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Vedoucí katedry: Doc.RNDr. Jindřich Palát, CSc.

ASYMPTOTIC PROPERTIES OF SOLUTIONS OF A CERTAIN THIRD-ORDER DIFFERENTIAL EQUATION WITH AN OSCILLATORY RESTORING TERM

JAN ANDRES

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1. Considering the equation

$$x''' + ax'' + g(x)x' + h(x) = p(t), \tag{1}$$

where $a > 0$ is a constant, $p(t) \in C\langle 0, \infty)$, $g(x), h(x) \in C^1(-\infty, \infty)$ and $h(x)$ is an oscillatory function in the whole interval $(-\infty, \infty)$ with isolated zero points $\bar{x}$, K.E.Swick [1] has proved under

$$\int_0^\infty |p(t)|\, dt < \infty \tag{2}$$

the following

<u>Theorem 0</u>. If there exist such positive constants b,c that the assumptions

1) $\frac{1}{x}\int_0^x g(s)ds \geq b$,

2) $h'(x) \leq c$ with $c < ab$,

3) $h(x) \operatorname{sgn} x \geq 0$

are fulfilled for all $x \in (-\infty, \infty)$, then all solutions $x(t)$ of (1) are bounded satisfying

$$\lim_{t\to\infty} x(t) = \bar{x}, \quad \lim_{t\to\infty} x'(t) = \lim_{t\to\infty} x''(t) = 0. \tag{3}$$

The aim of the present paper is to make the above result more precize in two directions, namely (i) condition 2) may be localized to the origin and (ii) $h(x) \operatorname{sgn} x$ may run bellow the x-axis for $g(x) \equiv b > 0$, both, when $|h(x)|$ is bounded everywhere.

2. Hence let us assume

$$\limsup_{|x|\to\infty} |h(x)| < \infty \tag{4}$$

and recall several well-known results at first.

Lemma 1. Let all solutions $x(t)$ of (1) be bounded together with their derivatives $x'(t), x''(t)$. If (3) is satisfied for those of autonomous equation (1) (i.e. $p(t) \equiv 0$), then (3) is also true for all solutions $x(t)$ of (1) with (2).

P r o o f. The above assertion is a direct consequence of the Markus-Opial-Yoshizawa theorem [2, p.59], when specified to (1).

Lemma 2. If there exists such an h-neighbourhood of the root $\bar{x}$ of $h(x)$ in (1) with $p(t) \equiv 0$ that conditions

2') $ag(x) - h'(x) \geq \delta > 0$ (δ -const.),

3') $h'(x) > 0$,

4) $g'(\bar{x}) = 0$

are satisfied for $0 < |x - \bar{x}| < h$, then $\bar{x}$ is asymptotically stable.

For the proof see [3] .

Remark 1. It can be readily checked that the basin of attractivity due to $\bar{x}$ is determined by $|g'(x)x'| \leq \delta_0$ (δ_0 - small enough constant), while for $g(x) \equiv b > 0$ even by $h(x)\operatorname{sgn}(x-\bar{x}) > 0$, because of the form of Liapunov's function employed in [3].

Lemma 3. If there exist such positive constants b,G that condition

1') $b \leq g(x) \leq G < a^2$

is satisfied for all $x \in (-\infty, \infty)$ together with (4) and

5) $\limsup_{t\to\infty} |p(t)| < \infty$,

6) $\limsup_{t\to\infty} \left|\int_0^t p(s)ds\right| < \infty$

then there exists also a constant D' such that all solutions $x(t)$ of (1) satisfy

$$\limsup_{t\to\infty} (|x'(t)| + |x''(t)|) < D' . \qquad (5)$$

For the proof see [4].

Lemma 4. If there exists (finite)

$$\lim_{t\to\infty} x(t)$$

of (1) satisfying (4), (5) and 5) of Lemma 3, then there is also

$$\lim_{t\to\infty} x'(t) = \lim_{t\to\infty} x''(t) = 0. \qquad (3')$$

P r o o f. This assertion follows directly from the theorem introduced in [5, p.141], because of $\limsup_{t\to\infty} |x'''(t)| < \infty$.

Lemma 5. If there exists (finite) $\lim_{t\to\infty} \int_0^t h(x(s))ds$ for $x(t)$ of (1), then

$$\lim_{t\to\infty} h(x(t)) = 0$$

and consequently $\lim_{t\to\infty} x(t) = \bar{x}$. (6)

P r o o f. This assertion immediately follows from the well--known lemma of Barbălat [6].

Lemma 6. Under the assumptions of Lemma 3 every bounded solution $x(t)$ of (1) either satisfies relation (3) or there exists such a root $\bar{x}$ of $h(x)$ that $(x(t) - \bar{x})$ oscillates.

P r o o f - can be performed just in the same way as in [7].

3. Assuming all solutions of (1) being bounded, we now will deduce several important consequences of the above statements.

Consequence 1. If $h(x)\operatorname{sgn} x \geq 0$ is satisfied for all x, then every bounded solution $x(t)$ of (1) either satisfies (6) or oscillates (i.e. $\limsup_{t\to\infty} |x(t)| > 0 = \liminf_{t\to\infty} |x(t)|$) under (5), 6).

P r o o f. If $x(t)$ is not oscillatory, then there is either $x(t) \geq 0$ or $x(t) \leq 0$ for t great, say $t \geq T$ and

$$\int_T^t h(x(s))ds$$

is a monotone function. Thus there exists finite (cf. (5), 6))

$$\lim_{t\to\infty} \int_0^t h(x(s))ds$$

and our assertion is implied by Lemma 5 immediately.

Consequence 2. Let the assumptions of Lemma 3 be fulfilled with conditions 5), 6) replaced by (2). If $h(x)\operatorname{sgn} x \geq 0$ is satisfied for all x and

2'') $ag(0) - h'(0) > 0$,

3'') $h'(0) > 0$,

4') $g'(0) = 0$,

then (3) is satisfied for every bounded solution x(t) of (1).

P r o o f. Consequence 1 says that every bounded solution x(t) of (1) either oscillates or satisfies (6). However, conditions 2''), 3''), 4') imply the existence of such an h-neighbourhood of the origin that assumptions of Lemma 2 are valid in it and therefore a trivial solution of autonomous equation (1) (i.e. $p(t) \equiv 0$) is asymptotically stable. Hence, any oscillatory solution must be attracted to the origin with respect to Remark 1 and Lemma 3 and so such a possibility is reduced to (6) with $\overline{x} = 0$ for $p(t) \equiv 0$.

Thus (3) is immediately implied by Lemma 3 and Lemma 4 and the same is true even for nonautonomous equation (1) in view of Lemma 1.

Consequence 3. Let $h'(\overline{x}) \neq 0$ be satisfied for all zero points of h(x). If

2') $ab - h'(x) \geq \delta > 0$ (δ -const.)

holds for all x and $a^2 > g(x) \equiv b > 0$, then every bounded solution x(t) of (1) obeys (3), provided (2) and (4).

P r o o f. Lemma 6 asserts that every bounded solution $x(t)$ of (1) either satisfies (3) or there exists a root $\overline{x}$ of $h(x)$ such that $(x(t) - \overline{x})$ oscillates, provided $p(t) \equiv 0$ and $a^2 > g(x) \equiv b > 0$ (i.e. assumptions of Lemma 3). However, assuming $h'(\overline{x}) \neq 0$ and $ab - h'(x) \geq \delta > 0$, the roots $\overline{x}$ of $h(x)$ with $h'(\overline{x}) > 0$ are asymptotically stable and consequently any nontrivial $x(t)$ of autonomous equation (1) is attracted to some $\overline{x}$ with $h'(\overline{x}) > 0$ (and therefore bounded as well) with respect to Remark 1. The remainder of the proof immediately follows from Lemma 1 and Lemma 4.

Remark 2. It is clear from the ideas introduced above that assumption $h'(\overline{x}) \neq 0$ of Consequence 3 can be replaced by a weaker one, namely $h(x)\operatorname{sgn}(x - \overline{x}) \neq 0$, in a suitable reduced neighbourhood of $\overline{x}$, but not $h(x)\operatorname{sgn} x < 0$.

4. In the final section boundedness results will be given.

Theorem 1. Under the assumptions of Consequence 2 all solutions of (1) are bounded satisfying (3).

P r o o f. If any solution $x(t)$ of (1) would not be bounded e.g.

$$\limsup_{t \to \infty} x(t) = \infty$$

(the case of $\liminf_{t \to \infty} x(t) = -\infty$

can be treated quite analogically), then integrating (1) from a suitable T to $t \geq T$ and using the above assumptions, we get the following inequality

$$b(x(t) - x(T))\operatorname{sgn} x \leq \left|\int_T^t p(s)ds\right| - \int_T^t h(x(s))\operatorname{sgn} x ds + a\left|x'(t) - x'(T)\right| + \left|x''(t) - x''(T)\right| \leq$$

$$\leq \left|\int_T^t p(s)ds\right| - \int_T^t \left|h(x(s))\right| ds + 2\max(a,1)D' ,$$

i.e. $|x(t)| \leq |x(T)| + \frac{1}{b}(2\max(a,1)D' + P)$,

where P is a constant implied by (2), contradictionally. The remaining part of our assertion is included in Consequence 2.

Theorem 2. Let the assumptions of Consequence 3 be fulfilled with $b < a^2/4$. If conditions (2) and (4) yield such constants H, P, P_0 that $|p(t)| \leq P$,

$$\left|\int_0^t p(s)ds\right| \leq P_0$$

for $t \geq 0$ and $|h(x)| \leq H$ for all $x \in (-\infty, \infty)$ together with

$$\min(d(\bar{x}_k,\bar{x}_{k+1}),\ d(\bar{x}_k,\bar{x}_{k-1})) > \frac{2(H + P)}{b}\left(\frac{2}{a} + \frac{a}{b}\right) + \frac{P_0}{b}, \qquad (7)$$

where $\bar{x}_k$ are the roots of $h(x)$ with $h'(\bar{x}_k) > 0$ and $\bar{x}_{k-1}, \bar{x}_{k+1}$ denote the couple of adjacent zero points of $\bar{x}_k$ ($k = 0, \pm 2, \pm 4, \ldots$), then all solutions of (1) are bounded satisfying (3).

P r o o f. The boundedness of all solutions of (1) can be verified quite analogously to [7]. Let us note that this follows directly from the assumptions of Consequence 3 for the autonomous equation even for $g(x) \not\equiv b$. Indeed, if any its solution would not be bounded i.e. $\limsup_{t\to\infty} x(t) = \infty$

or $\liminf_{t\to\infty} x(t) = -\infty$, then such a zero point of h(x) exists attracting x(t) asymptotically with respect to Remark 1 and Lemma 3, contradictionally. The remaining part of our assertion is included in Consequence 3.

Remark 3. Considering equation (1) with $a^2/4 > g(x) \equiv b > 0$ and (4), it is clear that Theorem 0 of Swick [1] can be generalized in the following way: under 2), 3) condition 1) takes the local form $ab - h'(0) > 0$ and under 1), 3) condition 2) can be replaced by much weaker assumption of oscillatory h(x) with (7), in general, but not $h(x)\operatorname{sgn} x < 0$ satisfied in reduced neighbourhoods of the zero points of h(x).

Remark 4. Further generalization could be certainly done if either ag(x) is great enough or $|g'(x)|$ is sufficiently small. This way is very important from the technical point of view, because of considering the phase-synchronization problem [8, 9].

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ASYMPTOTICKÉ VLASTNOSTI ŘEŠENÍ JISTÉ DIFERENCIÁLNÍ ROVNICE TŘETÍHO ŘÁDU S OSCILATORICKÝM OBNOVUJÍCÍM ČLENEM

Souhrn

V práci je upřesněn a doplněn Swickův výsledek [1], týkající se vlastnosti (3), kterou nabývají všechna řešení rovnice (1). Za předpokladu o ohraničenosti funkce h(x) je ukázáno, jak může být podmínka 2) jeho věty O lokalizována do počátku a zejména, že funkce h(x)sgn x může zabíhat i pod osu x.

АСИМПТОТИЧЕСКИЕ СВОЙСТВА РЕШЕНИЙ ОДНОГО ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ ТРЕТЬЕГО ПОРЯДКА С ОСЦИЛЛИРУЮЩИМ ВОССТАНАВЛИВАЮЩИМ ЧЛЕНОМ

Резюме

В работе уточняется и дополняется результат Свика [1], относящийся к свойству (3), которому подчиняются все решения уравнения (1). Ввиду предположения ограниченности функции h(x) показано, что условие 2) теоремы 0 можно локализовать в начало координат и доказано, что функция h(x)sgn x может находиться тоже под осью x.

Author's address:
RNDr. Jan Andres, CSc.
přírodovědecká fakulta
Univerzity Palackého
Gottwaldova 15
771 46 Olomouc
ČSSR /Czechoslovakia/

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