Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica

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Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica, Vol. 29 (1990), No. 1, 73--82

Persistent URL: http://dml.cz/dmlcz/120245

ACTA UNIVERSITATIS PALACKIANAE OLOMUCENSIS
FACULTAS RERUM NATURALIUM
1990 MATHEMATICA XXIX VOL. 97

Katedra matematické analýzy a numerické matematiky
přírodovědecké fakulty Univerzity Palackého v Olomouci
Vedoucí katedry: Doc.RNDr.Jindřich Palát, CSc.

ASSOCIATED DIFERENTIAL EQUATION TO A SELFADJOINT THIRD-ORDER LINEAR DIFFERENTIAL EQUATION

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(Received April 30, 1989)

In the theory of the second-order linear differential equations in theJacobian form

$$y'' = q(t)y, \qquad (q)$$

where $q \neq 0$, $q \in C^2(-\infty,\infty)$, there is of an importance the associated differential equation (see [1])

$$z'' = q_1(t)z, \qquad (q_1)$$

where

$$q_1 = q(t) + \sqrt{|q(t)|}.(1/\sqrt{|q(t)|})''.$$

The functions $z(t) = y'(t)/\sqrt{|q(t)|}$, where y is a solution of (q), are the solutions of the associated equation (q_1). The idea of the associated equation can be extended to the associated selfadjoint third-order linear differential equation.

1. Let us consider the third-order linear differential equations

$(\omega) y''' + 2\omega(t) y' + \omega'(t) y = 0$, $(\Omega) Y''' + 2\Omega(t) Y' + \Omega'(t) Y = 0$,

where ω, $\Omega \in C^3(j)$, $\omega' = d/dt(\omega(t))$, $\Omega' = d/dt(\Omega(t))$.

Let Y denotes the function

$$Y = \varrho(t).[\alpha u(t) + \beta u'(t) + \gamma u''(t)] ,$$

where u is a solution of the differential equation (ω). We are concerned with the question if it is possible to find ϱ such that the function Y be a solution of (Ω).

Let

$$v = \alpha u + \beta u' + \gamma u'' ,$$

then we have

$$v' = -\gamma \omega' u + (\alpha - 2\gamma\omega) u' + \beta u'' ,$$

$$v'' = (-\gamma\omega'' - \beta\omega') u + (-3\gamma\omega' - 2\beta\omega) u' + (\alpha - 2\gamma\omega) u'' ,$$

$$v''' = (-\gamma\omega''' - \beta\omega'' - \alpha\omega' + 2\gamma\omega\omega') u + (-4\gamma\omega'' - 3\gamma\omega' - 2\alpha\omega + 4\gamma\omega^2) u' + (5\gamma\omega' - 2\beta\omega) u'' .$$

Inserting now the function $Y = \varrho(t).v(t)$ and its derivatives into(Ω) we obtain

$$\varrho''' v + 3\varrho'' v' + 3\varrho' v'' + \varrho v''' + 2\Omega(\varrho' v + \varrho v') + \Omega \varrho v = 0 .$$

Inserting v, v', v'', v''' we obtain after arrangements

$$(\gamma\varrho''' + 3\beta\varrho'' + 3\alpha\varrho' - 6\gamma\omega\varrho' - 5\gamma\omega'\varrho - 2\beta\omega\varrho + 2\gamma\Omega\varrho' + 2\beta\Omega\varrho + \gamma\Omega'\varrho) u'' +$$
$$+ (\beta\varrho''' + 3\alpha\varrho'' - 6\gamma\omega\varrho'' - 9\gamma\omega'\varrho' - 6\beta\omega\varrho' - 4\gamma\omega''\varrho - 3\beta\omega'\varrho - 2\alpha\omega\varrho + 4\gamma\omega^2\varrho +$$
$$+ 2\beta\Omega\varrho' + 2\alpha\Omega\varrho - 4\gamma\omega\Omega\varrho + \beta\Omega'\varrho) u' + (\alpha\varrho''' - 3\gamma\omega'\varrho'' - 3\gamma\omega''\varrho' - 3\beta\omega'\varrho' -$$
$$- \gamma\omega'''\varrho - \beta\omega''\varrho - \alpha\omega'\varrho + 2\gamma\omega\omega'\varrho + 2\alpha\Omega\varrho' - 2\gamma\omega'\Omega\varrho + \alpha\Omega'\varrho) u = 0 .$$

This equality is satisfied for every u if the coefficients of u, u', u'' are equal to zero. We have

$$\gamma \varrho''' = -3\beta \varrho'' - 3\alpha \varrho' + 6\gamma\omega\varrho' + 5\gamma\omega'\varrho + 2\beta\omega\varrho - 2\gamma\Omega\varrho' - 2\beta\Omega\varrho - \gamma\Omega'\varrho ,$$

$$\beta \varrho''' = -3\alpha\varrho'' + 6\gamma\omega\varrho'' + 9\gamma\omega'\varrho' + 6\beta\omega\varrho' + 4\gamma\omega''\varrho + 3\beta\omega'\varrho + 2\alpha\omega\varrho - \\ -4\gamma\omega^2\varrho - 2\beta\Omega\varrho' - 2\alpha\Omega\varrho + 4\gamma\omega\Omega\varrho - \beta\omega'\varrho ,$$

$$\alpha \varrho''' = 3\gamma\omega'\varrho'' + 3\gamma\omega''\varrho' + 3\beta\omega'\varrho' + \gamma\omega'''\varrho + \beta\omega''\varrho + \alpha\omega'\varrho - \\ -2\gamma\omega\omega'\varrho - 2\alpha\Omega\varrho' + 2\gamma\omega'\Omega\varrho - \alpha\Omega'\varrho .$$

Multiplying the first equality by β and the second one by γ , resp. the first one by α and the third one by γ resp. the second one by β and the third one by γ , and subtracting them after arrangements we obtain:

$$(-2\beta^2 + 2\alpha\gamma - 4\gamma^2\omega)\varrho(\Omega - \omega) = (6\gamma^2\omega + 3\beta^2 - 3\alpha\gamma)\varrho'' + \\ + (9\gamma^2\omega' + 3\alpha\beta)\varrho' + (4\gamma^2\omega'' - 2\beta\gamma\omega')\varrho .$$

resp.

$$(2\gamma^2\omega' + 2\alpha\beta)\varrho(\Omega - \omega) = (-3\gamma^2\omega' - 3\alpha\beta)\varrho'' + \\ + (-3\gamma^2\omega'' - 3\beta\gamma\omega' + 6\alpha\gamma\omega - 3\alpha^2)\varrho' + \\ + (-\gamma^2\omega''' - \beta\gamma\omega'' + 4\alpha\gamma\omega')\varrho ,$$

resp.

$$(-2\alpha^2 + 4\alpha\gamma\omega - 2\beta\gamma\omega')\varrho(\Omega - \omega) = (3\alpha^2 - 6\alpha\gamma\omega + 3\beta\gamma\omega')\varrho'' - \\ -(9\alpha\gamma\omega' + 6\alpha\beta\omega - 3\beta\gamma\omega'' - 3\beta^2\omega')\varrho' - (2\alpha\beta\omega' + 4\alpha\beta\omega'' - \\ - \beta\gamma\omega''' - \beta^2\omega'')\varrho .$$

We get

$$\Omega - \omega + \frac{3}{2}\frac{\varrho'}{\varrho} = \frac{9\gamma^2\omega' + 3\alpha\beta}{-4\gamma^2\omega - 2\beta^2 + 2\alpha\gamma}\frac{\varrho'}{\varrho} + \frac{4\gamma^2\omega'' - 2\beta\gamma\omega'}{-4\gamma^2\omega - 2\beta^2 + 2\alpha\gamma} , \qquad (1)$$

resp.

$$\Omega - \omega + \frac{3}{2}\frac{\varrho'}{\varrho} = \frac{-3\gamma^2\omega'' - 3\beta\gamma\omega' + 6\alpha\gamma\omega - 3\alpha^2}{2\gamma^2\omega' + 2\alpha\beta}\frac{\varrho'}{\varrho} + \\ + \frac{-\gamma\omega''' - \beta\gamma\omega'' + 4\alpha\gamma\omega'}{2\gamma^2\omega' + 2\alpha\beta} , \qquad (2)$$

resp.

$$\Omega - \omega + \frac{3}{2}\frac{\varrho'}{\varrho} = \frac{9\alpha\gamma\omega' + 6\alpha\beta\omega - 3\beta\gamma\omega'' - 3\beta^2\omega'}{2\alpha^2 - 4\alpha\gamma\omega + 2\beta\gamma\omega'}\frac{\varrho'}{\varrho} + \\ + \frac{4\alpha\gamma\omega'' + 2\alpha\beta\omega' - \beta\gamma\omega''' - \beta^2\omega''}{2\alpha^2 - 4\alpha\gamma\omega + 2\beta\gamma\omega'} . \qquad (3)$$

The equality of the left sides of (1) and (2) implies the equality of the right sides and we obtain

$$\varrho'/\varrho = (2\alpha\beta^2\gamma\omega' + 5\alpha\beta\gamma^2\omega'' - 2\beta\gamma^3\omega'^2 + 4\gamma^4\omega'\omega'' + \alpha\gamma^3\omega''' - 4\alpha^2\gamma^2\omega' - 2\gamma^4\omega\omega''' - \\ - 2\beta\gamma^3\omega\omega'' + 8\alpha\gamma^3\omega\omega' - \beta^2\gamma^2\omega''' - \beta^3\gamma\omega'')/(-3\alpha\gamma^3\omega'' - 3\alpha^3\gamma + 6\gamma^4\omega\omega'' + \\ + 6\beta\gamma^3\omega\omega' + 12\alpha^2\gamma^2\omega - 12\alpha\gamma^3\omega^2 + 3\beta^2\gamma^2\omega'' + 3\beta^3\gamma\omega' - 6\alpha\beta^2\gamma\omega - 15\alpha\beta\gamma^2\omega' - 9\gamma^4\omega'^2).$$

The derivative of the denominator is equal to

$$-3\alpha\gamma^3\omega''' - 12\gamma^4\omega'\omega'' + 6\gamma^4\omega\,\omega''' + 6\beta\gamma^3\omega'^2 + 6\beta\gamma^3\omega\omega'' + 12\alpha^2\gamma^2\omega' - \\ - 24\alpha\gamma^3\omega\omega' + 3\beta^2\gamma^2\omega''' + 3\beta^3\gamma\omega'' - 6\alpha\beta^2\gamma\omega' - 15\alpha\beta\gamma^2\omega''$$

and we set that is (-3) multiple of the numerator. From here we get

$$\varrho = k_1 / \sqrt[3]{\{|(-3\gamma^3 + 3\beta^2\gamma^2)\omega'' + 6\gamma^4\omega\omega'' + 6\beta\gamma^3\omega\omega' - 9\gamma^4\omega'^2 + \\ + (3\beta^3\gamma - 15\alpha\beta\gamma^2)\omega' - 12\alpha\gamma^3\omega^2 + (12\alpha^2\gamma^2 - 6\alpha\beta^2\gamma)\omega - 3\alpha^3\gamma|\}}. \qquad (4)$$

The equality of the left sides of (1) and (3) implies the equality of the right sides and we obtain:

$$\varrho'/\varrho = (-16\beta\gamma^2\omega\omega' + 4\beta\gamma^3\omega\omega''' + 4\beta^2\gamma^2\omega\omega'' - 10\alpha\beta^2\gamma\omega'' - \\ - 4\alpha\beta^3\omega' + 2\beta^3\gamma\omega''' + 2\beta^4\omega'' + 8\alpha^2\beta\gamma\omega' - 2\alpha\beta^2\gamma\omega''' - \\ - 8\beta\gamma^3\omega'\omega'' + 4\beta^2\gamma^2\omega'^2)/(6\alpha^3\beta - 24\alpha^2\beta\gamma\omega + 18\beta\gamma^3\omega'^2 + \\ + 30\alpha\beta^2\gamma\omega' + 24\alpha\beta\gamma^2\omega^2 - 12\beta\gamma^3\omega\omega'' - 12\beta^2\gamma^2\omega\omega' + \\ + 12\alpha\beta^3\omega - 6\beta^3\gamma\omega'' - 6\beta^4\omega' + 6\alpha\beta\gamma^2\omega'').$$

The derivative of the denominator is equal to

$$-24\alpha^2\beta\gamma\,\omega' + 24\beta\gamma^3\omega'\omega'' + 3\alpha\beta^2\gamma\,\omega'' + 48\alpha\beta\gamma^2\omega\,\omega' - 12\beta\gamma^3\omega\,\omega''' -$$
$$-12\beta^2\gamma^2\omega'^2 - 12\beta^2\gamma^2\omega\,\omega'' + 12\alpha\beta^3\omega' - 6\beta^3\gamma\,\omega''' - 6\beta^4\omega'' + 6\alpha\beta\gamma^2\omega'''$$

and we see that it is (-3) multiple of the numerator. From here we get

$$\rho = k_2/\sqrt[3]{\{|(-6\beta^3\gamma + 6\alpha\beta\gamma^2)\omega'' - 12\beta\gamma^3\omega\omega'' - 12\beta^2\gamma^2\omega\omega' + 18\beta\gamma^3\omega'^2 +}$$
$$+ (30\alpha\beta^2\gamma - 6\beta^4)\omega' + 24\alpha\beta\gamma^2\omega^2 + (-24\alpha^2\beta\gamma + 12\alpha\beta^3)\omega + 6\alpha^3\beta|\}. \quad (5)$$

We see that the expressions for ρ in (4) and (5) differ by a multiplicative constant.

We arrive to a similar conclusion if we calculate ρ'/ρ from (2) and (3) too.

Let

$$a = 9\gamma^2\omega' + 3\alpha\beta\,, \quad b = 4\gamma^2\omega'' - 2\beta\gamma\omega'\,, \quad d = -4\gamma^2\omega - 2\beta^2 + 2\alpha\gamma,$$
$$A = -3\gamma^2\omega'' - 3\beta\gamma\omega' + 6\alpha\gamma\omega - 3\alpha^2, \quad B = -\gamma^2\omega''' - \beta\gamma\omega'' + 4\alpha\gamma\omega'\,, \quad (6)$$
$$D = 2\gamma^2\omega' + 2\alpha\beta\,.$$

Then from (1) and (2) we obtain

$$\Omega - \omega + \frac{3}{2}\frac{\rho''}{\rho} \Rightarrow \frac{a}{d}\frac{\rho'}{\rho} + \frac{b}{d} = \frac{A}{D}\frac{\rho'}{\rho} + \frac{B}{D} \quad (7)$$

and from here we have ρ'/ρ = (dB-Ad)/(aD-dA).

From (7) we obtain

$$\Omega - \omega + \frac{3}{2}\frac{\rho''}{\rho} = \frac{aB - bA}{aD - dA}\,.$$

Since $d/dt(\rho'/\rho) = (\rho''\rho - \rho'^2)/(\rho^2) = \rho''/\rho - (\rho'/\rho)^2$ we have

$$\Omega = \omega - \frac{3}{2}\left[\left(\frac{\rho'}{\rho}\right)' + \left(\frac{\rho'}{\rho}\right)^2\right] + \frac{aB - bA}{aD - dA} \quad (8)$$

where ρ'/ρ = (dB-bD)/(aD-dA).

The following theorem yields from the above mentioned investigations.

Theorem. There are given the third-order linear differential equations (ω) and (Ω). Let $u = u(t)$ be a solution of (ω). Let $\alpha, \beta, \gamma \in R$, $\alpha^2 + \beta^2 + \gamma^2 > 0$. Then the function

$$U = \varrho(t)[\alpha u(t) + \beta u'(t) + \gamma u''(t)]$$

is a solution of (Ω), where

$$\varrho(t) = \hat{k}_1 / \sqrt[3]{|(-3\alpha\gamma^2+3\beta^2\gamma)\omega''+(6\gamma^3+6\beta\gamma^2)\omega\omega' -}$$

$$\overline{- 9\gamma^3\omega'^2+(3\beta^3-15\alpha\beta\gamma)\omega'-12\alpha\gamma^2\omega^2+(12\alpha^2\gamma-6\alpha\beta^2)\omega -}$$

$$\overline{- 3\alpha^3|}, \quad \hat{k}_1 = k_1/\sqrt[3]{|\gamma|} .$$

The coefficient Ω in (Ω) is given by the formula (8), where the letters a, b, d, A, B, D stand for the expressions that they are given by the formulas (6).

Definition. The equation (Ω), in which the coefficient Ω is given by (8) and $\Omega' = d/dt\ (\Omega(t))$, is called the associated differential equation to a selfadjoint third-order linear differential equation (ω) at a basis (α, β, γ).

2. Example. In a special case with $\alpha = \gamma = 0$, $\beta = 1$, we get

$$\varrho = k/\sqrt[3]{|3\omega'|} . \tag{9}$$

The solution U of the differential equation (Ω) is given by the formula

$$U = u'/\sqrt[3]{|3\omega'|} ,$$

where u is a solution of (ω), and the coefficient Ω by the formula

$$\Omega = \omega - \tfrac{3}{2}(\varrho''/\varrho) . \tag{10}$$

If $\mathcal{U} = \omega$ we say that the equation (ω) is associated to itself at the basis (0,1,0).

(10) yields that it occurs in the case $\rho'' = 0$. According to (9) we get

$$(k/\sqrt[3]{|3\omega'|})'' = 0$$

hence

$$\omega = ((-1/2c_1).(1/(c_1 t + c_2)^2) + c_3 ,$$

where $c_i \in R$, $i = 1,2,3$.

SOUHRN

PRŮVODNÍ DIFERENCIÁLNÍ ROVNICE K SAMOADJUNGOVANÉ LINEÁRNÍ DIFERENCIÁLNÍ ROVNICI 3.ŘÁDU

MIROSLAV LAITOCH

V článku se definuje průvodní diferenciální rovnice při dané bázi k samoadjungované lineární diferenciální rovnici 3.řádu. Jde o rozšíření pojmu průvodní rovnice k lineární diferenciální rovnici 2.řádu Jacobiho typu [1].

РЕЗЮМЕ

СОПРСВОДИТЕЛЬНОЕ ДИФФЕРЕНЦИАЛЬНОЕ УРАВНЕНИЕ К САМОСОПРЯЖЕННОМУ ЛИНЕЙНОМУ ДИФФЕРЕНЦИАЛЬНОМУ УРАВНЕНИЮ 3-ЬЕГО ПОРЯДКА

М. ЛАИТОХ

В работе определяется сопроводительное дифференциальное уравнение с данным базисом к самосопряженному линейному дифференциальному уравнениь 3-ьего порядка.

Следует, речь идет о расширении понятия сопроводительногс уравнения к линейному дифференциальному уравнению 2-ого порядка формы Якоби /1/.

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Acta UPO, Fac.rer.nat., 97, Mathematica XXIX, 1990, 73 - 82.