( u(t) = \frac{\sin \alpha(t)}{\sqrt{\alpha'(t)}} \), \( v(t) = \frac{\cos \alpha(t)}{\sqrt{\alpha'(t)}} \), then \( u, v \) are independent solutions of \((Q)\), \( u(t_1) = u(t_1 + \mathcal{A}) = 0 \), \( v(t_2) = v(t_2 + \mathcal{A}) = 0 \). Thus \((29)\) holds for a \( \varphi \in \mathbb{C}, \varphi \neq 0 \) and since \( \alpha' \) is not a \( \mathcal{A} \)-periodic function, then - by Corollary 2 - we get \( q^2 \neq 1 \). Writing now \( t_2 \) and \( t_1 \) for \( t \) in the equations

\[
\frac{\sin \alpha(t + \mathcal{A})}{\sqrt{\alpha'(t + \mathcal{A})}} = \frac{\varphi \sin \alpha(t)}{\sqrt{\alpha'(t)}} , \quad \frac{\cos \alpha(t + \mathcal{A})}{\sqrt{\alpha'(t + \mathcal{A})}} = \frac{\varphi^{-1} \cos \alpha(t)}{\sqrt{\alpha'(t)}} ,
\]

respectively, we obtain

\[
\varphi = (-1)^{k_2 - n_2} \frac{\sqrt{\alpha'(t_2)}}{\sqrt{\alpha'(t_2 + \mathcal{A})}} \cdot (-1)^{k_1 - n_1} \frac{\sqrt{\alpha'(t_1 + \mathcal{A})}}{\sqrt{\alpha'(t_1)}} ,
\]

thus \((-1)^{k_2 - n_2} \frac{\sqrt{\alpha'(t_2)}}{\sqrt{\alpha'(t_2 + \mathcal{A})}} \cdot (-1)^{k_1 - n_1} \frac{\sqrt{\alpha'(t_1 + \mathcal{A})}}{\sqrt{\alpha'(t_1)}} \) (or also \((-1)^{k_2 - n_2} \frac{\sqrt{\alpha'(t_2 + \mathcal{A})}}{\sqrt{\alpha'(t_2)}} \cdot (-1)^{k_1 - n_1} \frac{\sqrt{\alpha'(t_1 + \mathcal{A})}}{\sqrt{\alpha'(t_1)}} \) ) are the characteristic multipliers of \((Q)\).

**Theorem 8.** Suppose \( \alpha \) is a phase of \((Q)\). This equation has independent solutions \( u, v \) satisfying \((2)\), where \( v \) has a zero on \( \mathbb{R} \) exactly if \( \alpha' \) is not a \( \mathcal{A} \)-periodic function and \( \alpha(t_1 + \mathcal{A}) = \alpha(t_1) + k \mathcal{A}, (-1)^{k} \frac{\sqrt{\alpha'(t_1 + \mathcal{A})}}{\sqrt{\alpha'(t_1)}} = \varphi \frac{\sqrt{\alpha'(t_1)}}{\sqrt{\alpha'(t_1 + \mathcal{A})}} \), where \( t_1 \in [0, \mathcal{A}) \), \( k \) being an integer.

**Proof.** \( \Rightarrow \) Let \((Q)\) have independent solutions \( u, v \) satisfying \((2)\) where \( v \) has a zero on \( \mathbb{R} \). It follows from \((2)\)
that there may be assumed without any loss on generality \( v(t_1) = 0 \) for \( t_1 \in [0, \hat{y}) \). If we put \( \beta(t) := \alpha(t) - \alpha(t_1) \) for \( t \in R \), then \( \beta \) is a phase of (Q), \( \beta(t_1) = 0 \) and \( v(t) = c \frac{\sin \beta(t)}{\sqrt{\beta'(t)}} \) for \( t \in R \), where \( 0 \neq c \in C \). Since \( v(t_1 + \hat{y}) = 0 \),

we have \( \beta(t_1 + \hat{y}) = k \hat{y}, k \) being an integer, hence
\[
\beta(t_1 + \hat{y}) - \beta(t_1) = \alpha(t_1 + \hat{y}) - \alpha(t_1) = k \hat{y}.
\]

Differentiating the equality \( \frac{\sin(\alpha(t + \hat{y}) - \alpha(t_1))}{\sqrt{\alpha'(t + \hat{y})}} = \frac{\sin(\alpha(t) - \alpha(t_1))}{\sqrt{\alpha'(t)}} \) and inserting \( t_1 \) in place of \( t \) in the resulting equality, we obtain \((-1)^k \sqrt{\alpha'(t_1 + \hat{y})} = \varphi \sqrt{\alpha'(t_1)} \).

Since it follows from (2) that every solution of (Q) is not a \( \hat{y} \)-periodic or \( \hat{y} \)-halfperiodic, then by Corollary 2, \( \alpha' \) is not a \( \hat{y} \)-periodic function, too.

\[(\Leftarrow) \] Let \( \alpha \) not to be a \( \hat{y} \)-periodic function and
\[
\alpha(t_1 + \hat{y}) = \alpha(t_1) + k \hat{y}, \ (-1)^k \sqrt{\alpha'(t_1 + \hat{y})} = \varphi \sqrt{\alpha'(t_1)} ,
\]
where \( t_1 \in [0, \hat{y}) \), \( k \) being an integer, and \( \varphi^2 = 1 \). Without any loss on generality there may be assumed \( \alpha(t_1) = 0 \). Putting \( v(t) := \frac{\sin \alpha(t)}{\sqrt{\alpha'(t)}} \) for \( t \in R \), then \( v \) is a solution of (Q),

\[
v(t_1) = v(t_1 + \hat{y}) = 0, \text{ thus } v(t + \hat{y}) = \tau v(t) \text{ for } t \in R,
\]
where \( \tau \in C \) is an appropriate number and from the equality
\[
(-1)^k \sqrt{\alpha'(t_1 + \hat{y})} = \varphi \sqrt{\alpha'(t_1)}\]
there follows \( \tau = \tau = 1 \). Since \( \alpha' \) is not a \( \hat{y} \)-periodic function, it follows from Corollary 2 that every solution of (Q) is not \( \hat{y} \)-periodic or \( \hat{y} \)-halfperiodic. Consequently it follows for (Q) from the Floquet theory that there exist such a solution \( u \) of (Q) that \( u, v \) are independent solutions of this equation and (2) holds.

**Remark 12.** If the assumptions of Theorem 7 or of Theorem 8 are satisfied, then there do not exist any \( \hat{y} \)-periodic solutions of the Riccati equation (11).
REFERENCES


Souhrn

Je vyšetřována diferenciální rovnice
\[ y'' = Q(t)y, \; Q(t+\tau) = Q(t), \; \text{Im} \; Q(t) \neq 0 \text{ pro } t \in \mathbb{R}, \]  
(Q)

kde Q je spojitá komplexní funkce na \(\mathbb{R}\). Z Floquetovy teorie plyne, že ke každé rovnici (Q) lze přiřadit čísla \(\varphi, \varphi^{-1}\), která se nazývají charakteristické multiplikátory rovnice (Q). Tato čísla jsou důležitá při vyšetřování kvalitativních vlastností řešení rovnice (Q). V práci je dán nový pohled na Floquetovu teorii rovnic typu (Q) z hlediska teorie fází. Zejména je dokázáno, jak lze hodnoty charakteristických multiplikátorů vyjádřit pomocí nějaké fáze rovnice (Q).

ТЕОРИЯ ФЛОКЕ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ
С КОМПЛЕКСНЫМ КОЭФФИЦИЕНТОМ ВЕЩЕСТВЕННОЙ ПЕРЕМЕННОЙ

Резюме

Изучается дифференциальное уравнение
\[ y'' = Q(t)y, \; Q(t+\tau) = Q(t), \; \text{Im} \; Q(t) \neq 0, \; t \in \mathbb{R}, \]  
(Q)

где Q непрерывная комплексная функция на \(\mathbb{R}\). Из теории
Флоке следует, что к каждому уравнению (Q) присоединяются числа \( \rho, \rho^{-1} \), которые называются характеристические мультипликаторы уравнения (Q). Эти числа важны при исследовании квазитривальных свойств решений уравнения (Q).

В этой работе приводится новый взгляд на теорию Флоке уравнений типа (Q) с точки зрения теории фаз. В особенности доказывается как значения характеристик мультипликаторов уравнения (Q) представить с помощью некоторой фазы уравнения (Q).