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ON EQUATION y"–q (t) y OF FINITE TYPE, 1–SPECIAL, WITH THE SAME CENTRAL DISPERSION OF THE FIRST KIND

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In the central dispersion theory of linear second order differential equations treated at length by O.Borůvka in [1] there were investigated the properties of bothsided oscillatory equations of the type

$$y'' = q(t)y \qquad (q)$$

having the same fundamental central dispersion of the first kind on their definition interval. By central dispersions of the first kind of (q) we mean thereby the functions mapping the zeros of an arbitrary solution y of (q) onto the zeros of the same solution.

In [3] and [4] there is introduced a certain generalization of concepts relating to central dispersions (q) of finite type $m \geq 2$, 1-special, on an open definition interval $j = (a,b)$, $q(t) \in C^{(0)}(j)$ (hereafter $(q^{(1)})$) by means of definitions of special central dispersions. There are also discussed conditions and properties of such generalizations. The generalization of the 1st kind central dispersions may be summed up in the following definition . The letter Z refers to a set of integers.

Definition 1.

By a (zm+k)-th central dispersion of the first kind for $z \in Z$, $k = 0,1,\dots,m-1$ of the equation $(q^{(1)})$ we mean a function

$$\Phi_{zm+k} = \begin{cases} \varphi_k & \text{for } t \in (a, a_{m-k}) \\ \varphi_{-(m-k)} & \text{for } t \in (a_{m-k}, b) \end{cases}$$

where $\varphi(t)$ is the first kind central dispersion in terms of the definition [1], and the points a_i are the zeros of the 1-fundamental solution of the equation $(q^{(1)})$ forming the 1-fundamental sequence $(a^{(1)})$ of this equation in the following ordering

$$a < a_1 < a_2 < \dots < a_{m-1} < b$$

The following text is devoted to a fuller account of the set of carriers of $(q^{(1)})$ with the same fundamental special central dispersion of the first kind $\Phi(t) = \Phi_1(t)$, i.e. we shall try to find a relation between the carriers $q, \overline{q}$ of $(q^{(1)})$, $(\overline{q}^{(1)})$ for which $\Phi(t) = \overline{\Phi}(t)$ is holding throughout the domain of definition of these functions. Evidently, this equality may be fulfilled only if the equations $(q^{(1)}$, $(\overline{q}^{(1)})$ are 1-special, of type m, $m \geq 2$ on a common definition interval $j = (a,b)$, where $-\infty \leq a < b \leq +\infty$, with the same 1-fundamental sequence $(a^{(1)})$.

In view to the fact that the functions $\Phi_k(t)$ form for $k = 0,1,\dots,m-1$ a finite cyclic group $G^{(1)}$ generated by the element $\Phi(t)$ on $S^{(1)}$, i.e. on the interval j except for the 1-fundamental sequence $(a^{(1)})$, it automatically also follows from the coincidence of the fundamental dispersions $\Phi(t)$ =

$= \overline{\Phi}(t)$ the equality of other dispersions $\Phi_k(t) = \overline{\Phi}_k(t)$ for all k considered. The function $\Phi_0(t)$ is coincident for all equations $(q^{(1)})$ with the same definition interval j. Because of the properties of the elements of the finite group, the converse implication of the coincidence of the fundamental dispersions is not always garanteed due to the coincidence of the higher order dispersions. The equality $\Phi_k(t) = \overline{\Phi}_k(t)$ implies the equality $\Phi(t) = \overline{\Phi}(t)$ only in such a case when each element of the group $G^{(1)}$ is expressible as a power of the element Φ_k, i.e. exactly if k is not a number with a common factor m.

The reply to the question what the magnitude of the set of carriers of equations $(q^{(1)})$ with the same special central dispersion looks like, follows from the statement of the following

Theorem 1

Let s be an arbitrary point of the interval j = (a,b), $m \geq 2$ be a natural number. Let next $\varphi(t)$ be a function defined on the interval (a,s) having the following properties:

$$\begin{aligned} &1) \quad \varphi(t) \in C^{(3)} \\ &2) \quad \varphi'(t) > 0 \\ &3) \quad \varphi(t) > t \qquad\qquad (1) \\ &4) \quad \lim_{t \to s^-} \varphi(t) = b \end{aligned}$$

5) on the interval (s,b) there is defined an inverse m-times composite function $\varphi^{-(m-1)}(t)$, for which $\lim_{t \to s^+} \varphi^{-(m-1)}(t) = a$ holds.

Then there exist to every function $\Phi(t)$ defined by

$$\Phi(t) = \begin{cases} \varphi(t) & \text{for } t \in (a,s) \\ \varphi^{-(m-1)}(t) & \text{for } t \in (s,b) \end{cases}, \qquad (2)$$

infinitely many solutions of the functional equation

$$\alpha[\phi(t)] = \begin{cases} \alpha(t) + \pi \operatorname{sign} \alpha' & \text{for } t \in (a,s) \\ \alpha(t) - (m-1)\pi \operatorname{sign} \alpha' & \text{for } t \in (s,b) \end{cases}, \qquad (3)$$

by means of the phase function α ($\alpha' \neq 0$, $\alpha \in C^{(3)}(j)$), which may be expressed as a 1-st phase of the equation $(q^{(1)})$.

Proof. Let us first assume $m > 2$. We will show that in this case the interval (a,s) may be expressed as a union of the disjunct intervals (a,a_1), $\langle a_1,a_2)$, ..., $\langle a_{m-2},s)$ such that

$$a_1 = \lim_{t \to a^+} \varphi(t), \quad a_2 = \varphi(a_1), \ldots, \quad a_{m-1} = s = \varphi(a_{m-2}),$$

and the function $\varphi^{-(m-1)}(t)$ realizes the schlicht mapping of the interval (s,b) onto (a,a_1).

Because of the properties 1) through 5) of the function $\varphi(t)$ the function $\varphi^{-(m-1)}(t)$ is increasing on the interval (s,b) belonging to the class $C^{(3)}$ and having a positive derivative. Next it holds that the function $\varphi^{-(m-1)}(t) < t$ and realizis the schlicht mapping of (s,b) onto (a,r), where $r = \lim \varphi^{-(m-1)}(t)$ for $t \to b^-$, $a < r < b$.

From the property 5) follows the existence of the limit and the validity of

$$\lim_{t \to a^+} \varphi^{(m-1)}(t) = s ,$$

and thus also the existence of

$$\lim_{t \to a^+} \varphi^{(i)}(t) = a_i \qquad \text{for } i = 1,2,\ldots,m-1 ,$$

where a_i is a real number, $a_{m-1} = s$. From the property 3 thereby follows the following ordering for the points a_i

$$a < a_i < a_{i+1} < b \qquad \text{for } i = 1,2,\ldots,m-1 .$$

From the existence of the limit and from the validity of

$$\lim_{t \to s^+} \varphi^{-(m-1)}(t) = a ,$$

follows also the existence of the limits and the validity of

$$\lim_{t \to s^+} \varphi^{-i} = \lim_{t \to s^+} \varphi^{(m-i-1)}[\varphi^{-(m-1)}] = \lim_{t \to a^+} \varphi^{(m-i-1)} = a_{m-i-1}$$

$i = 1,2,\dots,m-1$, where $a_0 = a$, $\varphi^0(t) = t$.

With respect to the continuity of the function $\varphi(t)$ on the interval (a,s) then follows

$$a_i = \lim_{t \to a^+} \varphi^i = \lim_{t \to a^+} \varphi(\varphi^{(i-1)}) = \lim_{t \to a_{i-1}^+} \varphi(t) = \varphi(a_{i-1})$$

for $i = 2,3,\dots,m-1$. It remains to prove $a_1 = r$.

From the validity of

$$\lim_{t \to a_1^-} \varphi^i(t) = \lim_{t \to a_1^-} \varphi[\varphi^{i-1}(t)] = \lim_{t \to a_1^-} \varphi(t) = a_{i+1}$$

for $i = 1,2,\dots,m-1$, where $a_m = b$, we obtain the equality

$$\lim_{t \to a_1^-} \varphi^{m-1}(t) = b ,$$

whence on account of the fact that

$$\lim_{t \to r^-} \varphi^{m-1}(t) = b$$

we get also the equality $a_1 = r$.

In assuming $m = 2$ we will show that the function $\varphi(t)$ intermediates the schlicht mapping of the interval (a,s) onto (s,b). Denoting again

$$a_1 = \lim_{t \to a^+} \varphi(t) , \qquad r = \lim_{t \to b^-} \varphi^{-1}(t) ,$$

we see that the function $\varphi(t)$ intermediates the schlicht mapping of the interval (a,s) onto (a_1,b), and the function $\varphi^{-1}(t)$ intermediates the schlicht mapping of the interval (s,b) onto (a,r). From this evidently follows $a_1 = s = r$.

The phase function satisfying the functional equation (3) for $m > 2$ may be hereby obtained by the following construction:

$$\alpha(t) = \begin{cases} f(t) & \text{for } t \in \langle a_{m-2}, s) \\ f(a_{m-2}) + \varepsilon\pi & \text{for } t = s \\ \alpha[\Phi(t)] - \varepsilon\pi & \text{for } t \in j_i \text{ pro } i=1,2,\dots,m-2 \\ \alpha[\Phi(t)] + \varepsilon(m-1)\pi & \text{for } t \in (s,b) \end{cases} \tag{4}$$

where $j_i = \langle a_{i-1}, a_i)$ for $i = 2,3,\dots,m-2$, $j_1 = (a,a_1)$, $f(t)$ is an arbitrary function defined on $\langle a_{m-2}, s)$, being everywhere increasing or everywhere decreasing on this interval, $f(t) \in C^{(3)} \langle a_{m-2}, s)$ with the derivative $f'(t) \neq 0$, where $\varepsilon = \operatorname{sign} f'(t)$, which in the left neighbourhood of the point s satisfies the properties

$$\lim_{t \to s^-} f(t) = f(a_{m-2}) + \varepsilon\pi ,$$

$$\lim_{t \to s^-} f'(t) = \frac{f'^{+}(a_{m-2})}{\Phi'(a_{m-2})} ,$$

$$\lim_{t \to s^-} f''(t) = \frac{1}{\Phi'(a_{m-2})} \cdot \left(\frac{f'^{+}(t)}{\Phi'(t)} \right)'^{+}_{a_{m-2}} ,$$

$$\lim_{t \to s^-} f'''(t) = \frac{1}{\Phi'(a_{m-2})} \cdot \left[\frac{1}{\Phi'(t)} \cdot \left(\frac{f'^{+}(t)}{\Phi'(t)} \right)'^{+} \right]'^{+}_{a_{m-2}} ,$$

In the case when $m = 2$ we have $a_{m-2} = a$. We thus need to consider the function $f(t)$ having the above properties on the interval (a,s) only, and instead of the functional values f(a), $\Phi'(a)$, we need to consider the limit of these functions at the point a on the right.

The phase function α obtained by the above construction (4) really represents the first phase of the equation $(q^{(1)})$ since the oscillation of the phase $O(\alpha)$ satisfies the required relation

$$O(\alpha) = \left|\lim_{t\to b^-} \alpha(t) - \lim_{t\to a^+} \alpha(t)\right| = \left|\lim_{t\to s^-} \alpha[\Phi(t)] - \right.$$

$$- \lim_{t\to s^+} \alpha[\Phi(t)] = \left|\lim_{t\to s^-} \alpha(t) + \varepsilon\pi - \lim_{t\to s^+} \alpha(t) - \right.$$

$$\left. + \varepsilon(m-1)\pi\right| = |\varepsilon m\pi| = m\pi .$$

It follows from the proof of Theorem 1 and from the properties of the central dispersions in terms of the definition stated in [1] that the function $\Phi(t)$ defined by (2) with the aid of the function $\varphi(t)$ with the properties 1) through 5)

(1) corresponds to the definition of the fundamental special dispersion of the first kind Φ relative to the equation $(q^{(1)})$ with the 1-fundamental sequence $a_1, a_2, \dots, a_{m-1} = s$. On account of the fact that through an arbitrary phase of $(q^{(1)})$ its carrier q is uniquely determined by

$$- \{\alpha, t\} - \alpha'^2(t) = q(t),$$

where $\{\alpha, t\}$ means the Schwarzian derivative of the function α at the point $t \in j$ in terms of the definition stated in [1], it becomes apparent that there exist infinitely many equations $(q^{(1)})$ with the domain of definition j, for with the function $\Phi(t)$ satisfying the assumptions of Theorem 1 represents the fundamental special central dispersion of the first kind.

Let us look now at the fundamental properties of the function $\Phi(t)$ defined by (2) with the aid of the function $\varphi(t)$ possessing the properties 1) trough 5) (1). There are the following properties fulfilled:

1) the domain of definition of the function $\Phi(t)$ forms $(a,s) \cup (s,b)$

2) the range of values of the function $\Phi(t)$ forms $(a,a_1) \cup (a_1,b)$

3) $\Phi(t)$ is on the intervals (a,s), (s,b) an increasing function belonging to the class $C^{(3)}$, with a derivative $\Phi'(t) \neq 0$

4) $\lim_{t \to s^-} \Phi(t) = b \qquad \lim_{t \to s^+} \Phi(t) = a$ (5)

5) the function $\Phi(t)$ intermediates the schlicht mapping of the intervals

(a_{i-1}, a_i) onto (a_i, a_{i+1}) for $i = 1,2,\dots,m-1$

(a_{m-1}, b) onto (a, a_1) ,

where

$$a = a_0 < a_1 = \lim_{t \to a^+} \Phi(t) < a_2 = \Phi(a_1) < \dots a_{m-1} = s =$$
$$= \Phi(a_{m-2}) < b = a_m$$

6) $\Phi^{m-1}\Phi(t) = t$ for all $t \in (s,b)$

The above properties are likewise sufficient properties that the function $\Phi(t)$ could be taken for a fundamental special central dispersion of the first kind of $(q^{(1)})$ as evidenced by the following

Corollary 1.

Suppose $a_1, a_2, \dots, a_{m-1}$ is an arbitrary point sequence of $j = (a,b)$ satisfying the following ordering

$$a < a_1 < a_2 < \dots < a_{m-1} < b \quad .$$

Then to every function $\Phi(t)$ satisfying on j the properties 1) through 6) of (5) there exist infinitely many solutions of the functional equation (3) through the phase function $\alpha(\alpha' \neq 0, \alpha \in C^{(3)})$, which may be interpreted as the first phase of the equation $(q^{(1)})$ on the interval j. Consequently, there exist infinitely many equations of type $(q^{(1)})$ on the interval j, for which the function $\Phi(t)$ represents the fundamental special central dispersion of the first kind.

Proof. Denote now

$$\Phi(t) = \begin{cases} \varphi(t) & \text{for} \quad t \in (a, a_{m-1}) \\ \varphi_{-(m-1)}(t) & \text{for} \quad t \in (a_{m-1}, b) \end{cases},$$

and show that the function $\Phi(t)$ satisfies the assumptions of Theorem 1, i.e. that the function $\varphi(t)$ possesses on the interval $(a, a_{m-1} = s)$ the properties 1) through 4) (1) and that $\varphi_{-(m-1)}(t) = \varphi^{-(m-1)}(t)$ holds on the interval (s,b) under the validity of $\lim \varphi_{-(m-1)}(t) = a$ for $t \to s^+$. However, the validity follows from a more comparison. Besides, there is a preassigned possibility of the appropriate separation of j.

Consider now a function $\Phi(t)$ satisfying the assumptions of Theorem 1. The symbol Q_Φ will denote a set of all carriers of equations $(q^{(1)})$ with the same fundamental special central dispersion equal to the function $\Phi(t)$. It follows from our foregoing consideration that the set Q_Φ is of the continuum magnitude.

The symbol I will denote a set of all solutions of these equations. We know that arbitrary solutions $y, \overline{y} \in I$ having one zero in common, have all their zeros in common. It also follows herefrom that all equations $(q^{(1)})$ for $q \in Q_\Phi$ have the same 1-fundamental sequence $(a^{(1)})$ coinciding with the sequence given at 5) (5). Consider next an arbitrary point $c \in j$. By a bundle of solutions I_c of the set Q_Φ with a node c we mean a set $I_c \subset I$ of all solutions of $(q^{(1)})$ for $q \in Q_\Phi$ having a zero at c. We know that all elements of the set I_c have all their zeros in common, which we will call the nodes of the bundle I_c. The set I_c is thus uniquely determined by its arbitrary.

In the sequel we will investigate the properties of the elements of the set I_c belonging to the set of all carriers

Q_Φ . We will derive these properties by means of results obtained in three lemmas below.

Lemma 1.

Consider a differential equation (q) on $j = (a,b)$, $q \in C^{(0)}(j)$ such that every its solution has at least $m \geq 2$ zeros in j. Let y be an arbitrary solution of this equation and the points $c_1, c_2, \dots, c_m$ be consecutive zeros of the solution y. Let the point x_i for $i = 0,1,\dots,m$ be arbitrary points from j, for which $y(x_i) \neq 0$, satisfying the following ordering

$$x_0 < c_1 < x_1 < c_2 < x_2 < \dots < c_{m-1} < x_{m-1} < c_m < x_m \ . \tag{6}$$

Let $g_m(t)$ be a function defined on the interval $\langle x_0, x_m \rangle$ except for the points $c_1, c_2, \dots, c_m$ as

$$g_m(t) = \frac{1}{y^2(t)} - \sum_{i=1}^{m} \frac{1}{y'^2(c_i)} \cdot \frac{1}{(t-c_i)^2} \ . \tag{7}$$

Then there exist Riemann integrals of this function

$$\int_{x_0}^{x_m} g_m(t)\, dt, \qquad \int_{c_1}^{c_m} g_m(t)\, dt$$

and it holds

$$\int_{x_0}^{x_m} g_m(t)dt = \sum_{i=1}^{m} \frac{1}{y'^2(c_i)} \left[-\operatorname{cotg} \alpha_i(x_i) + \operatorname{cotg} \alpha_i(x_{i-1}) \right] + \sum_{i=1}^{m} \frac{1}{y'^2(c_i)} \left[\frac{1}{c_i - x_0} + \frac{1}{x_m - c_i} \right] , \tag{8}$$

$$\int_{c_1}^{c_m} g_m(t)dt = \sum_{i=1}^{m-1} \frac{1}{y'^2(c_i)} \left[-\operatorname{cotg} \alpha_i(x_i) + \frac{1}{c_m - c_i} \right] + \sum_{i=1}^{m-1} \frac{1}{y'^2(c_{i+1})} \left[\operatorname{cotg} \alpha_{i+1}(x_i) - \frac{1}{c_1 - c_{i+1}} \right], \tag{9}$$

where α is the first phase of (q) determined by initial conditions

$$\alpha_i(c_i) = 0, \quad \alpha_i'(c_i) = 1, \quad \alpha_i''(c_i) = 0 . \tag{10}$$

Proof. The existence of the integral $\int_{x_0}^{x_m} g_m(t)dt$ and the validity of (8) is proved in 18 § 5 in [1]. It thus remains to prove the validity of (9). We proced hereby from the limit formulas derived in 18 § 5 [1]:

$$y'^2(c_1)\int_{x_0}^{c_1} \left[\frac{1}{y^2(t)} - \frac{1}{y'^2(c_1)} \cdot \frac{1}{(t-c_1)^2}\right] dt = \operatorname{cotg} \alpha(x_0) - \frac{1}{x_0-c_1} \tag{11}$$

$$y'^2(c_1)\int_{c_1}^{x_1} \left[\frac{1}{y^2(t)} - \frac{1}{y'^2(c_1)} \cdot \frac{1}{(t-c_1)^2}\right] dt = -\operatorname{cotg} \alpha(x_1) + \frac{1}{x_1-c_1} \tag{12}$$

where α is the first phase of (q) determinde by initial conditions

$$\alpha(c_1) = 0, \quad \alpha'(c_1) = 1, \quad \alpha''(c_1) = 0 .$$

The validity of (9) will be proved by an inductive method.

1) first we derive the value of the integral $\int_{c_1}^{c_2} g_2(t)dt$, where

$$g_2(t) = \frac{1}{y^2(t)} - \frac{1}{y'^2(c_1)\,(t-c_1)^2} - \frac{1}{y'^2(c_2)\,(t-c_2)^2} \quad .$$

Relations (11), (12) yield

$$\int_{c_1}^{x_1} g_2(t)dt = \frac{1}{y'^2(c_1)}\left[-\operatorname{cotg}\alpha_1(x_1) + \frac{1}{x_1-c_1}\right] + \frac{1}{y'^2(c_2)}\left[\frac{1}{x_1-c_2} - \frac{1}{c_1-c_2}\right],$$

$$\int_{x_1}^{c_2} g_2(t)dt = \frac{1}{y'^2(c_2)}\left[\operatorname{cotg}\alpha_2(x_1) - \frac{1}{x_1-c_2}\right] + \frac{1}{y'^2(c_1)}\left[\frac{1}{c_2-c_1} - \frac{1}{x_1-c_1}\right],$$

where α_1, α_2 are the first phases of (q) satisfying the initial conditions of (10).

From this directly follows the validity of

$$\int_{c_1}^{c_2} g_2(t)dt = \frac{1}{y'^2(c_1)}\left[-\operatorname{cotg}\alpha_1(x_1) + \frac{1}{c_2-c_1}\right] + \frac{1}{y'^2(c_2)}\left[\operatorname{cotg}\alpha_2(x_1) - \frac{1}{c_1-c_2}\right],$$

and thus also the validity of (9) for $m = 2$.

2) Assume the validity of (9) for the integral $\int_{c_1}^{c_{m-1}} g_{m-1}(t)dt$

and prove it for the integral $\int_{c_1}^{c_m} g_m(t)dt$.

It holds

$$\int_{c_1}^{c_m} g_m(t)dt = \int_{c_1}^{c_{m-1}} g_m(t)dt + \int_{c_{m-1}}^{x_{m-1}} g_m(t)dt + \int_{x_{m-1}}^{c_m} g_m(t)dt,$$

from which

$$\int_{c_1}^{c_{m-1}} g_m(t)dt = \sum_{i=1}^{m-2} \frac{1}{y'^2(c_i)} \left[- \operatorname{cotg}\alpha_i(x_i) + \frac{1}{c_{m-1}-c_i} \right] + \sum_{i=1}^{m-2} \frac{1}{y'^2(c_{i+1})} \left[\operatorname{cotg}\alpha_{i+1}(x_i) - \frac{1}{c_1-c_{i+1}} \right] + \frac{1}{y'^2(c_m)} \left[\frac{1}{c_{m-1}-c_m} - \frac{1}{c_1-c_m} \right] .$$

Applying (11) and (12) we obtain

$$\int_{c_{m-1}}^{x_{m-1}} g_m(t)dt = \frac{1}{y'^2(c_{m-1})} \left[- \operatorname{cotg}\alpha_{m-1}(x_{m-1}) + \frac{1}{x_{m-1}-c_{m-1}} \right] + \sum_{\substack{i=1 \\ i \neq m-1}}^{m} \frac{1}{y'^2(c_i)} \left[\frac{1}{x_{m-1}-c_i} - \frac{1}{c_{m-1}-c_i} \right] ,$$

$$\int_{x_{m-1}}^{c_m} g_m(t)dt = \frac{1}{y'^2(c_m)} \left[\operatorname{cotg}\alpha_m(x_{m-1}) - \frac{1}{x_{m-1}-c_m} \right] + \sum_{i=1}^{m-1} \frac{1}{y'^2(c_i)} \left[\frac{1}{c_m-c_i} - \frac{1}{x_{m-1}-c_i} \right]$$

from which we conclude

$$\int_{c_1}^{c_m} g_m(t)dt = \sum_{i=1}^{m-1} \frac{1}{y'^2(c_i)} \left[- \operatorname{cotg} \alpha_i(x_i) + \frac{1}{c_m-c_i}\right] + $$
$$+ \sum_{i=1}^{m-1} \frac{1}{y'^2(c_{i+1})} \left[\operatorname{cotg} \alpha_{i+1}(x_i) - \frac{1}{c_1-c_{i+1}}\right] ,$$

which was to be demonstrated.

Lemma 2.

Consider a differential equation (q) on $j = (a,b)$, $q \in C^{(0)}(j)$, such that each its solution has at least m zeros $m \geq 2$ in j. Let $\varphi_k(t)$ for $k = 0,\pm1,\ldots,\pm m-1$ be the k-th central dispersion of the first kind in terms of the definition stated in [1]. Let y be an arbitrary solution of (q), x be an arbitrary point at the interval j such that $y(x) \neq 0$ and let, respectively, to the right and to the left of it lie at least in case 1) k zeros, in case 2) $|k|+1$ zeros of the solution y. Let c be a zero of the solution y satisfying the inequality $x < c < \varphi(x)$ and $\varphi_{-1}(x) < c < x$, respetively. Define the function

$$f_k(t) = \frac{y'^2(c)}{y^2(t)} - \sum_{i=0}^{k} \frac{\varphi_i'(c)}{(t-\varphi_i(c))^2} \tag{13}$$

on the interval j except for the zeros of the solution y for $k = 1,2,\ldots,m-1$ or $k = -1,-2,\ldots,-(m-1)$. Then there exist Riemann integrals

$$1) \quad \int_{x}^{\varphi_k(x)} f_{k-\varepsilon}(t)dt , \tag{14}$$

$$2) \quad \int_{c}^{\varphi_k(c)} f_k(t)dt , \tag{15}$$

where ε = sign k and it holds

1) $$\int_x^{\varphi_k(x)} f_{k-\varepsilon}(t)\,dt = \sum_{i=0}^{k-\varepsilon} \varphi_i'(c)\left[\frac{1}{\varphi_i(c)-x} - \frac{1}{\varphi_i(c)-\varphi_k(x)}\right], \quad (16)$$

2) $$\int_c^{\varphi_k(c)} f_k(t)\,dt = -\frac{1}{2}\,\frac{\varphi_k''(c)}{\varphi_k'(c)} + \sum_{i=0}^{k-\varepsilon} \frac{\varphi_i'(c)}{\varphi_k(c)-\varphi_i(c)} - \frac{\varphi_{i+\varepsilon}'(c)}{\varphi_0(c)-\varphi_{i+\varepsilon}(c)} \,. \quad (17)$$

Proof. 1) To derive (16), we proceed from (8). With respect to the assumptions given, there are satisfied the conditions of the preceding Lemma for $x_0 = x$, $x_k = \varphi_k(x)$, $c_1 = c$ at sign $k = 1$ or $x_k = x$, $x_0 = \varphi_k(x)$, $c_k = c$ at sign $k = -1$.

Thus, for positive, with the following ordering of points

$$x < c < \varphi(x) < \varphi(c) < \dots < \varphi_k(x)$$

there holds

$$\int_x^{\varphi_k(x)} \left[\frac{1}{y^2(t)} - \sum_{i=0}^{k-1} \frac{1}{y'^2(\varphi_i(c))(t-\varphi_i(c))^2}\right] dt =$$

$$= \sum_{i=0}^{k-1} \frac{1}{y'^2(\varphi_i(c))}\left[\frac{1}{\varphi_i(c)-x} - \frac{1}{\varphi_i(c)-\varphi_k(x)}\right] +$$

$$+ \sum_{i=0}^{k-1} \frac{1}{y'^2(\varphi_i(c))}\left[-\operatorname{cotg}\alpha_i(\varphi_{i+1}(x)) + \right.$$

$$\left. + \operatorname{cotg}\alpha_i(\varphi_i(x))\right] , \quad (18)$$

where α_i is the first phase of (q) satisfying the initial conditions

$$\alpha_i(\varphi_i(c)) = 0, \quad \alpha_i'(\varphi_i(c)) = 1, \quad \alpha_i''(\varphi_i(c)) = 0 \,. \quad (19)$$

From this making use of (5) from 3 § 13 [1] to express the derivatives of the central dispersions at the zero c of, the solution y

$$\varphi_k'(c) = \frac{y'^2(c)}{y'^2(\varphi_k(c))} \tag{20}$$

and also the fact that the points $\varphi_{i+1}(x)$, $\varphi_i(x)$ are always 1-conjugate, we come directly to (16).

For a negative value of k the sequence

$$\varphi_k(x) < \varphi_{k+1}(c) < \varphi_{k+1}(x) < \dots < \varphi_{-1}(c) < \varphi_{-1}(x) < c < x$$

corresponds to the assumptions of the foregoing Theorem. Consequently

$$\int_{\varphi_k(x)}^{x} \left[\frac{1}{y^2(t)} - \sum_{i=k+1}^{0} \frac{1}{y'^2(\varphi_i(c))(t-\varphi_i(c))^2}\right] dt =$$

$$= \sum_{i=k+1}^{0} \frac{1}{y'^2(\varphi_i(c))}\left[\frac{1}{\varphi_i(c) - \varphi_k(x)} - \frac{1}{\varphi_i(c) - x}\right]. \tag{21}$$

Applying (20) and on multilying the equation by -1 we come with the value $\varepsilon = -1$ to the equality (16).

2) Similarly, in proving (17) we proceed from (9).
For k being positive, we may express (9) as

$$\int_{c}^{\varphi_k(c)} \left[\frac{1}{y^2(t)} - \sum_{i=0}^{k} \frac{1}{y'^2(\varphi_i(c))(t-\varphi_i(c))^2}\right] dt =$$

$$= \sum_{i=0}^{k-1} \frac{1}{y'^2(\varphi_i(c))}\left[- \operatorname{cotg} \alpha_i(\varphi_{i+1}(x)) + \right.$$

$$+ \frac{1}{\varphi_k(c) - \varphi_i(c)} \Big] +$$

$$+ \sum_{i=0}^{k-1} \frac{1}{y'^2(\varphi_{i+1}(c))} \Big[\operatorname{cotg} \alpha_{i+1}(\varphi_{i+1}(x)) -$$

$$- \frac{1}{\varphi_0(c) - \varphi_{i+1}(c)} \Big] \ .$$

Using (20) to express $\varphi_i'(c)$ we deduce the relation

$$\int_c^{\varphi_k(c)} \Big[\frac{y'^2(c)}{y^2(t)} - \sum_{i=0}^{k} \frac{\varphi_i'(c)}{(t-\varphi_i(c))^2} \Big] dt =$$

$$= \sum_{i=0}^{k-1} \varphi_i'(c) \Big[- \operatorname{cotg} \alpha_i(\varphi_{i+1}(x)) +$$

$$+ \frac{1}{\varphi_k(c) - \varphi_i(c)} \Big] +$$

$$+ \sum_{i=0}^{k-1} \varphi_{i+1}'(c) \Big[\operatorname{cotg} \alpha_{i+1}(\varphi_{i+1}(x)) -$$

$$- \frac{1}{\varphi_0(c) - \varphi_{i+1}(c)} \Big] \ ,$$

which, in expressing $\varphi_{i+1}'(t)$ in the form of derivative of a composite function at the point t = c, goes over into the form

$$\int_c^{\varphi_k(c)} f_k(t)dt = \sum_{i=0}^{k-1} \varphi_i'(c) \Big[- \operatorname{cotg} \alpha_i(\varphi_{i+1}(x)) +$$

$$+ \varphi'(\varphi_i(c)) \operatorname{cotg} \alpha_{i+1}(\varphi_{i+1}(x)) \Big] +$$

$$+ \sum_{i=0}^{k-1} \Big[\frac{\varphi_i'(c)}{\varphi_k(c)-\varphi_i(c)} - \frac{\varphi_{i+1}'(c)}{\varphi_0(c)-\varphi_{i+1}(c)} \Big] \quad (22)$$

where α_i is the first phase of (q) satisfying the initial conditions (19). From this and from the Abel functional equation $\alpha_{i+1}(\varphi_{i+1}) = \alpha_{i+1}(\varphi_i) + \pi$ we see that

$$\alpha_{i+1}(\varphi_i(c)) = -\pi, \quad \alpha'_{i+1}(\varphi_i(c)) = \varphi'(\varphi_i(c)),$$
$$\alpha''_{i+1}(\varphi_i(c)) = \varphi''(\varphi_i(c)) \tag{23}$$

Hence, by relation (4) § 7 [1] for the expression of the first phase of (q) determined by the initial conditions, there follows from (19) and (23) for the phases α_i, α_{i+1}, where $i \in \{0,1,\dots$ $\dots,k-1\}$, and for all $t \in j$ the following relation

$$\operatorname{tg} \alpha_{i+1}(t) = \frac{-\varphi'(\varphi_i(c)) \operatorname{tg} \alpha_i(t)}{-1 + \frac{1}{2} \frac{\varphi''(\varphi_i(c))}{\varphi'(\varphi_i(c))} \operatorname{tg} \alpha_i(t)} ,$$

which may by rearranged into the form

$$-\operatorname{cotg} \alpha_i(t) + \varphi'(\varphi_i(c)) \operatorname{cotg} \alpha_{i+1}(t) = -\frac{1}{2} \frac{\varphi''(\varphi_i(c))}{\varphi'(\varphi_i(c))} . \tag{24}$$

Applying (24) in (22) and expressing the quotient as

$$\frac{\varphi'_i(t)\ \varphi''(\varphi_i(t))}{\varphi'(\varphi_i(t))} = \frac{\varphi''_{i+1}(t)}{\varphi'_{i+1}(t)} - \frac{\varphi''_i(t)}{\varphi'_i(t)}$$

we come to the validity of (17) proved.

For k being negative, the sequence

$$\varphi_k(c) < \varphi_k(x) < \varphi_{k+1}(c) < \varphi_{k+1}(x) < \dots < \varphi_{-1}(c) < \varphi_{-1}(x) < c < x$$

satisfies the assumptions of the preceding Lemma. Hence, it holds

$$\int_{\varphi_k(c)}^{c}\left[\frac{1}{y^2(t)} - \sum_{i=k}^{0}\frac{1}{y'^2(\varphi_i(c))(t-\varphi_i(c))^2}\,dt\right] =$$

$$= \sum_{i=k+1}^{0}\frac{1}{y'^2(\varphi_{i-1}(c))}\left[-\operatorname{cotg}\alpha_{i-1}(\varphi_{i-1}(x)) + \right.$$

$$\left. + \frac{1}{\varphi_0(c) - \varphi_{i-1}(c)}\right] +$$

$$+ \sum_{i=k+1}^{0}\frac{1}{y'^2(\varphi_i(c))}\left[\operatorname{cotg}\alpha_i(\varphi_{i-1}(x)) - \right.$$

$$\left. - \frac{1}{\varphi_k(c) - \varphi_i(c)}\right] ,$$

whence, applying (20) and expressing $\varphi'_{i-1}(t)$ in the form of a derivative of a composite function at the point c, we obtain

$$\int_{\varphi_k(c)}^{c} f_k(t)dt = -\sum_{i=0}^{k+1}\varphi'_i(c)\left[-\operatorname{cotg}\alpha_i(\varphi_{i-1}(x)) + \right.$$

$$\left. + \varphi'_{-1}(\varphi_i(c))\operatorname{cotg}\alpha_{i-1}(\varphi_{i-1}(x)) - \right.$$

$$- \sum_{i=0}^{k+1}\left[\frac{\varphi'_i(c)}{\varphi_k(c)-\varphi_i(c)} - \frac{\varphi'_{i-1}(c)}{\varphi_0(c)-\varphi_{i-1}(c)}\right] ,$$

(25)

where α_i is the first phase of (q) determined by the initial conditions (19). From the above conditions and applying the Abel functional equation we obtain

$$\alpha_{i-1}(\varphi_i(c)) = \pi , \quad \alpha'_{i-1}(\varphi_i(c)) = \varphi'_{-1}(\varphi_i(c)),$$

$$\alpha''_{i-1}(\varphi_i(c)) = \varphi''_{-1}(\varphi_i(c)) \qquad (26)$$

By relation (4) § 7 [1] for the expression of the first phase of (q) determined by the initial conditions, there follows from (19) and (26) for the phases α_i, α_{i-1}, where $i \in \{0,-1,\dots,-(k-1)\}$, $t \in j$, the following relation

$$\operatorname{tg}\alpha_{i-1}(t) = \frac{-\varphi'_{-1}(\varphi_i(c))\ \operatorname{tg}\alpha_i(t)}{-1 + \frac{1}{2}\frac{\varphi''_{-1}(\varphi_i(c))}{\varphi'_{-1}(\varphi_i(c))}\ \operatorname{tg}\alpha_i(t)} \ .$$

This relation rearranged into the form

$$-\operatorname{cotg}\alpha_i(t) + \varphi'_{-1}(\varphi_i(c))\ \operatorname{cotg}\alpha_{i-1}(t) = $$

$$= -\frac{1}{2}\frac{\varphi''_{-1}(\varphi_i(c))}{\varphi'_{-1}(\varphi_i(c))}\ , \qquad (27)$$

with afterwards expression of the quotient

$$\frac{\varphi'_i(t)\ \varphi''_{-1}(\varphi_i(t))}{\varphi'_{-1}(\varphi_i(t))} = \frac{\varphi''_{i-1}(t)}{\varphi'_{i-1}(t)} - \frac{\varphi''_i(t)}{\varphi'_i(t)}\ ,$$

will be used to the final formation of (25). So, we come to the expression

$$\int_{\varphi_k(c)}^{c} f_k(t)dt = \frac{1}{2}\frac{\varphi''_k(c)}{\varphi'_k(c)} - \sum_{i=0}^{k+1}\Big[\frac{\varphi'_i(c)}{\varphi_k(c) - \varphi_i(c)} - \frac{\varphi'_{i-1}(c)}{\varphi_0(c) - \varphi_{i-1}(c)}\Big]\ ,$$

which on multiplying by -1 for $\varepsilon = -1$ proves the validity of (17).

<u>Lemma 3</u>.

Consider the equation $(q^{(1)})$ of finite type $m \geq 2$, 1-special on the interval $j = (a,b)$. Let $\Phi_i(t)$ be the i-th special central dispersion of $(q^{(1)})$, for $i = 0,1,\dots,m-1$. Let y be an arbitrary solution of this equation, c be its zero. Let x be an arbitrary point in the interval (a, a_{m-k}) such that $x < c < \Phi(x)$ or an arbitrary point in the interval (a_{m-k}, b) such that $\Phi_{m-1}(x) < c < x$. Then there exist Riemann integrals on the left sides of formulas (28) and (29), for $c \neq a_{m-k}$ formulas (30) and (31) and there is fulfilled the validity of

$$\int_x^{\Phi_k(x)} \left[\frac{y'^2(c)}{y^2(t)} - \sum_{i=0}^{k-1} \frac{\Phi_i'(c)}{(t - \Phi_i(c))^2} \right] dt =$$

$$= \sum_{i=0}^{k-1} \Phi_i'(c) \left[\frac{1}{\Phi_k(x) - \Phi_i(c)} - \frac{1}{\Phi_0(x) - \Phi_i(c)} \right] \tag{28}$$

$$\text{for } x \in (a, a_{m-k}),$$

resp.

$$\int_x^{\Phi_k(x)} \left[\frac{y'^2(c)}{y^2(t)} - \sum_{i=k+1}^{m} \frac{\Phi_i'(c)}{(t - \Phi_i(c))^2} \right] dt =$$

$$= \sum_{i=k+1}^{m} \Phi_i'(c) \left[\frac{1}{\Phi_k(x) - \Phi_i(c)} - \frac{1}{\Phi_m(x) - \Phi_i(c)} \right] \tag{29}$$

$$\text{for } x \in (a_{m-k}, b),$$

$$\int_c^{\Phi_k(c)} \left[\frac{y'^2(c)}{y^2(t)} - \sum_{i=0}^{k} \frac{\Phi_i'(c)}{(t - \Phi_i(c))^2} \right] dt =$$

$$= -\frac{1}{2} \frac{\Phi_k''(c)}{\Phi_k'(c)} + \sum_{i=0}^{k-1} \left[\frac{\Phi_i'(c)}{\Phi_k(c) - \Phi_i(c)} - \right.$$

$$\left. - \frac{\Phi_{i+1}'(c)}{\Phi_0(c) - \Phi_{i+1}(c)} \right] \tag{30}$$

$$\text{for } c \in (a, a_{m-k}),$$

$$\int_{c}^{\Phi_k(c)} \left[\frac{y'^2(c)}{y^2(t)} - \sum_{i=k}^{m} \frac{\Phi_i'(c)}{(t-\Phi_i(c))^2} \right] dt =$$

$$= -\frac{1}{2}\frac{\Phi_k''(c)}{\Phi_k'(c)} + \sum_{i=k+1}^{m} \left[\frac{\Phi_i'(c)}{\Phi_k(c) - \Phi_i(c)} - \right.$$

$$\left. - \frac{\Phi_{i-1}'(c)}{\Phi_m(c) - \Phi_{i-1}(c)} \right] \qquad (31)$$

$$\text{for } c \in (a_{m-k}, b)$$

where $\Phi_m = \Phi_0 = t$ for all $t \in j$.

Proof. For every equation of type $(q^{(1)})$, for its arbitrary solution y and for the points x,c of the mentioned properties there are fulfilled the assumptions of Lemma 2, i.e. it holds:

1) For all $x \in (a, a_{m-k})$, $c \in (x, \phi(x))$ such that $\varphi(x) \neq 0$, $y(c) = 0$ there always exist in the interval j the points $x_i = \varphi_i(x)$ for $i = 1,2,\dots,k$ and the points $c_i = \varphi_i(c)$ for $i = 1, 2,\dots,k-1$, where the function $\varphi_i(t)$ is the i-th central dispersion in terms of the definition stated in [1]. There are thus fulfilled the assumptions of Lemma 2 and the Riemann integral $\int_x^{\varphi_k(x)} f_{k-1}(t)dt$ exist for $x \in (a, a_{m-k})$, $k = 1,2,\dots,m-1$, whose value is given by (16). With respect to the definition of the special central dispersions there is satisfied $\varphi_i(x) = \Phi_i(x)$ for $i = 1,2,\dots,k$, $\varphi_i(c) = \Phi_i(c)$ for $i = 1,2,\dots,k-1$, at the points x and c, from which clearly follows the validity of (28).

2) For all $x \in (a_{m-k}, b)$, $c \in (\Phi_{m-1}(x), x)$ such that $y(x) \neq 0$, $y(c) = 0$ there always exist in j the points $x_i = \varphi_{-i}(x)$ for $i = 1,2,\dots,k$ and the point $c_i = \varphi_{-i}(c)$ for $i = 1,2,\dots,k-1$, where $\varphi_{-i}(t)$ is the -i-th central dispersion in terms of the definition stated in [1]. So, with respect to the validity $\Phi_{m-1}(x) = \Phi_{-1}(x) = \varphi_{-1}(x)$ there are satisfied the assumptions

of Lemma 2 and the Riemann integral $\int_x^{\varphi_{-k}(x)} f_{-k+1}(t)dt$ exists for $x \in (a_{m-k}, b)$, $k = 1,2,\ldots,m-1$, whose value is given by (16). With respect to the definition of the special central dispersions and to the fact that they form a finite cyclic group of order m, there is fulfilled $\varphi_{-i}(x) = \Phi_{m-i}(x)$ for $i = 1,2,\ldots,k$, and $\varphi_{-i}(c) = \Phi_{m-i}(c)$ for $i = 1,2,\ldots,k-1$ at the points x and c, from which, with the equality $\Phi_0(t) = \Phi_m(t) = t$ for $t \in j$, there follows also the validity of (29).

3) For all $c \in (a, a_{m-k})$, $x \in j$ such that $y(c) = 0$, $y(x) \neq 0$, $x < c < \Phi(x)$ there always exist in j the points $x_i = \varphi_i(x)$, $c_i = \varphi_i(c)$, for $i = 1,2,\ldots,k$, where $\varphi_i(t)$ is the i-th central dispersion in terms of the definition stated in [1]. So, the assumptions of Lemma 2 are fulfilled and the Riemann integral $\int_c^{\varphi_k(c)} f_k(t)dt$ exists for $c \in (a, a_{m-k})$, $k = 1,2,\ldots,m-1$, whose value is given by (17). With respect to the validity of $\varphi_i(x) = \Phi_i(x)$, $\varphi_i(c) = \Phi_i(c)$ for $i = 1,2,\ldots,k$ there clearly follows also the validity of (30).

4) For all $x \in j$, $c \in (a_{m-k}, b)$ such that $y(x) \neq 0$, $y(c) = 0$, $\Phi_{m-1}(x) < c < x$ there always exist in j points $x_i = \varphi_i(x)$, $c_i = \varphi_{-i}(c)$ for $i = 1,2,\ldots,k$, where $\varphi_{-i}(t)$ is the -i-th central dispersion in terms of the definition stated in [1]. So, with respect to the validity of $\Phi_{m-1}(x) = \Phi_{-1}(x) = \varphi_{-1}(x)$ there are fulfilled the assumptions of Lemma 2 and the Riemann integral $\int_c^{\varphi_{-k}(c)} f_k(t)dt$ exists for $c \in (a_{m-k}, b)$, $k = 1,2,\ldots,m-1$, whose value is given by (17). With respect to the validity of $\varphi_{-i}(x) = \Phi_{m-i}(x)$, $\varphi_{-i}(c) = \Phi_{m-i}(c)$ for $i = 1,2,\ldots,k$ with the equality $\Phi_0(t) = \Phi_m(t) = t$ for $t \in j$, there also follows the validity of (31).

Let us now return to the set Q_Φ of all carriers of the equations $(q^{(1)})$ with the same fundamental special central dispersion of the first kind $\Phi(t)$ and to the bundle I_c of all solutions y of these equations, having the zero c in common.

On the basis of the validity of the foregoing three Lemmas we are able to express a Theorem collecting together the common properties of elements from the bundle of integrals I_c.

Theorem 2

1) For all elements y of the bundle I_c belonging to the set Q_{Φ} there exist Riemann integrals stated on the left sides of formulas (28), (29), (30), (31) whose values are invariant with respect to the elements of the bundle I_c.

2) The quotient of derivatives $y'/\bar{y}'$ of two arbitrary elements $y, \bar{y} \in I_c$ is at all modes of the bundle equal to the same constant λ.

3) For any two elements $y, \bar{y} \in I_c$ and for all $t \in j$, $t \neq a_{m-k}$ there is fulfilled the relation

$$\int_t^{\Phi_k(t)} \left[\frac{y'^2(c)}{y^2(\tau)} - \frac{\bar{y}'^2(c)}{\bar{y}^2(\tau)} \right] d\tau = 0 \qquad (32)$$

Proof.

1) The first part of this statement immediately follows from (28), (29), (30), (31), of Lemma 3. The right sides of these relations do not depend on a concrete element $y \in I_c$ but merely on the values of here presented special central dispersions and their derivatives, whereby no dispersion has its point of discontinuity on the corresponding interval (a, a_{m-k}) or (a_{m-k}, b) and all dispersions represent the functions of class $C^{(3)}$ on the given intervals.

2) The second part of this statement follows from (12) [4] giving expression to the derivatives of the special central dispersions of the first kind at the zero of the solution y. This evidently implies

$$\frac{y'(c)}{y'(\Phi_k(c))} = \frac{\bar{y}'(c)}{\bar{y}'(\Phi_k(c))} ,$$

for $c \neq a_{m-k}$, $k = 0,1,\ldots,m-1$ and thus also

$$\frac{y'(c)}{\overline{y}'(c)} = \frac{y'(\overline{c})}{\overline{y}'(\overline{c})} = \lambda$$

for arbitrary two modes $c, \overline{c}$ of the bundle I_c.

3) From the inequalities (28), (30) or (29), (31) then follows the validity of (32) for all $t \in (a, a_{m-k})$ satisfying the inequality $t \leq c < \Phi(t)$ or for all $t \in (a_{m-k}, b)$ satisfying the inequality $\Phi_{m-1}(t) < c \leq t$. For the other $t \in (a, a_{m-k})$ there always exists a zero $\overline{c}$ of the solution y such that the inequality $t \leq \overline{c} < \Phi(t)$ is valid and also for the other $t \in (a_{m-k}, b)$ there always exists a zero $\overline{\overline{c}}$ of the solution y such that the inequality $\Phi_{m-1}(t) < \overline{\overline{c}} \leq t$ is valid. Thus relation (32) is satisfied again for the mode $\overline{c}$ or $\overline{\overline{c}}$. With respect to the expression $y'^2(\overline{c}) = \lambda_1 y'^2(c)$, $\overline{y}'^2(\overline{c}) = \lambda_1 \overline{y}'^2(c)$ or $y'^2(\overline{\overline{c}}) = \lambda_2 y'^2(c)$, $\overline{y}'^2(\overline{\overline{c}}) = \lambda_2 \overline{y}'^2(c)$, where λ_1, λ_2 are constants, the equality (32) is true for all $t \in j$, $t \neq a_{m-k}$.

The common properties of arbitrary elements y, $\overline{y}$ of the same bundle of solutions I_c corresponding to the set Q_Φ are collected together in Theorem 2. Now our interest will centre upon the question whether the given properties of the couple of solutions y, $\overline{y}$ with the same zero c of the corresponding equations $q^{(1)}$, $(\overline{q}^{(1)})$ ensure the carriers q, $\overline{q}$ to belong to the same set Q_Φ. This question is replied by the following

Theorem 3.

Consider differential equations $(q^{(1)})$, $(\overline{q}^{(1)})$ 1-special of type m on the interval j. Let y and $\overline{y}$ be arbitrary solutions of the equations $(q^{(1)})$ and $(\overline{q}^{(1)})$, respectively. Suppose these solutions have all their zeros in common and that the quotient of the derivatives $y' : \overline{y}'$ at these zeros is equal to the same constant λ. Suppose further that at least one of the following relations

$$\int_t^{\Phi(t)} \left[\frac{\lambda}{y^2(\tau)} - \frac{1}{\lambda}\frac{1}{\bar{y}^2(\tau)}\right] d\tau = 0 , \qquad (34)$$

$$\int_t^{\bar{\Phi}(t)} \left[\frac{\lambda}{y^2(\tau)} - \frac{1}{\lambda}\frac{1}{\bar{y}^2(\tau)}\right] d\tau = 0 , \qquad (35)$$

is satisfied for all $t \in (a, a_{m-1})$. Then the fundamental central dispersions $\Phi(t)$, $\bar{\Phi}(t)$ and thus also the functions $\Phi_k(t)$, $\bar{\Phi}_k(t)$ for $k \in \{0,1,\dots,m-1\}$ relating to these equations coincide in their whole domain of definition.

Proof. If y and $\bar{y}$ are solutions of the equations $(q^{(1)})$ and $(\bar{q}^{(\bar{1})})$, respectively, and these solutions have all their zeros in common, then $\Phi_k(c) = \bar{\Phi}_k(c)$ holds, where is an arbitrary zero of them, $c \neq a_{m-k}$, $k \in \{0,1,\dots,m-1\}$. Some we need to prove the equality $\Phi_k(t) = \bar{\Phi}_k(t)$ for t different from the zeros of the solution y, $t \neq a_{m-k}$.

1) In proving the equality $\Phi(t) = \bar{\Phi}(t)$ on the interval (a, a_{m-1}) we proceed from (18), Lemma 2. The validity of $\varphi(t) = \Phi(t)$ on this interval implies the validity of

$$\int_t^{\Phi(t)} \left[\frac{1}{y^2(\tau)} - \frac{1}{y'^2(c)(\tau - c)^2}\right] d\tau =$$

$$= \frac{1}{y'^2(c)}\left[\frac{1}{c-t} - \frac{1}{c-\Phi(t)}\right] +$$

$$+ \frac{1}{y'^2(c)}\left[-\operatorname{cotg}\alpha_0[\Phi(t)] + \operatorname{cotg}\alpha_0(t)\right] , \qquad (36)$$

where α_0 is the first phase of the equation $(q^{(1)})$ satisfying the initial conditions

$$\alpha_0(c) = 0, \quad \alpha_0'(c) = 1, \quad \alpha_0''(c) = 0 .$$

Assuming next, say, the equality (34), we obtain the equality of integrals

$$\int_t^{\Phi(t)} \left[\frac{y'^2(c)}{y^2(\tau)} - \frac{1}{(\tau-c)^2} \right] d\tau = \int_t^{\Phi(t)} \left[\frac{\bar{y}'^2(c)}{\bar{y}^2(\tau)} - \frac{1}{(\tau-c)^2} \right] d\tau \qquad (37)$$

for $t \in (a, a_{m-1})$, the zero $c \in (t, \Phi(t))$ and, with the respect to the validity of (36), also the equality

$$- \operatorname{cotg} \alpha_0(t) + \operatorname{cotg} \alpha_0[\Phi(t)] = - \operatorname{cotg} \bar{\alpha}_0(t) + \operatorname{cotg} \bar{\alpha}_0[\Phi(t)] \qquad (38)$$

where α_0 and $\bar{\alpha}_0$ are the first phases of the equations $(q^{(1)})$ and $(\bar{q}^{(1)})$ satisfying the above conditions and it holds $\Phi(t) \in (c, \Phi(c))$ and thus also $\Phi(t) \in (c, \bar{\Phi}(c))$. On account of the fact that the points t and $\Phi(t)$ are 1-conjugate points of the equation $(q^{(1)})$, the left side and thus also the right side of (38) are equal to zero. The points t, $\Phi(t)$ are again 1-conjugate points of $(\bar{q}^{(1)})$ and with respect to the validity of $\Phi(t) \in (c, \bar{\Phi}(c))$ we have $\bar{\Phi}(t) = \Phi(t)$ for all $t \in (a, a_{m-1})$. With respect to the validity of $\Phi_k(t) = \Phi^k(t)$, $\bar{\Phi}_k(t) = \bar{\Phi}^k(t)$ on the interval (a, a_{m-k}), there is also satisfied $\Phi_k(t) = \bar{\Phi}_k(t)$.

2) In proving the equality $\Phi(t) = \bar{\Phi}(t)$ on the interval (a_{m-1}, b) we could analogously proceed from a relation corresponding to (18) for a negative k. This, however, is no more necessary. From the validity $\Phi(t) = \bar{\Phi}(t)$ on the interval (a, a_{m-1}) there namely follows the validity $\Phi_{-1}(t) = \bar{\Phi}_{-1}(t)$ on the interval (a_1, b), whence on account of the fact that $\Phi_k(t) = \Phi_{-(m-k)}(t) = \Phi_{-1}^{(m-k)}(t)$, $\bar{\Phi}_k(t) = \bar{\Phi}_{-(m-k)}(t) = \bar{\Phi}_{-1}^{(m-k)}(t)$ for $t \in (a_{m-k}, b)$ also the validity $\Phi_k(t) = \bar{\Phi}_k(t)$ on the interval (a_{m-k}, b).

Our further consideration will be directed to investigating the mutual relation between the carriers of $(q^{(1)})$, $(\bar{q}^{(1)})$ belonging to the same set Q_{Φ}. In analogy with [1] we will consider here the following quotient function

$$p(t) = \begin{cases} \dfrac{\bar{y}(t)}{y(t)} & \text{for } t \in j \text{ different from all modes } c \text{ of the bundle } I_c \\ \dfrac{\bar{y}'(t)}{y'(t)} & \text{at all modes } c \text{ of the bundle } I_c \end{cases}$$

for two arbitrary elements y, $\bar{y}$ of the bundle I_c and will investigate the properties of the function so defined.

1) The function $p(t)$ is everywhere positive or everywhere negative in j according as $\bar{y}'(c) : y'(c) > 0$ or < 0.

Proof. The property follows from the fact that the solutions $y(t)$ and $\bar{y}(t)$ have all their zeros in common. The solutions y, $\bar{y}$ have between any two neightbouring zeros either the values with always the same or opposite signs, accordina as $\bar{y}'(c) : y'(c) > 0$ or < 0.

2) It holds $p[\Phi_k(t)] = p(t)$ for all $t \in j$, $t \neq a_{m-k}$, where $k \in \{0,1,\ldots,m-1\}$, $a_m = b$.

Proof. The validity immediately follows from (39) and (12) [4] giving expression to the derivatives of the special dispersions of the first kind.

3) The function $p(t)$ belongs to the class $C^{(2)}$ for all $t \in j$.

Proof. The function $p(t)$ is continuous for all $t \in j$. The continuity for t different from the zeros c follows from definition (39), whereby we find on making use of L'Hospital's rule that the limit of the function $p(t)$ is equal to the functional value $p(c)$ at every zero c of the solutions y, $\bar{y}$. Evidently, the function $p(t)$ (except for zeros c) is continuously twice differentiable and it holds

$$p' = \frac{w}{y^2}\,, \tag{40}$$

$$p'' = (\bar{q} - q)p - 2\,\frac{y'}{y}\,p'\,, \tag{41}$$

for $t \neq c$, where $w = y\bar{y}' - y'\bar{y}$. On making use of L'Hospital's rule, we obtain

$$\lim_{t \to c} p'(t) = 0, \quad \lim_{t \to c} p''(t) = \frac{1}{3}\left[\bar{q}(c) - q(c)\right]p(c) \tag{42}$$

which, however, corresponds to the values of the derivatives of the function $p(t)$ at the points c expressed in (39). Consequently, the function $p(t)$ belongs to $C^{(2)}(j)$.

4) It holds $p(c) = 0$.

Proof. The validity immediately follows from the expresion of the derivative $p'(t)$ at the zero c of the solutions y, $\bar{y}$.

5) The function $p(t)$ satisfies throughout the interval j except for $t \neq a_{m-k}$ the following relation

$$\int_t^{\Phi_k(t)} \left[\frac{1}{p^2(t)} - \frac{1}{p^2(c)}\right] \frac{1}{y^2(t)}\, dt = 0 \tag{43}$$

where the function under the integral sign is everywhere continuous and its limit at the mode c has the value $-p''(c)$: $: \left[p^3(c)y'^2(c)\right]$.

Proof. The validity of (43) directly follows from (32) of Theorem 2. On making twice use of L'Hospital's rule we come to the limit and to the continuity of the function under the integral sign.

By means of the quotient function $p(t)$ of the properties 1) through 5) and by an element q of the set Q_Φ we may express another element $\bar{q}$ of the set Q_Φ as it is stated in the following Theorem. The sufficient properties of the function $p(t)$ to dering the general form of the carrier $\overline{q}$ by means of the known carrier q may be presented here in a somewhat weakened form, as apposed to the properties 1) through 5).

Theorem 4.

Suppose $\Phi(t)$ is the fundamental special central dispersion of the equation $(q^{(1)})$, that y(t) is an arbitrary solution of this equation and that c is an arbitrary zero of the solution y from the interval (a, a_{m-1}). Then all carriers $\overline{q}$ of the equations $(\overline{q}^{(1)})$ with the same fundamental special central dispersion equal to the function $\Phi(t)$ are determined by the relation

$$\overline{q} = q + \frac{p''}{p} + \frac{2y'}{p}\frac{p'}{y} , \tag{44}$$

where p is an arbitrary function with the properties 1) through 5) given in (45) below, and where the value of the last summand $2y'p':(py)$ at the point c is given by the quotient $2p''(c):p(c)$.

$$\begin{aligned}
&1)\quad p(t) \neq 0 \qquad \text{for } t \in j \\
&2)\quad p[\Phi(t)] = p(t) \qquad \text{for } t \in (a, a_{m-1}) \\
&3)\quad p(t) \in C^{(2)}(j) \\
&4)\quad p'(c) = 0 \\
&5)\quad \int_c^{\Phi(c)} \left[\frac{1}{p^2(t)} - \frac{1}{p^2(c)}\right] \frac{1}{y^2(t)}\, dt = 0 \qquad \text{for } c \in (a, a_{m-1})
\end{aligned} \tag{45}$$

Proof. 1) If the carriers q, $\overline{q}$ are likewise the elements of the same set Q_Φ and the solutions y, $\overline{y}$ likewise the elements of the bundle I_c belonging to the set Q_Φ, then the function p(t) defined by (39) evidently satisfies the equations 1) through 5) from (45). Then the validity of (44) directly follows from the equality (41).

2) Suppose conversely the function p(t) with the properties 1) through 5) stated in (45). Then the function $\overline{y}(t) = p(t)y(t)$ represents a solution of the equation $(\overline{q}^{(1)})$ determined by the initial conditions $\overline{y}(c) = 0$, $\overline{y}'(c) = p(c)y'(c)$. The property 1)

ensures hereby the coincidence of the zeros of the solutions y, $\overline{y}$ and along with the properties 2) and 4), utilizing the relation $\overline{y}' = p'y + py'$, it ensures the same value of the quotient of the derivatives $\overline{y}'(c) : y'(c) = p(c)$ in all zeros c relative to these solutions. Denote now by F(t) the function under the integral sign of 5) from (45). We known from the foregoing that this function is continuous in the interval j, its value is determined by the product $p''(c)p^{-3}(c)y^{1-2}(c)$ at the point c, and for $t \neq a_{m-1}$ it satisfies the relation

$$F\left[\Phi(t)\right]\ \Phi'(t) \quad = \quad F(t).$$

From this we obtain

$$\left[\int_t^{\Phi(t)} F(\tau)d\tau\right]' = 0 \qquad \text{for } t \in (a, a_{m-1}),$$

and from the validity 5) in (45) also

$$\int_t^{\Phi(t)} F(\tau)d\tau = \int_c^{\Phi(c)} F(\tau)d\tau = 0 .$$

Thus the relation

$$\int_t^{\Phi(t)} \left[\frac{\lambda}{y^2(\tau)} - \frac{1}{\lambda}\,\frac{1}{\overline{y}^2(\tau)}\right] d\tau = 0$$

is fulfilled in the interval (a, a_{m-1}), where $\lambda = 1/p(c)$. So, the assumptions of the Theorem 3 are fulfilled. From its validity we then have $\overline{q} \in Q_{\Phi}$.

Example.

Consider the differential equation

$$Y'' = \frac{1-m^2}{(1+t^2)^2}\ Y \qquad (46)$$

where $m \geq 2$ is a natural number, $t \in (-\infty, +\infty)$. Inserting

$$Y_1 = \sqrt{t^2 + 1}\,\sin(m \operatorname{arctg} t)$$

$$Y_2 = \sqrt{t^2 + 1}\,\cos(m \operatorname{arctg} t)$$

into (46), we see that the functions Y_1, Y_2 are independent solutions of (46) for $t \in (-\infty, +\infty)$.

We know from [2] that this is a 1-special equation of finity type m on the interval $(-\infty, +\infty)$, with the 1-fundamental sequence

$$\operatorname{cotg}\frac{(m-1)\pi}{m}, \quad \operatorname{cotg}\frac{(m-2)\pi}{m}, \ldots, \operatorname{cotg}\frac{\pi}{m},$$

and that the function $\Phi_k(t)$ defined by

$$\Phi_k(t) = \operatorname{tg}(\operatorname{arctg} t + \frac{k\pi}{m})$$

for $k \in \{0, 1, \ldots, m-1\}$ maps the zero c of an arbitrary solution Y onto the left lying k-th zero of the same solution, if any. In the contrary case it maps c onto the right lying (m-k)-th zero of the same solution. It holds thereby

$$\lim_{t \to (\operatorname{cotg}\frac{k\pi}{m})^-} \Phi_k(t) = +\infty, \qquad \lim_{t \to (\operatorname{cotg}\frac{k\pi}{m})^+} \Phi_k(t) = -\infty,$$

$$\lim_{t \to +\infty} \Phi_k(t) = \lim_{t \to -\infty} \Phi_k(t) = \operatorname{cotg}\frac{(m-k)\pi}{m}$$

The functions $\Phi_k(t)$ satisfy the properties 1) through 6) in (5), they form a finite cyclic group and represent the special central dispersions of the equation (46). On the basis of this example we may formulate the following

Corollary 2.

All carriers $\bar{q}$ of equations $(\bar{q}^{(1)})$ defined in the interval

$(-\infty,+\infty)$ with the same special central dispersion equal to the function

$$\Phi(t) = \mathrm{tg}(\mathrm{arctg}\ t + \frac{\pi}{m})$$

are determined by the relation

$$\bar{q}(t) = \frac{1-m^2}{(1+t^2)^2} + \frac{p''(t)}{p(t)} + \tag{47}$$

$$+ \frac{2\,p'(t)}{p(t)}\left[\frac{1}{(t^2+1)}\,(t + m\ \mathrm{cotg}(m\ \mathrm{arctg}\ t - m\ \mathrm{arctg}\ c))\right]$$

where c is an arbitrary point of the interval $(-\infty,\ \mathrm{cotg}(\pi/m))$ and $p(t)$ is an arbitrary function having the following properties

1) $p(t) \neq 0$ for $t \in (-\infty,+\infty)$

2) $p\left[\mathrm{tg}(\mathrm{arctg}\ t + \frac{\pi}{m})\right] = p(t)$ for $t \in (-\infty, \mathrm{cotg}\ \frac{\pi}{m})$

3) $p(t) \in C^{(2)}$ for $t \in (-\infty,+\infty)$

4) $p'(c) = 0$

5) $$\int_c^{\mathrm{tg}(\mathrm{arctg}\ c+\frac{\pi}{m})} \left[\frac{1}{p^2(t)} - \frac{1}{p^2(c)}\right] \frac{1}{(t^2+1)\sin^2(m\ \mathrm{arctg}\ t-m\ \mathrm{arctg}\ c)}\,dt = 0$$

Proof. The general solution of the equation (46) is expressible in the form

$$Y = c_1\sqrt{t^2+1}\ \sin(m\ \mathrm{arctg}\ t + c_2)\ ,$$

where c_1, c_2 are real numbers.

Then

$$Y' = \frac{c_1}{\sqrt{t^2+1}}\left[t\ \sin\ m(\mathrm{arctg}\ t+c_2)+m\ \cos(m\ \mathrm{arctg}\ t+c_2)\right]$$

$$Y'' = \frac{c_1}{\sqrt{t^2+1}^3}\left[\sin m(\mathrm{arctg}\ t+c_2)(1-m^2)\right] .$$

Choosing a particular solution y of the equation (46) in the form

$$y = \sqrt{t^2+1}\ \sin(m\ \mathrm{arctg}\ t - m\ \mathrm{arctg}\ c) ,$$

we see that it is a solution satisfying the initial conditions

$$y(c) = 0 , \quad y'(c) = \frac{m}{\sqrt{c^2+1}} ,$$

and the quotient $y'(t) : y(t)$ is given by the relation

$$\frac{y'(t)}{y(t)} = \frac{1}{(t^2+1)}\left[t + m\ \mathrm{cotg}(m\ \mathrm{arctg}\ t - m\ \mathrm{arctg}\ c)\right] .$$

The validity directly follows from the statement of the preceding Theorem 4.

Remark.

In the assumptions of Theorem 3 and 4 there were utilized the definition and the properties of the fundamental special central dispersion of the 1st kind relative to $(q^{(1)})$ on the interval (a, a_{m-1}), only, where the function $\Phi(t)$ coincides with the fundamental central dispersion $\varphi(t)$ in terms of the definition stated in [1]. The assumption of a one-to-one reverse mapping of the interval (a_{m-1}, b) onto the interval (a, a_1) is not utilized here. For this reason, it is evidently possible to utilize the wording of the above theorems also for the equations (q) of finite type, which are not 1-special in their interval of definition.

ROVNICE y" = q(t)y KONEČNÉHO TYPU, 1-SPECIÁLNÍ, S TOUŽ SPECIÁLNÍ CENTRÁLNÍ DISPERZÍ 1.DRUHU

Souhrn

V teorii centrálních disperzí lineárních diferenciálních rovnic 2.řádu, podrobně rozpracované v monografii [1] O.Borůvky byly vyšetřovány vlastnosti oboustranně oscilatorických rovnic typu

$$y'' = q(t)y \qquad (q)$$

které mají na svém definičním intervalu tutéž základní centrální disperzi 1.druhu. V článcích [3] a [4] zavedla autorka jisté zobecnění pojmů centrálních disperzí pro rovnice (q) konečného typu $m \geq 2$, 1-speciální na definičním intervalu $j = (a,b)$, $q(t) \in C^0(j)$, (značené $(q^{(1)})$) prostřednictvím definic speciálních centrálních disperzí jednotlivých druhů a diskutovala podmínky a vlastnosti takovýchto zobecnění.

Text tohoto článku je věnován bližšímu určení množiny nosičů rovnic $(q^{(1)})$ s touž základní speciální centrální disperzí 1.druhu $\Phi(t)$. Jsou uvedeny postačující vlastnosti obecné funkce, aby tato mohla představovat základní speciální centrální disperzi 1.druhu, je hledán vztah mezi nosiči q, $\overline{q}$ rovnic $(q^{(1)})$ $(\overline{q}^{(1)})$, pro které platí $\Phi(t) = \overline{\Phi}(t)$ na celém definičním oboru těchto funkcí.

Резюме

ОБ УРАВНЕНИЯХ y"=q(t)y КОНЕЧНОГО ТИПА, 1-СПЕЦИАЛЬНЫХ С ТОЙ ЖЕ САМОЙ СПЕЦИАЛЬНОЙ ЦЕНТРАЛЬНОЙ ДИСПЕРСИЕЙ 1-ого РОДА

В теории центральных дисперсий для линейных дифференциальных уравнений 2-ого порядка основанной на литературе [1]

расследованы свойства уравнений

$$y''=q(t)\,y \qquad (q)$$

с осцилирующими решениями и с той же самой центральной дисперсией 1-ого рода. В статьях [3] и [4] введены обобщения понятий центральных дисперсий для уравнения (q) конечного типа $m \geq 2$ специального на промежутке определения $j = (a,b)$, $q(t) \in C^0(j)$, посредством определений специальных центральных дисперсий отдельных родов и расследованы свойства этих обобщений.

Текст этой статьи намерен к определению множества уравнений с той же самой основной специальной дисперсией 1-ого рода $\phi(t)$.

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