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ON A FUNDAMENTAL CENTRAL DISPERSION OF THE FIRST KIND AND THE ABEL FUNCTIONAL EQUATION IN STRONGLY REGULAR SPACES OF CONTINUOUS FUNCTIONS

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The present paper deals with a distribution problem of zeros of functions from a strongly regular space of continuous functions of infinite type and bothsided oscillatory.

The distribution of zeros is described by a function called a fundamental central dispersion of the first kind. Relations between a fundamental central dispersion of the first kind and a phase of an ordered pair of functions from a strongly regular space expressed by the Abel functional equation are found.

The results obtained are applied to spaces of solutions of second order linear differential equations of a general form

$$y'' + a(t)y' + b(t)y = 0, \qquad \text{(ab)}$$

where $a, b \in C^{(0)}(j)$ and of Sturm form

$$(p(t)y')' + q(t)y = 0, \qquad (pq)$$

where $p, q \in C^{(0)}(j)$, $py' \in C^{(1)}(j)$, $p(t) \neq 0$ in j, here $C^{(0)}(j)$ and $C^{(1)}(j)$ denote respectively a set of all continuous functions and a set of all functions with a continuous first derivative, on an interval j.

Let R be a field of real numbers and j be an open interval in R. Suppose S is a two-dimensional linear space of continuous functions with a definition interval j and with a basis (y_1, y_2) over the field R. Let functions y_1, y_2 be oscillatory provided that their zeros separate and the only cluster points of these zeros are just end points of the interval j. If t_i, $i = 0, \pm 1, \pm 2, \dots$ are zeros of the function $y_1(t)$ then we require

$$\lim_{t \to t_i+} \frac{y_2(t)}{y_1(t)} = -\infty \quad \text{and} \quad \lim_{t \to t_i-} \frac{y_2(t)}{y_1(t)} = +\infty$$

in the case of $\frac{y_2(t)}{y_1(t)}$ is increasing by parts and

$$\lim_{t \to t_i+} \frac{y_2(t)}{y_1(t)} = +\infty \quad \text{and} \quad \lim_{t \to t_i-} \frac{y_2(t)}{y_1(t)} = -\infty$$

in the case of $\frac{y_2(t)}{y_1(t)}$ is decreasing by parts.

Then the space S is called a strongly regular space of continuous functions of infinite type bothsided oscillatory with the definition interval j. (See [2], [3].)

Obviously the space S^* with the basis $(\cos s, \sin s)$, $s \in J$, $J = (-\infty, \infty)$ may serve as an example of such a space.

Remark. In [2] the space S^* is named a canonical two-dimensional space of continuous functions.

It is known ([2]) that there exists a global transformation of the space S onto the space S^*, that is, there exists a bijection $h: j \to (-\infty, \infty)$, $h \in C^{(0)}(j)$,

a function $f \in C^{(0)}(j)$, $f(t) \neq 0$ for $t \in j$,

a matrix $A = \| a_{ik} \|$, $a_{ik} \in R$, $i,k = 1,2$, $\det A \neq 0$,

such that the equality

$$y(t) = A\, f(t)\, Y[h(t)] \tag{1}$$

holds for every $t \in j$, where $y = (y_1, y_2)^T$, $Y = (Y_1, Y_2)^T$ and $Y_1 = \cos s$, $Y_2 = \sin s$, $s \in (-\infty, \infty)$.

Let $t \in j$ be an arbitrary point. The set of all functions $u \in S$, such that $u(t) = 0$, is characterized by the following facts: the functions are dependent on j, all their zeros are common and the zeros are isolated besides.

Definition 1. The function $\varphi_1 = \varphi_1(t)$, $t \in j$, mapping every number t on the first to the right of t lying zero $\varphi_1(t)$ of the functions $u \in S$ vanishing in t, is called a fundamental central dispersion of the first kind of the space S.

Suppose the space S is a strongly regular space of continuous functions of infinite type bothsided oscillatory with the definition interval j, in all the following theorems.

Theorem 1. The fundamental central dispersion of the first kind $\varphi_1 = \varphi_1(t)$, $t \in j$, of the space S satisfies the following assertions

1. φ_1 is defined on j,
2. φ_1 is increasing in j,
3. φ_1 is attains to every value from j,
4. φ_1 is continuous in j.

Proof. Ad 1. The assertion follows from the Definition 1 at once.

Ad 2. We show that for every two points t_1, $t_2 \in j$, such that $t_1 < t_2$, the inequality $\varphi_1(t_1) < \varphi_1(t_2)$ holds. If $\varphi_1(t_1) \geq \varphi_1(t_2)$ for some points $t_1, t_2 \in j$, $t_1 < t_2$, then zeros of the functions of the space S coming through the points t_1, $\varphi_1(t_1)$ and zeros of the functions of the space S coming through the points t_2, $\varphi_1(t_2)$ would not separate. This contradicts the assumption of the separation of zeros of functions from the space S.

Ad 3. Both end points of the interval j are cluster points of zeros of every function from S according the assumption. So there exists a point $t_1 \in j$ for every $t_2 \in j$, so that $t_1 < t_2$ and $\varphi_1(t_1) = t_2$. The proof of the third assertion is complete.

Ad 4. Let $\varepsilon > 0$ and $t_0 \in j$ be an arbitrary point. It is sufficient to show that φ_1 is a continuous function in t_0. Let $\varphi_1(t_0) = t_1$, $(t_1 - \varepsilon) \in j$ and $(t_1 + \varepsilon) \in j$. Since φ_1 increases in j and attains every value from j so there exists a point $t_0 - \delta_1 (< t_0)$, $\delta_1 > 0$ to the point $t_1 - \varepsilon (< t_1)$ such that $\varphi_1(t_0 - \delta_1) = t_1 - \varepsilon$ and there exists a point $t_0 + \delta_2 (> t_0)$, $\delta_2 > 0$ to the point $t_1 + \varepsilon (> t_1)$ such that $\varphi_1(t_0 + \delta_2) = t_1 + \varepsilon$. Let $\delta = \min(\delta_1, \delta_2)$. Then

$$t_1 - \varepsilon = \varphi_1(t_0 - \delta_1) \leqq \varphi_1(t_0 - \delta) < \varphi_1(t) < \varphi_1(t_0 + \delta) \leqq$$
$$\leqq \varphi_1(t_0 + \delta_2) = t_1 + \varepsilon$$

is valid for every $t \in (t_0 - \delta, t_0 + \delta)$. From this we obtain

$$|\varphi_1(t) - t_1| < \varepsilon$$

or

$$|\varphi_1(t) - \varphi_1(t_0)| < \varepsilon \qquad \text{for } t \in (t_0 - \delta, t_0 + \delta),$$

thus φ_1 is continuous in every point $t_0 \in j$.

Theorem 2. Suppose the strongly regular space S globally transforms itself onto the space S^* by the formula (1). Then the fundamental central dispersion $\varphi_1 = \varphi_1(t)$ of the space S and the bijection h = h(t) are related by

$$[h\,\varphi_1(t)] - h(t) = \varepsilon\pi,$$

where $\varepsilon = 1$ if the function h increases in j and $\varepsilon = -1$ if the function h decreases in j.

Note that $h \in C^{(0)}(j)$, $\varphi_1 \in C^{(0)}(j)$.

P̲r̲o̲o̲f̲. Let $k_1, k_2 \in R$, $K_1 = \sqrt{k_1^2+k_2^2}$ and K_2 be defined by the formulas: $\cos K_2 = k_1/K_1$, $\sin K_2 = k_2/K_1$. Multiplying (1) by a vector $k = (k_1,k_2)$ we get

$$k_1y_1(t) + k_2y_2(t) = K_1f(t)\cos[h(t) - K_2] . \qquad (3)$$

If $t_1 < t_2$ are two neighbouring zeros of the function $k_1y_1(t) + k_2y_2(t) \in S$ then the difference between the values of arguments of the cosine function on the right-hand side of (3) is equal to π, i.e.

$$|h(t_2) - h(t_1)| = \pi . \qquad (4)$$

For $t_1 < t_2$ we have

$$t_2 = \varphi_1(t_1).$$

If $h = h(t)$ increases in j, then $h[\varphi_1(t)] - h(t) = \pi$, if $h = h(t)$ decreases in j, then $h[\varphi_1(t)] - h(t) = -\pi$, which yields from (4). The proof is complete.

Recall now that in [2] there is defined the first phase (briefly phase) α of an ordered pair of independent functions (y_2,y_1) of a strongly regular space S with a definition interval j as every continuous function in j satisfying

$$\mathrm{tg}\,\alpha(t) = y_2(t)/y_1(t).$$

A countable system of phases α_k, k integral number, $\alpha_k(t) = \alpha(t) + k\pi$, $\alpha_0 = \alpha$, is defined by this formula. Then there is found the relation between the phase and the bijection h by which is arranged a global transformation of the space S^* onto the space S and shown that

$$\alpha_k(t) = h(t) + k\pi, \quad k \text{ integral number}, \quad \alpha_0 = \alpha(t).$$

See [2], Theorem 2.7.

Theorem 3. Let $\varphi_1 = \varphi_1(t)$ be the fundamental central dispersion of the first kind of the space S, $\alpha = \alpha(t)$ be the phase of basis $(y_2, y_1) \in S$. Then there holds

$$\alpha[\varphi_1(t)] - \alpha(t) = \varepsilon\pi , \qquad (5)$$

where $\varepsilon = 1$ or $\varepsilon = -1$ according as the phase α increases or decreases in j.

The formula (5) is known as the Abel functional equation, compare [1].

Proof. From the definition of the phase and from the Theorem 2.7 in [2] we have that the phase α of the ordered pair of functions (y_2, y_1) satisfies

$$\mathrm{tg}\ \alpha(t) = \frac{y_2(t)}{y_1(t)}$$

and increases or decreases according as the quotient y_2/y_1 increases or decreases. From the formula for α_k we obtain

$$\alpha_k(t) = h(t) + k\pi ,$$

where h is the bijection given by (1).

Taking account of (2) we have

$$\varepsilon\pi = h[\varphi_1(t)] - h(t) = \alpha_k[\varphi_1(t) - k\pi - \alpha_k(t) + k\pi =$$

$$= \alpha_k[\varphi_1(t)] - \alpha_k(t) .$$

If α denote an arbitrary phase of the basis (y_2, y_1) then

$$\alpha[\varphi_1(t)] - \alpha(t) = \pi \text{ or } \alpha[\varphi_1(t)] - \alpha(t) = -\pi$$

according as α increases or decreases in j.

The fundamental central dispersion of the first kind and the Abel functional equation in the space $S_{ab}(S_{pq})$

Let $S_{ab}(S_{pq})$ be the space of solutions of differential equations (ab) ((pq)) of infinite type and bothsided oscillatory with the definition interval j.

From the definition of these spaces follows that the spaces globally transform themselves onto the space S^* with the definition interval $(-\infty, \infty)$. That is, there exist to the basis $(y_1, y_2) \in S_{ab}(S_{pq})$

a bijection $h : j \longrightarrow (-\infty, \infty)$, $h \in C^{(2)}(j)$,

a function $f \in C^{(2)}(j)$, $f(t) \neq 0$ for $t \in j$,

a matrix $A = \| a_{ik} \|$, $i,k = 1,2$

such that the equality

$$y(t) = A\, f(t)\, Y[h(t)] ,$$

holds for every $t \in j$, where $y = (y_1, y_2)^T$, $Y = (Y_1, Y_2)^T$, $Y_1 = \cos s$, $Y_2 = \sin s$, $s \in (-\infty, \infty)$.

Suppose φ_1 is the fundamental central dispersion of the first kind of the space $S_{ab}(S_{pq})$.

Theorem 4. The fundamental central dispersion of the first kind φ_1 of the space $S_{ab}(S_{pq})$ satisfies the following assertions

1. φ_1 is defined on j,
2. φ_1 is increasing in j,
3. φ_1 attains to every value from j,
4. $\varphi_1 \in C^{(0)}(j)$.

Proof. The theorem is a modification of Theorem 1 for the space $S_{ab}(S_{pq})$.

Theorem 5. Suppose the space $S_{ab}(S_{pq})$ globally transforms itself onto the space S^* by the formula (1): $y(t) =$

$= Af(t)Y[h(t)]$. Then the fundamental central dispersion of the first kind $\varphi_1 = \varphi_1(t)$ of the space $S_{ab}(S_{pq})$ and the bijection $h = h(t)$ satisfying the equality (1) are related by

$$h[\varphi_1(t)] - h(t) = \varepsilon\pi , \tag{6}$$

where $\varepsilon = 1$ or $\varepsilon = -1$ according as the function h increases or decreases in j.

Proof. The theorem is a modification of Theorem 2 for the space $S_{ab}(S_{pq})$.

Theorem 6. Suppose φ_1 is the fundamental central dispersion of the first kind of the space $S_{ab}(S_{pq})$. Then $\varphi_1 \in C^{(2)}(j)$.

Proof. The assertion yields from the formula (6).

Theorem 7. Suppose $\varphi_1 = \varphi_1(t)$ is the fundamental central dispersion of the first kind of the space $S_{ab}(S_{pq})$, $\alpha = \alpha(t)$ is the phase of the basis $(y_2, y_1) \in S_{ab}(S_{pq})$. Then the Abel functional equation

$$\alpha[\varphi_1(t)] - \alpha(t) = \varepsilon\pi$$

holds in j, where $\varepsilon = 1$ or $\varepsilon = -1$ according as the phase α increases or decreases in j.

If α increases in j then

$$\varphi_1(t) = \alpha^{-1}[\alpha(t) + \pi]$$

for $t \in j$.

Proof. The theorem is a modification of Theorem 3 for the space $S_{ab}(S_{pq})$.

Summary

The paper deals with a distribution problem of zeros of functions from a strongly regular space of continuous functions of infinite type and bothsided oscillatory. The distribution of zeros is described by a function φ_1 called a fundamental central dispersion of the first kind.

Relations between the fundamental central dispersion of the first kind and the phase α of an ordered pair of functions of a strongly regular space are found. They are expressed by the Abel functional equation

$$\alpha[\varphi_1(t)] - \alpha(t) = \varepsilon\pi .$$

The results obtained are applied to spaces of solutions of second order linear differential equations of general and Sturm forms.

Souhrn

ZÁKLADNÍ CENTRÁLNÍ DISPERSE 1.DRUHU A ABELOVA FUNKČNÍ ROVNICE V SILNĚ REGULÁRNÍM PROSTORU SPOJITÝCH FUNKCÍ

V článku se zabýváme otázkou rozložení nulových bodů funkcí silně regulárního prostoru spojitých funkcí dimenze 2, který je nekonečného typu a oscilatorický. Rozložení nulových bodů je popsáno funkcí φ_1 nazývanou základní centrální disperse 1.druhu.

Jsou nalezeny vztahy mezi základní centrální dispersí 1. druhu a fází α uspořádané dvojice funkcí silně regulárního prostoru, které jsou vyjádřeny Abelovou funkční rovnicí

$$\alpha[\varphi_1(t)] - \alpha(t) = \varepsilon\pi .$$

Výsledky se aplikují v prostorech oboustranně oscilatorických řešení nekonečného typu lineárních diferenciálních rovnic 2.řádu obecného a Sturmova tvaru.

Р е з ю м е

**ОСНОВНАЯ ЦЕНТРАЛЬНАЯ ДИСПЕРСИЯ 1-ого РОДА
И ФУНКЦИОНАЛЬНОЕ УРАВНЕНИЕ АБЕЛЯ
В СИЛЬНО РЕГУЛЯРНОМ ПРОСТРАНСТВЕ НЕПРЕРЫВНЫХ ФУНКЦИЙ**

В работе занимаемся вопросом распределения нулей функций сильно регулярного двумерного пространства непрерывных функций, который есть бесконечного вида и колебающихся. Распределение нулей описано функцией φ_1 называемой основная центральная дисперсия 1-ого рода.

Находятся отношения между основной центральной дисперсией 1-ого рода и фазой α упорядоченной пары функций сильно регулярного пространства, которые выражены функциональным уравнением Абеля

$$\alpha\left[\varphi_1(t)\right] - \alpha(t) = \varepsilon\pi.$$

Результаты применяются к пространствам колебающихся решений бесконечного вида линейных дифференциальных уравнений 2-ого порядка общей и Штурмовой формы.

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