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Quadratic splines interpolating derivatives


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Abstract: There are described the algorithms for computing appropriate parameters of quadratic splines interpolating prescribed values of the first or the second derivative in the points of interpolation, which are different from the knots of the spline in general. The relations between various types of linear and quadratic splines are mentioned.

Key words: Spline functions, quadratic splines, interpolation.

MS Classification: 41A05, 41A15

1. Introduction

Quadratic spline interpolating given function values have been studied by many authors ([3], [5], [6], [7], [9]). There are very simple two-term recurrence relations between parameters of such a spline in case of coinciding knots and points of interpolation. But there are also some unpleasant features connected with such splines (error propagation without damping, un-
symmetry of boundary conditions, existence questions). It was soon recognized, that some of that features escape, when separate knots of spline and points of interpolation are used (error propagation with damping - see [6]). We can also find approaches to use splines interpolating first or second derivatives in the area of "shape-preserving approximations" ([4]) or in solution of differential equations ([1]). The first results for quadratic splines interpolating the first derivative on equidistant mesh can be found in [8].

The aim of our contribution is to give more detailed results concerning the quadratic splines interpolating the first or the second derivative in case of nonequidistant or separated meshes and to mention the relations between various types of such simple splines.

2. Simple knot set

Let us have an increasing set of spline knots

\((\Delta x) = \{x_i; i = 0(1)n+1\}\).

We call a quadratic spline on the set \((\Delta x)\) the function \(s(x)\) fulfilling conditions

\[\begin{align*}
1^0 & \quad s(x)\in C^1[x_0,x_{n+1}]; \\
2^0 & \quad s(x)\text{ is a quadratic polynomial on every interval } [x_i,x_{i+1}], i = 0(1)n.
\end{align*}\]

Let us denote \(S(2,\Delta x)\) the linear space of functions fulfilling conditions \(1^0, 2^0\).

Statement of the problem

Given real numbers \(m_i, i = 0(1)n+1\), we have to find \(s \in S(2,\Delta x)\) such that \(s'(x_i) = m_i, i = 0(1)n+1\) hold (e.g. we have to find spline interpolating the first derivatives at the knots of the spline).

Simple calculation shows that we have together \(3n+3\) spline parameters and only \(3n+2\) connecting continuity and interpolation conditions - one free parameter is also at our disposal.
Theorem 1

Quadratic spline $s \in S(2, \Delta x)$ is uniquely determined by conditions

1. $s'(x_i) = m_i$, $i = 0(1)n+1$ (conditions of interpolation), (2)
2. $s(x_0) = s_0$ (or $s(x_k) = s_k$, $k \in \{0,1,...,n+1\}$, initial condition). (2)

Proof

A spline $s \in S(2, \Delta x)$ can be written as

$$s(x) = (1-t)^2s_i + t^2s_{i+1} + h_i t(1-t)m_i \text{ for } x \in [x_i,x_{i+1}],$$

for $i = 0(1)n$, with $h_i = x_{i+1} - x_i$, $t = (x - x_i)/h_i$,

$s_i = s(x_i)$.

We have further

$$s'(x) = 2t(s_{i+1} - s_i)/h_i + (1-2t)m_i.$$ 

The continuity condition on $s'(x)$ in the knot $x = x_i$ leads to the recurrence relation

$$s_i - s_{i-1} = h_{i-1}(m_{i-1} + m_i)/2, \quad i = 1(1)n+1.$$ (4)

Given data (2), we can use (4) to the computation of all values $s_i$, $i \neq k$. Then we know all parameters needed for using (3) to compute $s(x)$.

Remarks

1. The computation of values $m_i$ from given $s_i$ using (4) (interpolation of function values) is known to be a little unstable; in [11] the composed relation on equidistant mesh

$$m_{i+1} - m_i = 2(s_{i+1} - 2s_i + s_{i-1})/h$$

with $s_0$, $s_1$ given is used.

2. We cannot prescribe the values of the second derivative on the boundary for spline interpolating the first derivatives (as can be done when interpolating function values). Given
m_i = s'(x_i), i = 0(1)n+1, it follows that s''(x_i + 0) = M_i =
= (m_{i+1} - m_i)/h_i is determined and constant in interval [x_i, x_{i+1}]
(the second derivative has discontinuities in knots x_i).

3. The quadratic spline described in Theorem 1 has similar
unpleasing error propagation features - the error in free para-
meter s_k or in parameter m_i is propagated over the whole interval
[x_0, x_{n+1}] without damping, as can be deduced from the relation

\[ s_i = s_0 + \sum_{j=0}^{i-1} h_j (m_j + m_{j+1})/2. \]  

For example, error \( v_0 = s_0 - \bar{s}_0 \) in the value \( s_0 \) results in
\( s_i - \bar{s}_i = v_i \). The isolated error \( e_i = m_i - \bar{m}_i \) in the value \( m_i \) is
propagated as \( s_i - \bar{s}_i = h_{i-1} e_i/2, s_{i+1} - \bar{s}_{i+1} = e_i (h_{i-1} + h_i)/2, \ldots. \)

4. It is possible to use another notation for \( s(x) \) instead
of (3) - for example

\[ s(x) = s_i + h_i c_i t + (m_i - c_i) h_i t(1-t), \quad x \in [x_i, x_{i+1}], \]  

where \( c_i = (s_{i+1} - s_i)/h_i \) (the slopes).

Continuity conditions can be now written as

\[ 2 c_{i-1} = m_{i-1} + m_i, \quad i = 1(1)n. \]  

We can use it in a similar way for construction of the spline
interpolating the function values or the first derivatives.

3. Mesh with separated knots and points of interpolation

3.1 Spline representation

Let us have a mesh of knots \( x_i \) and points of interpola-
tion \( t_i \)

\((\Delta x \Delta t): x_0 \leq t_0 < x_1 < t_1 < \ldots < t_n \leq x_{n+1}\)

and denote \( h_i = x_{i+1} - x_i, \quad d_i = (t_i - x_i)/h_i, \quad m_i = s'(t_i), \)

\[ i = 0(1)n, \quad s_i = s(x_i), \quad i = 0(1)n+1; \]

\[ t = (x - x_i)/h. \]

We have to find now again the spline \( s \in S(2, \Delta x) \) determined by
the conditions
\[ s'(t_i) = m_i, \; i = 0(1)n, \; m_i \text{ given real numbers.} \] (10)

Counting and comparing the number of parameters and continuity conditions in the knots \( x_i, \; i = 1(1)n \) of the spline, we recognize existence of two free parameters, which can be prescribed for unique determination of the spline. Some specific feature of the problem (10) can be found in the fact, that the case \( t_i = (x_i + x_{i+1})/2 \), which is the most popular and quite regular in function values interpolation, must be treated separately now.

Lemma 1

The solution of the problem (10) is given by the quadratic spline \( s \in S(2, \Delta x) \), which can be written for \( x \in [x_i, x_{i+1}] \)

a) in case of \( d_i \neq 1/2 \) as

\[ s(x) = A(t) s_i + B(t) s_{i+1} + h_i C(t) m_i, \; t \in [0,1], \] (11)

with functions

\[
A(t) = \frac{(t^2 - 2td_i)/(2d_i - 1) + 1 - B(t) + 1}{2d_i - 1}, \\
B(t) = - \frac{t(t - 2d_i)/(2d_i - 1)}{2d_i - 1}, \\
C(t) = \frac{t(t - 1)/(2d_i - 1)}{2d_i - 1};
\]

b) in case of \( d_i = 1/2 \), denoting \( s_1^* = s'(x_i) \), as

\[ s(x) = s_i + s_i^*(x - x_i) + (s_{i+1}^* - s_i^*)(x - x_i)^2/(2h_i), \] (12)

where we have

\[ m_i = (s_i^* + s_{i+1}^*)/2. \] (13)

Proof follows from the properties of functions \( A(t), B(t), C(t) \) and representation (11) in case \( d_i \neq 1/2 \). In case of \( d_i = 1/2 \) we recognize the Taylor formula in (12).

3.2 Computation of parameters \( s_i \)

The continuity condition for \( s \in S(2, \Delta x) \) at \( x = x_i \) is realized implicitly in our notation \( s_i = s(x_i - 0) = s(x_i + 0) = s(x_i) \).

We have further
Continuity condition for \( s'(x) \) at \( x = x_i \), \( i = 1(1)n \) in case \( d_i \neq 1/2, i = 0(1)n \) results in the relations

\[
\frac{1-d_i}{1-2d_i-1} s_{i-1} - \left[ \frac{1-d_i}{1-2d_i-1} - \frac{p_i d_i}{1-2d_i} \right] s_i + \frac{p_i d_i}{1-2d_i} s_{i+1} = \frac{1}{2} h_i \left[ \frac{m_i-1}{1-2d_i-1} + \frac{m_i}{1-2d_i} \right],
\]

for \( x \in [x_i, x_{i+1}] \).

Continuity condition for \( s'(x) \) at \( x = x_i \), \( i = 1(1)n \) in case \( d_i \neq 1/2, i = 0(1)n \) results in the relations

\[
-\frac{1-d_i-1}{1-2d_i-1} s_{i-1} + \left[ \frac{1-d_i-1}{1-2d_i-1} - \frac{p_i d_i}{1-2d_i} \right] s_i + \frac{p_i d_i}{1-2d_i} s_{i+1} = \frac{1}{2} h_i \left[ \frac{m_i-1}{1-2d_i-1} + \frac{m_i}{1-2d_i} \right],
\]

where \( p_i = \frac{h_{i-1}}{h_i} \). Denoting further

\[
a_i = \frac{(1-d_i-1)}{(1-2d_i-1)}, \quad b_i = \frac{p_i d_i}{1-2d_i},
\]

\[
f_i = \frac{1}{2} h_i \left( \frac{m_i-1}{1-2d_i-1} + \frac{m_i}{1-2d_i} \right), \quad i = 1(1)n,
\]

we can rewrite the system of continuity conditions (15) as

\[
- a_i s_{i-1} + (a_i - b_i) s_i + b_i s_{i+1} = f_i, \quad i = 1(1)n.
\]

In the special case of \( h_1 = h, d_1 = \lambda, i = 1(1)n \) we obtain the result of [8].

3.2.1 Boundary conditions \( s_0, s_{n+1} \)

Choosing \( s_0 = s(x_0), s_{n+1} = s(x_{n+1}) \) as two free spline parameters mentioned in 3.1, we recognize (17) to be the tridiagonal system of linear equations for determining parameters \( s_i, i = 1(1)n : \)

\[
\begin{bmatrix}
    a_1-b_1, & b_1, & & \\
    -a_2, & a_2-b_2, & b_2, & \\
    & \ldots & \ldots & \\
    -a_{n-1}, & a_{n-1}-b_{n-1}, & b_{n-1}, & \\
    -a_n, & a_n-b_n
\end{bmatrix}
\begin{bmatrix}
    s_1 \\
    s_2 \\
    \vdots \\
    s_{n-1} \\
    s_n
\end{bmatrix}
=
\begin{bmatrix}
    f_1+a_1 s_0 \\
    f_2 \\
    \vdots \\
    f_{n-1} \\
    f_n-b_n s_{n+1}
\end{bmatrix}.
\]

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The determinants $D_n$ of the matrix of system (18) fulfill recurrence relation

$$
D_i = (a_i - b_i)D_{i-1} + a_i b_{i-1}D_{i-2} \quad \text{(19)}
$$

Hence,

$$(a_i - b_i)a_{i-1}b_{i-1} > 0, \ i = 3(1)n, \quad D_1D_2 > 0$$

is sufficient for $D_n \neq 0$.

In case of the equidistant mesh with $h_i = h$, $d_i \neq 1/2$
we have $p_i = 1,$

$$a_i = a(1-d)/(1-2d), \quad a_i - b_i = 1, \quad \text{(20)}$$

$$b_i = d/(1-2d) = a - 1, \quad f_i = (m_{i-1} + m_1)(2(1-2d)).$$

The matrix $M_n$ of the system (18) and its determinant are

$$M_n = \begin{bmatrix}
1, & a-1, \\
-a, & 1, & a-1, \\
& \ddots & \ddots & \ddots \\
& & -a, & 1, & a-1
\end{bmatrix}, \quad \det(M_n) = \begin{cases}
1 & \text{for } n = 1, \\
1 + a(a - 1) & \text{for } n = 2, \\
1 + 2a(a - 1) & \text{for } n = 3
\end{cases}.$$

The recurrence relation (19) is now

$$\det(M_1) = \det(M_{i-1}) + a(a - 1)\det(M_{i-2}). \quad \text{(22)}$$

From positivity of $a(a - 1) = d(1-d)/(1-2d)^2$ for $d \in (0,1)\{1/2\}$
follows, that we have $\det(M_n) > 0$ generally in our case. Let us summarize our discussion in the following theorem.

Theorem 2

Under boundary conditions $s(x_0) = s_0, \ s(x_{n+1}) = s_{n+1}$ with
given numbers $s_0, \ s_{n+1}$ there exists a unique quadratic spline
$s(x)$ interpolating prescribed values of the first derivatives
$m_i = s'(t_i)$ on the mesh $(\Delta x \Delta t)$

a) on equidistant mesh with $h_i = h, \ d_i = d \neq 1/2, \ i = 0(1)n;$
b) on general mesh \((\Delta x \Delta t)\) with \(0, D_1 D_2 > 0, (a_1 - b_1) a_i b_{i-1} > 0, i = 3(1)n, d_{\frac{1}{2}} \neq 1/2\).

The values \(s_i = s(x_i)\) can be computed from the system (18) for the use of spline representation (11).

3.2.2 Another types of boundary conditions

There is possible to prescribe the boundary conditions on the first derivatives \(s'(x_0) = s'_0, s'(x_{n+1}) = s'_{n+1}\) for the spline fulfilling conditions (10). Suppose \(d_0, d_{\frac{1}{2}} \neq 1/2;\) then using (14) we obtain

\[
\begin{align*}
  s'_0 &= -\frac{2d_0}{h_0(2d_0 - 1)} (s_0 - s_1) - \frac{m_0}{2d_0 - 1}, \\
  s'_{n+1} &= \frac{2(1 - d_n)}{h_n(1 - 2d_n)} (s_{n+1} - s_n) - \frac{m_n}{1 - 2d_n}.
\end{align*}
\]

Denote

\[
\begin{align*}
  a_0 &= -b_0 = d_0/(1 - 2d_0), \\
  a_{n+1} &= -b_{n+1} = (1 - d_n)/(1 - 2d_n);
\end{align*}
\]

than we can join (23) with (17) to obtain the needed system of linear equations

\[
\begin{bmatrix}
  a_0 & b_0 \\
  -a_1 & a_1 - b_1 & b_1 \\
  \ddots & \ddots & \ddots \\
  -a_n & a_n - b_n & b_n \\
  -a_{n+1} & a_{n+1} - b_{n+1} & b_{n+1}
\end{bmatrix}
\begin{bmatrix}
  s_0 \\
  s_1 \\
  \vdots \\
  s_n \\
  s_{n+1}
\end{bmatrix}
= \begin{bmatrix}
  \frac{1}{2}h_0(s'_0 + m_0/(2d_0 - 1)) \\
  f_1 \\
  \vdots \\
  f_n \\
  \frac{1}{2}h_n(s'_{n+1} + m_{n+1}/(2d_n - 1))
\end{bmatrix}.
\]

According to the defining relations (24) we have

\[
\begin{align*}
  a_0 + b_0 &= 0, \\
  -a_{n+1} + b_{n+1} &= 0; \\
  -a_i + (a_i - b_i) + b_i &= 0, i = 1(1)n.
\end{align*}
\]

It indicates singularity of the matrix of the system (25) and we couldn't obtain the solution with every \(s'_0, s'_{n+1}\).

Similarly, we obtain the system with singular matrix when we try to determine the parameters \(s_i\) for the spline with boundary conditions \(s''(x_0) = M_0, s''(x_{n+1}) = M_{n+1}\).
It could be seen from (14) that
\[ s''(x) = \frac{2}{h_1(2d_1 - 1)} \frac{s_i - s_{i+1}}{h_1} + \frac{2}{d_1 - 1} \frac{m_i}{h_1} \text{ for } x \in [x_i, x_{i+1}] . \]

With \( i = 0, n \) we obtain the equations of boundary conditions with zero sum of coefficients.

Similar problems are involved in boundary conditions
\[ s''(t_0) = M_0, \quad s''(t_n) = M_n, \]
or when the first (resp. the second) derivatives on the boundary are approximated by some numerical differentiation formula using the values \( s_i \).

3.2.3 Mesh with \( d_1 = 1/2 \)

Let us consider the case \( d_1 = 1/2, i = 0(1)n, m_i = s'(t_i) \) given. The continuity relations are now
\[ s_{i+1} - s_i = h_i(s_i + s_{i+1})/2 = h_i m_i, \quad s_j = s(x_j), \quad s'(x_j) = s'_j. \] (27)

Given \( s_0 = s(x_0) \) as the first free parameter, it is possible recursively calculate the values \( s_i \):
\[ s_{i+1} = s_i + h_i m_i, \quad i = 0(1)n . \]

But the spline \( s \in S(2, \Delta x) \) is not uniquely determined by the parameters \( s_i, m_i \) only. Given \( s_0 = s'(x_0) \) as the second free parameter, we can find remaining values \( s_i \) using (27) as
\[ s'_{i+1} = 2m_i - s'_i, \quad i = 0(1)n . \]

Now the spline \( s \in S(2, \Delta x) \) is completely determined. More generally, it could be possible to choose the parameters \( s_k, s'_j \) and use (27) for determining other values \( s_i, s'_i \). It is not possible to determine spline \( s \in S(2, \Delta x) \) with the free parameters \( (s_k, s'_j) \), or \( (s'_k, s'_j) \).

Remark

For the given spline \( s(x) \) let us denote \( s'(x) = s_1(x) \) the
first degree spline (polygon). The problem (2) can be reformulated now as to find \( s_1(x) \) given by interpolation conditions 
\[
s_1(t_i) = s'(t_i) = m_i.
\]
It is now easy to see, that given \( s_1(x_0) = s_0 \), the spline \( s_1(x) \) is uniquely determined. For exact determination of \( s(x) = \int s_1(x) dx \) we have to prescribe yet the integration constant - the value \( s(x) \) in some point \( (x_0) \) in case 3.2.2). We can see as well, that it is not possible generally to prescribe two values \( s_1(x_0) = s'(x_0), s_1(x_{n+1}) = s'(x_{n+1}) \) (see 3.2.2). Another results of 3.2.3 could be interpreted similarly.

3.3 Error propagation

3.3.1 Errors in boundary values

Let us have equidistant mesh with \( d_1 = d = (t_i - x_i)/h \neq 1/2 \), and consider splines \( s, \tilde{s} \in S(2, \Delta x) \) determined by the same conditions of interpolation derivatives at points \( t_i \), \( i = 0(1)n \), but by different boundary conditions: the spline \( s \) by boundary values \( s_0, s_{n+1} \) and \( \tilde{s} \) by values \( \tilde{s}_0, \tilde{s}_{n+1} \). The difference \( \zeta = s - \tilde{s} \) belongs to \( S(2, \Delta x) \) and fulfills the interpolation conditions 
\[
s'(t_i) = 0, i = 0(1)n \text{ and boundary conditions}
\]
\[
\tilde{s}_0 = e_0 = s_0 - \tilde{s}_0, \quad \tilde{s}_{n+1} = e_{n+1} = s_{n+1} - \tilde{s}_{n+1}.
\]

The system of continuity and boundary conditions can be written now (see 3.2.1) as 
\[
-a\tilde{s}_{i-1} + \tilde{s}_i + (a - 1)\tilde{s}_{i+1} = 0, \quad \tilde{s}_0 = e_0, \quad \tilde{s}_{n+1} = e_{n+1}.
\]

(28)

We can consider it also as boundary value problem for the second order difference equation and to solve it explicitly. The characteristic equation has roots \( r_1 = 1, r_2 = 1 - 1/d \).

For \( d \in (0, 1/2) \) there is \( r_2 \in (-\infty, -1) \), 
for \( d \in (1/2, 1) \) we have \( r_2 \in (-1, 0) \).

General solution of equation (28) could be now written as 
\[
s_j = c_1 + c_2(1 - 1/d)^j.
\]

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Constants $c_1, c_2$ are determined by boundary conditions $e_0, e_{n+1}$:

$$c_1 = (e_0 r_2^{n+1} - e_{n+1})/(r_2^{n+1} - 1), \quad c_2 = (e_{n+1} - e_0)/(r_2^{n+1} - 1).$$

The particular solution of the problem (28) is therefore

$$\tilde{s}_j = \frac{1}{r^{n+1} - 1} \left[(e_0 r_2^{n+1} - e_{n+1}) + (e_{n+1} - e_0)r_2^j\right]. \quad (30)$$

For large $n$ we can write it with respect to (29) approximatively

$$\tilde{s}_j \approx e_0 + (e_{n+1} - e_0)r_2^{j-n-1} \quad \text{for} \quad d \in (0, 1/2) \quad (|r_2| > 1),$$

$$\tilde{s}_j \approx e_{n+1} + (e_0 - e_{n+1})r_2^j \quad \text{for} \quad d \in (1/2, 1) \quad (|r_2| < 1). \quad (31)$$

### 3.3.2 Isolated errors in derivatives

Let us suppose that some isolated error of magnitude $e$ occurs in the value $m_k : m_k = m_k - e$. Then, by (16), it causes the errors in the right hand sides $f_k$ with indices $k, k+1$ only:

$$f_k = f_k + \frac{1}{2}eh/(1-2d) = f_k - E, \quad f_{k+1} = f_{k+1} - E, \quad (E = \frac{1}{2}eh/(1-2d)).$$

Denoting $\tilde{s} = s - \bar{s}$ the difference of splines differing in the values $m_k, \bar{m}_k$ only, the system (18) looks now

$$-a\tilde{s}_{i-1} + \tilde{s}_i + (a-1)\tilde{s}_{i+1} = f_i, \quad \tilde{s}_1 = \begin{cases} 0 & \text{for } i \neq k, k+1 \\ E & \text{for } i = k, k+1, \end{cases} \quad \tilde{s}_0 = \tilde{s}_{n+1} = 0. \quad (32)$$

The solution of this boundary value problem can be written as

$$\tilde{s}_j = \sum_{i=1}^{n} g_{ji} \tilde{f}_i, \quad (33)$$

where $g_{ji}$ are values of the Green's function of the problem (see [2]) defined as
So we have for example
\[ s_k = -dE(1-r_2^k)(1-r_2^{n+1})/(1-r_2^n) + dE(1-r_2^{-n-1})/(1-r_2^{-n}) \]
with \[ s_k = 0(E) \text{ for } d \in (0,1) \backslash \frac{1}{2} . \]

Therefore we can see that in both cases 3.3.1, 3.3.2 we have no damping of errors in data \( m_i \) or in boundary conditions among splines \( S(2, Ax) \) interpolating the first derivatives.

4. Quadratic splines interpolating the second derivatives

Let us have a mesh \((AxAt)\) and denote \( s_i = s(t_i) \), \( M_i = s''(t_i) \), \( i = 0(1)n \) for \( s \in S(2, Ax) \). These quantities satisfy (see [5]) continuity relations

\[ a_i M_{i-1} + b_i M_i + c_i M_{i+1} = f_i, \quad i = 1(1)n-1, \quad (34) \]

where \( h_i = x_{i+1} - x_i \), \( k_i = t_{i+1} - t_i \),
\[ a_i = \left( (x_i - t_{i-1})/k_{i-1} \right)^2 k_{i-1}^{-1}/(k_{i-1} + k_i) , \]
\[ b_i = (t_i - x_i) \left[ 1 + (x_i - t_{i-1})/k_{i-1} \right] + (x_{i+1} - t_i) \left[ 1 + (t_i - x_{i+1})/k_{i+1} \right] / (k_{i-1} + k_i) , \]
\[ c_i = \left( (t_{i+1} - x_{i+1})/k_{i} \right)^2 k_i/(k_{i-1} + k_i) , \]
\[ f = 2[(s_{i+1} - s_i)/k_i + (s_i - s_{i-1})/k_{i-1}]/(k_{i-1} + k_i) = 2[t_{i-1}, t_i, t_{i+1}] s \]

\((a_i, b_i, c_i)\) are determined by the mesh only; \( f_i \) depends also on the data \( s_i \); the symbol for divided difference is used).

4.1 With \( M_i, i = 0(1)n \) and \( s_0 = s(t_0), s_1 = s(t_1) \) given, we can calculate all values \( s_i, i = 2(1)n \) with the help of three-term recurrence relation (34). The value \( m_i = s'(t_i) \) can be then calculated from the relations (see 5).
\[ m_i k_i = s_{i+1} - s_i - \left[ (x_{i+1} - t_i) (t_{i+1} - x_{i+1} + k_i) M_i - (t_{i+1} - x_{i+1})^2 M_{i+1} + k_i \right] / 2. \]  

On an equidistant mesh with \( h_1 = h, t_i = (x_i + x_{i+1})/2 \) we have simply

\[ m_i = (s_{i+1} - s_i)/h - h(3M_i - M_{i+1})/8. \]  

In both cases we can then use the Taylor representation of the spline \( s \):

\[ s(x) = s_i + m_i (x - t_i) + \frac{1}{2} M_i (x - t_i)^2. \]  

4.2 When we prescribe boundary conditions \( s_0, s_n \) as free parameters, the system (34) can be then written as system of linear equations with symmetric regular matrix for unknown values \( s_i \):

\[ \frac{1}{k_{i-1}} s_{i-1} - \frac{1}{k_i} s_i + \frac{1}{k_{i+1}} s_{i+1} = \frac{1}{2} (k_{i-1} + k_i) (a_{i-1} M_i + b_i M_i + c_i M_{i+1}). \]  

In case of the equidistant mesh mentioned above it is

\[ s_{i-1} - 2s_i + s_{i+1} = \frac{1}{8} h^2 (M_{i-1} + 3M_i + M_{i+1}), \quad i = 1(n-1). \]  

In both cases, we have uniquely determined parameters of such spline \( s \in S(2, \Delta x) \) for any \( M_i, \quad i = 0(1)n \). Its first derivatives \( m_i = s'(t_i) \) can be found using (35) or (36); values \( s(x) \) through (37).

We can similarly make an analysis of error propagation in system (38) in case of equidistant mesh. For example, when the value \( s_0, s_n \) are perturbed by errors \( e_0, e_n \), the error \( e_j \) in value \( s_j \) is given by

\[ e_j = e_0 + j(e_n - e_0)/n. \]  

It means that an isolated error is propageted with very slow damping with growing distance from the place of perturbation.
4.3 When the boundary conditions \( s(x_0), s(x_{n+1}) \) are given on the mesh with \( x_0 < t_0, t_n < x_{n+1} \), we can apply

\[
s(t_0) = s(x_0) - m_0(x_0 - t_0) - \frac{1}{2} M_0(x_0 - t_0)^2
\]

with \( m_0 \) given by (35). In this way we obtain relation between \( s(t_0) \) and \( M_0, M_1 \), which we add to the system (38) or (39) as the first equation. Similarly we obtain the last equation from the condition on \( s(x_{n+1}) \); alltogether we have now \( n+1 \) equations for \( n+1 \) unknowns \( s_i = s(t_i), i = 0(1)n \).

But the zero sum of coefficients in all rows of the matrix of the system indicates its singularity. It means, we have no solution of this system for general data \( M_1 \) and boundary conditions of this kind - such a spline \( s \in S(2, \Delta x) \) need not exist!

We can also try some another alternative for free parameters:

- \( s(x_0), s'(x_0) \) (more generally: \( s(x_k), s'(x_k) \));
- \( s(x_j), s'(x_k) \);
- \( s(t_j), s'(t_k) \).

For example, by given value of \( s'(x_k) \) the polygon \( s_1(x) = s(x) \) with \( M_1 = s_1'(t_1) \) is uniquely determined. Integrating \( s_1(x) \), the constant of integration is determined by any value \( s(x_j) \). For calculation of \( s_1, m_1 \) we can then use (34), (35).

REFERENCES


[7] M a k a r o v, V.L. and Ch l o b y s t o v, V.V.: Spline-Approximation of Functions (in Russian), Nauka, Moscow, 1983.

[8] S a l l a m, S. and E l - T a r a z i, M.N.: Quadratic spline interpolation on uniform meshes, in [9], 145-150.


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