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ASYMPTOTIC PROPERTIES OF SOLUTIONS OF A CERTAIN THIRD-ORDER DIFFERENTIAL EQUATION WITH AN OSCILLATORY RESTORING TERM

JAN ANDRES

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1. Considering the equation

\[ x''' + ax'' + g(x)x' + h(x) = p(t), \]

where \( a > 0 \) is a constant, \( p(t) \in C(0, \infty) \), \( g(x), h(x) \in C^1(-\infty, \infty) \) and \( h(x) \) is an oscillatory function in the whole interval \((-\infty, \infty)\) with isolated zero points \( \tilde{x} \), K.E. Swick [1] has proved under

\[ \int_0^\infty |p(t)| \, dt < \infty \]

the following

**Theorem 0.** If there exist such positive constants \( b, c \) that the assumptions
1) \( \frac{1}{x} \int_{0}^{x} g(s) \, ds \geq b, \)

2) \( h''(x) \leq c \) with \( c < ab, \)

3) \( h(x) \text{sgn } x \geq 0 \)

are fulfilled for all \( x \in (-\infty, \infty), \) then all solutions \( x(t) \) of (1) are bounded satisfying

\[
\lim_{t \to \infty} x(t) = \bar{x}, \quad \lim_{t \to \infty} x'(t) = \lim_{t \to \infty} x''(t) = 0. \tag{3}
\]

The aim of the present paper is to make the above result more precise in two directions, namely (i) condition 2) may be localized to the origin and (ii) \( h(x) \text{sgn } x \) may run below the x-axis for \( g(x) \equiv b > 0, \) both, when \( |h(x)| \) is bounded everywhere.

2. Hence let us assume

\[
\limsup_{|x| \to \infty} |h(x)| < \infty \tag{4}
\]

and recall several well-known results at first.

**Lemma 1.** Let all solutions \( x(t) \) of (1) be bounded together with their derivatives \( x'(t), x''(t). \) If (3) is satisfied for those of autonomous equation (1) (i.e. \( p(t) = 0 \)), then (3) is also true for all solutions \( x(t) \) of (1) with (2).

**Proof.** The above assertion is a direct consequence of the Markus-Opial-Yoshizawa theorem [2, p.59], when specified to (1).

**Lemma 2.** If there exists such an \( h \)-neighbourhood of the root \( \bar{x} \) of \( h(x) \) in (1) with \( p(t) \equiv 0 \) that conditions

2') \( a g(x) - h'(x) \geq \delta > 0 \) \( (\delta \text{-const.}), \)

3') \( h''(x) > 0, \)

4) \( g'(\bar{x}) = 0 \)
are satisfied for $0 < |x - \bar{x}| < h$, then $\bar{x}$ is asymptotically stable.

For the proof see [3].

Remark 1. It can be readily checked that the basin of attraction due to $\bar{x}$ is determined by $|g^*(x)x^*| - \delta_n \leq \delta_0$ (small enough constant), while for $g(x) \equiv b > 0$ even by $h(x)\text{sgn}(x-x^*) > 0$, because of the form of Liapunov’s function employed in [3].

Lemma 3. If there exist such positive constants $b, G$ that condition

\begin{equation}
1') b \leq g(x) \leq G < a^2
\end{equation}

is satisfied for all $x \in (-\infty, \infty)$ together with (4) and

5) $\limsup_{t \to \infty} |p(t)| < \infty$,

6) $\limsup_{t \to \infty} \int_0^t \left| \int_0^s p(s) \, ds \right| < \infty$.

then there exists also a constant $D'$ such that all solutions $x(t)$ of (1) satisfy

\begin{equation}
\limsup_{t \to \infty} (|x^{\prime}(t)| + |x^{\prime\prime}(t)|) < D'.
\end{equation}

For the proof see [4].

Lemma 4. If there exists (finite)

\begin{equation}
\lim_{t \to \infty} x(t)
\end{equation}

of (1) satisfying (4), (5) and 5) of Lemma 3, then there is also

\begin{equation}
\lim_{t \to \infty} x'(t) = \lim_{t \to \infty} x''(t) = 0. \quad (3')
\end{equation}
Lemma 5. If there exists (finite) \( \lim_{t \to \infty} \int_0^t h(x(s))\,ds \) for \( x(t) \) of (1), then

\[
\lim_{t \to \infty} h(x(t)) = 0
\]

and consequently \( \lim_{t \to \infty} x(t) = \bar{x} \). (6)

Lemma 6. Under the assumptions of Lemma 3 every bounded solution \( x(t) \) of (1) either satisfies relation (3) or there exists such a root \( \bar{x} \) of \( h(x) \) that \( (x(t) - \bar{x}) \) oscillates.

- can be performed just in the same way as in [7].

3. Assuming all solutions of (1) being bounded, we now will deduce several important consequences of the above statements.

Consequence 1. If \( h(x) \text{sgn } x \geq 0 \) is satisfied for all \( x \), then every bounded solution \( x(t) \) of (1) either satisfies (6) or oscillates (i.e. \( \lim_{t \to \infty} \sup |x(t)| > 0 = \lim_{t \to \infty} \inf |x(t)| \)) under (5), (6).

If \( x(t) \) is not oscillatory, then there is either \( x(t) \geq 0 \) or \( x(t) \leq 0 \) for \( t \) great, say \( t \geq T \) and

\[
\int_T^t h(x(s))\,ds
\]

is a monotone function. Thus there exists finite (cf. (5), (6))
\[
\lim_{t \to \infty} \int_{0}^{t} h(x(s)) ds
\]

and our assertion is implied by Lemma 5 immediately.

**Consequence 2.** Let the assumptions of Lemma 3 be fulfilled with conditions 5), 6) replaced by (2). If \( h(x) \operatorname{sgn} x \geq 0 \) is satisfied for all \( x \) and

\[
\begin{align*}
2'' & \quad \alpha g(0) - h'(0) > 0, \\
3'' & \quad h'(0) > 0, \\
4' & \quad g'(0) = 0,
\end{align*}
\]

then (3) is satisfied for every bounded solution \( x(t) \) of (1).

**Proof.** Consequence 1 says that every bounded solution \( x(t) \) of (1) either oscillates or satisfies (6). However, conditions 2''), 3''), 4'') imply the existence of such an \( h \)-neighbourhood of the origin that assumptions of Lemma 2 are valid in it and therefore a trivial solution of autonomous equation (1) (i.e. \( p(t) \equiv 0 \)) is asymptotically stable. Hence, any oscillatory solution must be attracted to the origin with respect to Remark 1 and Lemma 3 and so such a possibility is reduced to (6) with \( x = 0 \) for \( p(t) \equiv 0 \).

Thus (3) is immediately implied by Lemma 3 and Lemma 4 and the same is true even for nonautonomous equation (1) in view of Lemma 1.

**Consequence 3.** Let \( h'(x) \neq 0 \) be satisfied for all zero points of \( h(x) \). If

\[
2' \quad ab - h'(x) \geq \delta > 0 \quad (\delta - \text{const.})
\]

holds for all \( x \) and \( a^2 > g(x) \equiv b > 0 \), then every bounded solution \( x(t) \) of (1) obeys (3), provided (2) and (4).
Lemma 6 asserts that every bounded solution $x(t)$ of (1) either satisfies (3) or there exists a root $\bar{x}$ of $h(x)$ such that $(x(t) - \bar{x})$ oscillates, provided $p(t) = 0$ and $a^2 > b > 0$ (i.e. assumptions of Lemma 3). However, assuming $h'(\bar{x}) \neq 0$ and $ab - h'(x) \geq 0$, the roots $\bar{x}$ of $h(x)$ with $h'(\bar{x}) > 0$ are asymptotically stable and consequently any nontrivial $x(t)$ of autonomous equation (1) is attracted to some $\bar{x}$ with $h'(\bar{x}) > 0$ (and therefore bounded as well) with respect to Remark 1. The remainder of the proof immediately follows from Lemma 1 and Lemma 4.

Remark 2. It is clear from the ideas introduced above that assumption $h'(\bar{x}) \neq 0$ of Consequence 3 can be replaced by a weaker one, namely $h(x)\text{sgn}(x - \bar{x}) \neq 0$, in a suitable reduced neighbourhood of $\bar{x}$, but not $h(x)\text{sgn} x \leq 0$.

4. In the final section boundedness results will be given.

Theorem 1. Under the assumptions of Consequence 2 all solutions of (1) are bounded satisfying (3).

Proof. If any solution $x(t)$ of (1) would not be bounded e.g.

$$\limsup_{t \to \infty} x(t) = \infty$$

(the case of $\liminf_{t \to \infty} x(t) = -\infty$ can be treated quite analogically), then integrating (1) from a suitable $T$ to $t \geq T$ and using the above assumptions, we get the following inequality
\[ b(x(t) - x(T)) \text{sgn } x \leq \left| \int_{T}^{t} p(s) \, ds \right| - \int_{T}^{t} h(x(s)) \text{sgn } x \, ds + a |x'(t) - x'(T)| + \left| \int_{T}^{t} x''(t) - x''(T) \, dt \right| \]
\[ - \int_{T}^{t} |p(s)\, ds| - \int_{T}^{t} |h(x(s))| \, ds + 2\max(a,1)D', \]

i.e. \( |x(t)| \leq |x(T)| + \frac{1}{b}(2\max(a,1)D' + P), \)

where \( P \) is a constant implied by (2), contradictionally. The remaining part of our assertion is included in Consequence 2.

**Theorem 2.** Let the assumptions of Consequence 3 be fulfilled with \( b < \frac{a^2}{4}. \) If conditions (2) and (4) yield such constants \( H, P, P_0 \) that \( |p(t)| \leq P, \)
\[ \int_{0}^{t} p(s) \, ds \leq P_0 \]

for \( t \geq 0 \) and \( |h(x)| \leq H \) for all \( x \in (-\infty, \infty) \) together with
\[ \min(d(\overline{x}_k, \overline{x}_{k+1}), d(\overline{x}_k, \overline{x}_{k-1})) > \frac{2(H + P)}{b} \left( \frac{2}{b} + \frac{P}{b} \right) + \frac{P_0}{b}, \]  
(7)

where \( \overline{x}_k \) are the roots of \( h(x) \) with \( h'(\overline{x}_k) > 0 \) and \( \overline{x}_{k-1}, \overline{x}_{k+1} \) denote the couple of adjacent zero points of \( \overline{x}_k \) (\( k = 0, \pm 2, \pm 4, \ldots \)), then all solutions of (1) are bounded satisfying (3).

\( P \equiv \alpha \equiv f. \) The boundedness of all solutions of (1) can be verified quite analogously to [7]. Let us note that this follows directly from the assumptions of Consequence 3 for the autonomous equation even for \( g(x) \not\equiv b. \) Indeed, if any its solution would not be bounded i.e. \( \lim \sup x(t) = \infty \)
for \( t \to \infty \)
or \( \lim \inf x(t) = -\infty \), then such a zero point of \( h(x) \) exists attracting \( x(t) \) asymptotically with respect to Remark 1 and Lemma 3, contradictionally. The remaining part of our assertion is included in Consequence 3.

**Remark 3.** Considering equation (1) with \( a^2/4 > g(x) \equiv b > 0 \) and (4), it is clear that Theorem 0 of Swick [1] can be generalized in the following way: under 2), 3) condition 1) takes the local form \( ab - h'(0) > 0 \) and under 1), 3) condition 2) can be replaced by much weaker assumption of oscillatory \( h(x) \) with (7), in general, but not \( h(x) \text{sgn} x < 0 \) satisfied in reduced neighbourhoods of the zero points of \( h(x) \).

**Remark 4.** Further generalization could be certainly done if either \( a g(x) \) is great enough or \( |g'(x)| \) is sufficiently small. This way is very important from the technical point of view, because of considering the phase-synchronization problem [8, 9].

**REFERENCES**


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Souhrn

V práci je upřesněn a doplněn Swickův výsledek [1], týkající se vlastnosti (3), kterou nabývají všechna řešení rovnice (1). Za předpokladu o ohraničenosti funkce h(x) je ukázáno, jak může být podmínka 2) jeho věty 0 lokalizována do počátku a zejména, že funkce h(x)sgn x může zabíhat i pod osu x.

ACIMPTOTICHESKIE SVOISTVA RESHENIJ
ODNOGO DIFERENCIAL'NOGO URAVNEIIENIa TRET'YEO PORYDA
C OSCIILIIRUJUSHIM VOSSTANAVLIVAJUSHIM CHLENOm

Резюме

209
В работе уточняется и дополняется результат Свика [1], относящийся к свойству (3), которому подчиняются все решения уравнения (1). Ввиду предположения ограниченности функции $h(x)$ показано, что условие 2) теоремы 0 можно локализовать в начало координат и доказано, что функция $h(x) \text{sgn } x$ может находиться тоже под осью $x$.

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