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NATURAL TRANSFORMATIONS OF THE SECOND TANGENT FUNCTOR AND SOLDERED MORPHISMS

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Abstract. In [3], all natural transformations of the second order prolongation functor TT into itself were found. We shall show here another method of obtaining similar results using $GL(V)$-equivariant maps of double vector space $VxVxV$ with TT-soldering, and avoiding coordinates where possible.

Key words: Double vector space, double linear morphism, soldering, natural transformation.

MS Classification : 53C05

Given a vector space V, let 1_V denote the identity on V, and $Aut(V)$ the linear automorphisms group of V.

Lemma 1. *Let V be an n-dimensional space over a field K with char $K \neq 2$. Let $f: V \longrightarrow V$ be a map satisfying*

(1) $\qquad \varphi f = f \varphi \qquad$ *for all* $\varphi \in Aut(V)$.

Then there exists a unique $\lambda \in \mathfrak{K}$ such that $f = \lambda . 1_V$.

Proof. First we shall show that f is an endomorphism. By our assumption, f commutes with $\varphi=\mu.1_V$ for $\mu\neq0$, i.e. $f(\mu v)=\mu f(v)$ for all $v\in V$, $\mu\neq0$. This equality holds even for $\mu=0$. Therefore f is homogenous. In the case r=1, f is obviously additive. So suppose $r\geq2$, and assume v_1,v_2 from V linearly independent. We shall prove $f(v_1+v_2)=f(v_1)+f(v_2)$. Choose a basis $\{e_1,\dots,e_n\}$ in V such that $e_1=v_1$, $e_2=v_2$ .We can write $f(v)=\sum_{k=1}^{n}f_k(v)e_k$. Let $i\neq j$ be two different indexis, $i,j\in\{1,\dots,n\}$. We define $\varphi^*\in Aut(V)$ by

$$\varphi^*(e_i)=e_j \quad , \quad \varphi^*(e_j)=e_i \quad , \quad \varphi^*(e_k)=e_k \qquad \text{for } k\neq i,j.$$

Using (1) and comparing the corresponding coefficients in the expressions of $\varphi^*f(e_i)$ and $f\varphi^*(e_i)$ gives

$$f_i(e_j)=f_j(e_i) \quad , \quad f_i(e_i)=f_j(e_j) \ .$$

Similarly, an evaluation of $\varphi^{**}f(e_1)$ and $f\varphi^{**}(e_1)$ where φ^{**} is given by $\varphi^{**}(e_1)=e_1+e_2$, $\varphi^{**}(e_2)=e_1-e_2$, $\varphi^{**}(e_k)=e_k$ for $k\geq2$ yields $f_1(e_1+e_2)=f_1(e_1)+f_1(e_2)$. An application of φ' with $\varphi'(e_1)=-e_1+e_2$, $\varphi'(e_2)=e_1+e_2$, $\varphi'(e_k)=e_k$ for $k>2$ and comparison of $\varphi'f(e_2)$ with $f\varphi'(e_2)$ gives $f_2(e_1+e_2)=f_2(e_1)+f_2(e_2)$. If $n=2$, the proof is finished. Suppose $n>2$, and choose a fixed index $i>2$. Define φ by $\varphi e_1=e_2+e_1$, $\varphi e_2=e_1+e_i$, $\varphi e_i=e_1+e_2$, $\varphi e_k=e_k$ for $k\neq1,2,i$. Comparing coefficients in $\varphi f(e_i)$ and $f\varphi(e_i)$,we find $f_i(e_1+e_2)=f_i(e_1)+f_i(e_2)$ which proves the additivity of f. Hence f is an endomorphism of V commuting with all automorphisms of V.

Now let $v\in V$ be a non-zero vector. Choose a basis $\{e_1,\dots,e_n\}$ in V with $e_1=v$, and define $\varphi\in Aut(V)$ by $\varphi e_1=e_1,\varphi e_k=\nu e_k$ for $\nu\neq0,1$, $\nu\in\mathfrak{K}$, $k\neq1$. Since $\varphi f(e_1)=f\varphi(e_1)$ we have $f_k(e_1)=0$ for $k>1$. Thus there exists a unique function $\lambda:V-\{0\}\longrightarrow\mathfrak{K}$ such that $f(v)=\lambda(v).v$ for all $v\in V$, $v\neq0$. Let $v_1,v_2\in V$ be non-zero vectors, and $\varphi^*\in Aut(V)$ sends v_1 onto v_2. Then we obtain

$$\lambda(v_2)v_2=f(v_2)=f\varphi^*(v_1)=\varphi^*f(v_1)=\lambda(v_1).\varphi^*(v_1)=\lambda(v_1).v_2$$

which proves that λ is a constant function. Since $f(0)=0$ the equality $f(v)=\lambda v$ is fullfilled for all $v\in V$, and the unicity of λ is obvious.

Consider a trivial double vector space $C=A\times B\times C\longrightarrow A\times B$ where A,B,V are finite-dimensional vector spaces over reals.

Any **automorphism** $\varphi \in Aut(C)$ can be **identified** with a quadruple $(\varphi_1, \varphi_2, \varphi_3, \sigma)$ where $\varphi_1 \in Aut(A)$, $\varphi_2 \in Aut(B)$, $\varphi_3 \in Aut(V)$ are the underlying linear morphisms, and $\sigma \in Hom(AxB, V)$ is bilinear. It holds $\varphi(a,b,v)=(\varphi_1(a), \varphi_2(b), \sigma(a,b)+\varphi_3(v))$, [4]. A map $f:C \longrightarrow C$ will be expressed by means of it components f_1, f_2, f_3.

Proposition 1. *Let $f:C \longrightarrow C$ be a continuous map such that*

(2) $$\varphi f = f \varphi \quad \text{for all } \varphi \in Aut(C).$$

Then there are uniquely determined $\lambda, \mu \in \mathfrak{K}$ satisfying

$$f(a,b,v)=(\lambda a, \mu b, \lambda\mu v)=\lambda._1(\mu._2(a,b,v)).$$

Proof.

Using components of f and φ, we rewrite (2) as follows:

(3) $$\varphi_1(f_1(a,b,v))=f_1(\varphi_1(a), \varphi_2(b), \sigma(a,b)+\varphi_3(v)),$$

(4) $$\varphi_2(f_2(a,b,v))=f_2(\varphi_1(a), \varphi_2(b), \sigma(a,b)+\varphi_3(v)),$$

(5) $$\varphi_3(f_3(a,b,v))+\sigma(f_1(a,b,v), f_2(a,b,v))= f_3(\varphi_1(a), \varphi_2(b), \sigma(a,b)+\varphi_3(v)).$$

In (3), let us fix the vectors b, v, and set $\varphi_2=1_B$, $\varphi_3=1_V$, $\sigma=0$. We obtain a map $f_1(-,b,v):A \longrightarrow A$ satisfying the condition (1) of L.1. Hence there is $\lambda(b,v)\in\mathfrak{K}$ such that

$$f_1(a,b,v)=\lambda(b,v).a.$$

This formula defines a continuous function $\lambda:BxV \longrightarrow \mathfrak{K}$.
A substitution $\varphi_1=1_A$, $\sigma=0$ in (3) shows that λ is constant on a dense subset $\{(b,v) \mid b\neq 0, v\neq 0\}$ of BxV. Since λ is continuous, it is constant on the whole BxV. Thus $f_1(a,b,v)=\lambda a$. The existence of μ can be proved similarly. Further, a substitution $\varphi_1=1_A$, $\varphi_2=1_B$, $\sigma=0$ in (5) yields a continuous function $\nu:AxB \longrightarrow \mathfrak{K}$ satisfying $f_3(a,b,v)=\nu(a,b)v$. Using $\varphi_1=1_A$, $\varphi_1=1_B$, $\varphi_3=1_V$ in (5) gives $\nu(a,b)=\lambda\mu$. The unicity of λ, μ is obvious.

The TT-soldered double vector space VxVxV.

Let C, $\pi:C \longrightarrow AxB$ be a double vector space with the kernel V, [4]. A *TT*-soldering on C is a couple of linear isomorphisms

$$\chi_1:V \longrightarrow A \ , \qquad \chi_2:V \longrightarrow B.$$

The *space C* with a *TT*-soldering will be called *TT-soldered*.
A double linear *morphism* $\varphi:C \longrightarrow C'$ of two *TT*-soldered $\mathfrak{D}\mathfrak{L}$-spaces

with TT-solderings χ_1, χ_2, or χ'_1, χ'_2 respectively will be called ***TT-soldered*** if the underlying linear maps φ_1, φ_2, φ_3 satisfy

$$\chi'_1\varphi_3=\varphi_1\chi_1 \quad \text{and} \quad \chi'_2\varphi_3=\varphi_2\chi_2 .$$

From now on, V will denote an n-dimensional vector space over reals, with the usual topology and differentiable structure. Assume a trivial double vector space $C^{\circ}=VxVxV \longrightarrow VxV$ with a TT-soldering $\chi_1=\chi_2=1_V$. A double linear automorphism of C°, φ, is TT-soldered if and only if $\varphi_1=\varphi_2=\varphi_3$. A TT-soldered automorphism of C°, $\Phi=(\varphi,\varphi,\varphi,\sigma)$, will be called *strongly soldered* if the bilinear map σ is symmetric.

In this part, we shall investigate differentiable maps $f:C^{\circ}\longrightarrow C^{\circ}$ commuting with all TT-soldered (or strongly TT-soldered, respectively) automorphisms of C°.

Assume a fixed continuous map $f:V^{m}\longrightarrow V$ commuting with all $\varphi\in Aut(V)$. We shall need some of its properties:

Lemma 2. *Let $v_1,\dots,v_m$ be a set of linearly independent vectors in V. Then there exist uniquely determined reals $f_k(v_1,\dots,v_m)$, $k=1,\dots,m$ such that*

$$f(v_1,\dots,v_m)=\sum_{k=1}^{m} f_k(v_1,\dots,v_m)v_k .$$

Proof.

The unicity is obvious. To prove the existence, choose $\lambda\neq 0$, and consider $\varphi=\lambda.1_V$. By the above assumption,

$$\lambda f(v_1,\dots,v_m)=f(\lambda v_1,\dots,\lambda v_m).$$

Since f is continuous, this equality holds also for $\lambda=0$, i.e. $f(0,\dots,0)=0$. Let us add $n-m$ vectors so as $\{v_1,\dots,v_m,v_{m+1},\dots,v_n\}$ would be a basis in V. We can write

$$f(v_1,\dots,v_m)=\sum_{k=1}^{m} f_k(v_1,\dots,v_m)v_k .$$

Using $\varphi^{*}\in Aut(V)$, $\varphi^{*}(v_k)=\lambda v_k$ with $\lambda\neq 1$ for $k=1,\dots,m$, $\varphi^{*}(v_k)=v_k$ for $k=m+1,\dots,n$, we get

$$f_k(\lambda v_1,\dots,\lambda v_m)=f_k(v_1,\dots,v_m) \quad \text{for } k=m+1,\dots,n.$$

Further, $\quad f_k(v_1,\dots,v_m)=\lim_{\lambda\to 0} f_k(\lambda v_1,\dots,\lambda v_m)=f_k(0,,,,0)=0$

for $k=m+1,\dots,n$ which finishes the proof.

Lemma 3. *There are uniquely determined functions*

$$g_k : V-\{0\} \longrightarrow \mathfrak{K},$$

$k=1,\dots,m$ *such that for linearly independent* $v_1,\dots,v_m \in V$,

$$f(v_1,\dots,v_m) = \sum_{k=1}^{m} g_k(v_k)v_k \quad .$$

Proof .

Let $i\in\{1,\dots,m\}$ be fixed. Let $v_1,\dots,v_m$ and $v'_1,\dots,v'_m$ be two independent sets of vectors from V with $v_i=v'_i$.
By L.2., there are uniquely determined numbers $f_k(v_1,\dots,v_m)$, $f'_k(v'_1,\dots,v'_m)$; $k=1,\dots,m$ satisfying

(6) $$f(v_1,\dots,v_m) = \sum_{k=1}^{m} f_k(v_1,\dots,v_m)v_k \ ,$$

$$f(v'_1,\dots,v'_m) = \sum_{k=1}^{m} f'_k(v'_1,\dots,v'_m)v'_k .$$

Applying suitable automorphisms we get

$$f'_i(v'_1,\dots,v'_i,v_i,v'_{i+1},\dots,v'_m) = f_i(v_1,\dots,v_i,\dots,v_m) =$$

$$= f_i(0,\dots,v_i,\dots,0) = 0 \quad .$$

Let $v\in V-\{0\}$. Choose a linearly independent set $v_1,\dots,v_m$ in V with $v_i=v$. We can use (6) to define the function

$$g_i(v) = f_i(v_1,\dots,v_m),$$

$i=1,\dots,m$ having the required properties.

Proposition 2. *Let* $m\le\dim V$, *and let* $f: V^m \longrightarrow V$ *satisfy*

(7) $$\varphi f = f\varphi \qquad \text{for all} \quad \varphi\in Aut(V) \quad .$$

Then there exist unique $\lambda_1,\dots,\lambda_m \in \mathfrak{K}$ *such that*

(8) $$f(v_1,\dots,v_m) = \sum_{k=1}^{m} \lambda_k v_k \quad \text{for any} \quad v_1, \dots, v_m \in V.$$

Proof. Choose $v\in V-\{0\}$, and $v_1,\dots,v_m\in V$ independent with $v_i=v$. By L.2. and (7) , $g_i(\varphi(v)) = g_i(v)$ *for any* $\varphi\in Aut(V)$. Therefore $g_i : V-\{0\} \longrightarrow \mathfrak{K}$ is a constant function with a value denoted by λ_i, and the equality (8) holds for any independent set $v_1,\dots,v_m$. By continuity, this formula is true for any m-tuple from V^m.

The above proposition does not involve some useful cases as $\dim V=1$, $m=2,3$, or $\dim V=2$, $m=3$. So we shall slightly modify our assumptions.

Proposition 3. *Let $f:V^m \longrightarrow V$ satisfy* (7), *and has a differential at a point $0\in V^m$. Then there exist uniquely determined reals $\lambda_1,\dots,\lambda_m$ satisfying* (8).

Proof. Since f has a differential at 0, we can write

$$f(v_1,\dots,v_m)=(Tf)_0(v_1,\dots,v_m)+g(v_1,\dots,v_m)$$

where

$$\lim_{v\to 0}\frac{g(v)}{||v||}=0,\quad v=(v_1,\dots,v_m),$$

and $||\ ||$ is any norm on V^m.

For $\lambda\neq 0$, $\lambda f(v_1,\dots,v_m)=\lambda(Tf)_0(v_1,\dots,v_m)+\lambda g(v_1,\dots,v_m)$. On the other hand, $\lambda f(v_1,\dots,v_m)=f(\lambda v_1,\dots,\lambda v_m)=(Tf)_0(\lambda v_1,\dots,\lambda v_m)$ $+g(\lambda v_1,\dots,\lambda v_m)=\lambda(Tf)_0(v_1,\dots,v_m)+g(\lambda v_1,\dots,\lambda v_m)$.

Hence $\lambda g(v_1,\dots,v_m)=g(\lambda v_1,\dots,\lambda v_m)$. Further, for any $v\neq 0$,

$$0=\lim_{\lambda\to 0^+}\frac{g(\lambda v)}{||\lambda v||}=\lim_{\lambda\to 0^+}\frac{\lambda g(v)}{\lambda||v||}=\frac{g(v)}{||v||}$$

which implies $g(v)=0$ for any $v\neq 0$. Since g has a differential at 0, it is continuous, and $g(0)=0$. Therefore $f=(Tf)_0$, f is linear, and $f(v_1,\dots,v_m)=\sum_{i=1}^{m} g_i(v_i)$ where $g_i:V\longrightarrow V$ are given by $g_i(v)=f(0,\dots,v_i,\dots,0)$, $i=1,\dots,m$. By (7), any g_i commutes with all automorphisms of V, and by **L.1.**, there exists $\lambda_i\in\mathfrak{R}$ such that $g_i(v)=\lambda_i v$ for any $v\in V$, i.e. (8) is satisfied. The unicity is obvious.

Let us return to our problem. Among the maps $f:C^o\longrightarrow C^o$ having differential at $0\in C^o=VxVxV$, we shall distinguish such ones that commute with all soldered (or strongly soldered, resp.) automorphisms $\Phi=(\varphi,\varphi,\varphi,\sigma)$ of C^o. The equality $\Phi f=\Phi f$ can be rewritten by means of components in the form (3), (4), (5) with $\varphi_i=\varphi$, $i=1,2,3$. If we choose $\sigma=0$, **Prop. 3.** guarantees the existence of a set of real numbers λ_{ij}; $i,j=1,2,3$ satisfying (8) with $m=3$. A substitution of (8) into (3) and (4) gives

$$\lambda_{13}\sigma(v_1,v_2)=0,\qquad \lambda_{23}\sigma(v_1,v_2)=0$$

for any bilinear (or symmetric bilinear, respectively) map $\sigma:VxV\longrightarrow V$; therefore $\lambda_{13}=0$, $\lambda_{23}=0$. Similarly, substituing (8) into (5), we obtain

(9) $\qquad \lambda_{11}\lambda_{21}=0,\quad \lambda_{12}\lambda_{22}=0,\quad \lambda_{11}\lambda_{22}=\lambda_{33},\quad \lambda_{12}\lambda_{21}=0$

(and $\lambda_{11}\lambda_{22}+\lambda_{12}\lambda_{21}=\lambda_{33}$, respectively).

Obviously,(9) can be fullfilled in three (or four, resp.) ways:

I. If $\lambda_{11}=\lambda_{12}=0$, then f is of the form $f_1(v_1,v_2,v_3)=0$, $f_2(v_1,v_2,v_3)=\lambda_{21}v_1+\lambda_{22}v_2$, $f_3(v_1,v_2,v_3)=\lambda_{31}v_1+\lambda_{32}v_2$.

II. If $\lambda_{12}=\lambda_{21}=0$, then $f_1(v_1,v_2,v_3)=\lambda_{11}v_1$, $f_2(v_1,v_2,v_3)=\lambda_{22}v_2$, $f_3(v_1,v_2,v_3)=\lambda_{31}v_1+\lambda_{32}v_2+\lambda_{11}\lambda_{22}v_3$.

III. In the case $\lambda_{21}=\lambda_{22}=0$, we have $f(v_1,v_2,v_3)=\lambda_{11}v_1+\lambda_{12}v_2$, $f_2(v_1,v_2,v_3)=0$, $f_3(v_1,v_2,v_3)=\lambda_{31}v_1+\lambda_{32}v_2$.

(IV. If $\lambda_{11}=\lambda_{22}=0$, then f is of the form $f_1(v_1,v_2,v_3)=\lambda_{12}v_2$, $f_2(v_1,v_2,v_3)=\lambda_{21}v_1$, $f_3(v_1,v_2,v_3)=\lambda_{31}v_1+\lambda_{32}v_2+\lambda_{12}\lambda_{21}v_3$.)

On the set $Z(C^{\circ})$ of all differentiable maps of the double linear space $C^{\circ}=VxVxV$ into itself, we can define usual composition, and addition in the following cases:

$$f +_1 g \quad \text{if } \pi_1 f=\pi_1 g, \qquad f +_2 g \quad \text{if } \pi_2 f=\pi_2 g \ ,$$

$$f+g \quad \text{if } g(C^{\circ})\subset V, \quad f,g\in Z(C^{\circ}) \ .$$

Denote by $Z_s(C^{\circ})$ (or $Z_{ss}(C^{\circ})$,respectively) the subset of all $f\in Z(C^{\circ})$ satisfying $\Phi f=f\Phi$ for any TT-soldered (or strongly TT-soldered) double linear automorphism $\Phi: C^{\circ} \longrightarrow C^{\circ}$. $Z_s(C^{\circ})$ *as well as* $Z_{ss}(C^{\circ})$ are closed with respect to the above operations. It can be verified the following:

Proposition 4. *By means of the above operations, the set* $Z_s(C^{\circ})$ (or $Z_{ss}(C^{\circ})$, respectively) *is generated by the following maps:*

(10) $(v_1,v_2,v_3) \longrightarrow \lambda._1(\mu._2(v_1,v_2,v_3))$, $\lambda,\mu\in\mathfrak{R}$

(11) $(v_1,v_2,v_3) \longrightarrow (v_1+v_2,0,0)$

(12) $(v_1,v_2,v_3) \longrightarrow (0,v_1+v_2,0)$

(13) $(v_1,v_2,v_3) \longrightarrow (0,0,v_1+v_2)$

(and (14) $(v_1,v_2,v_3) \longrightarrow (v_2,v_1,v_3)$).

The **maps** of the type (10) **commute** even with all $\mathcal{DL}$-**automorphisms**.

Natural transformations of TT into itself.

The second order lifting functor TT will be here regarded as a covariant functor from the category of n-dimensional differentiable manifolds and their diffeomorphisms to the category of fibred manifolds and morphisms. TT assigns a double linear fibration TTM to a differentiable manifold M, and for any diffeomorphism $\varphi: M \longrightarrow N$, the assigned map $TT\varphi: TTM \longrightarrow TTN$ is a double linear morphism. All three underlying vector fibrations are identified with TM.

Consider a natural transformation $\psi: TT \longrightarrow TT$. Let $\alpha: \mathfrak{R}^n \longrightarrow \mathfrak{R}^n$ be a diffeomorphism with $\alpha(0)=0$. The space $TT_0\mathfrak{R}^n$ is canonically $\mathcal{DL}$-isomorphic with the trivial $\mathcal{DL}$-space $\mathfrak{R}^n x \mathfrak{R}^n x \mathfrak{R}^n$. The map $TT_0\alpha$ regarded as a double linear automorphism has the components

(15) $$TT_0\alpha = (T_0\alpha, T_0\alpha, T_0\alpha, \dot{\sigma}_\alpha)$$

where $T_0\alpha$ is a differential of α at $0 \in \mathfrak{R}^n$, and σ_α is its second differential at 0. Clearly, (15) is a strongly soldered $\mathcal{DL}$-automorphism of the trivial $\mathcal{DL}$-space $\mathfrak{R}^n x \mathfrak{R}^n x \mathfrak{R}^n$, and it depends only on the 2-jet of α at 0. This fact enables us to define a map $\nu: L^2_n \longrightarrow Aut_o(\mathfrak{R}^n x \mathfrak{R}^n x \mathfrak{R}^n)$ by $\nu(j^2_0\alpha) = (T_0\alpha, T_0\alpha, T_0\alpha, \sigma_\alpha)$ where L^2_n denotes the group of all invertible 2-jets (2-jets of local diffeomorphisms) on $\mathfrak{R}^n$ with source and target 0, and Aut_o is the group of all strongly soldered $\mathcal{DL}$-automorphisms. It can be verified that L^2_n is a semidirect product of L^1_n and the Abelian group $Hom_s(\mathfrak{R}^n x \mathfrak{R}^n, \mathfrak{R}^n)$ of all symmetric bilinear maps; $j^2_0\alpha$ corresponds to the couple $(T_0\alpha, \sigma_\alpha)$. Expressing L^2_n via this semidirect product, we find that ν is a group isomorphism.

The following diagram is commutative :

$$\begin{array}{ccc} TT_0\mathfrak{R}^n & \xrightarrow{\;TT_0\alpha\;} & TT_0\mathfrak{R}^n \\ \psi_0\mathfrak{R}^n \big\downarrow & & \big\downarrow \psi_0\mathfrak{R}^n \\ TT_0\mathfrak{R}^n & \xrightarrow{\;TT_0\alpha\;} & TT_0\mathfrak{R}^n . \end{array}$$

Therefore $\psi_0\mathfrak{R}^n$ commutes with all strongly soldered

$\mathcal{DL}$-automorphisms of the $\mathcal{DL}$-space $TT_0\mathfrak{R}^n$, i.e. $\psi_0\mathfrak{R}^n \in Z_{ss}(TT_0\mathfrak{R}^n)$. Further, any natural transformation ψ is fully determined by $\psi_0\mathfrak{R}^n$. In fact, choose a map $\varphi: U \longrightarrow \mathfrak{R}^n$ in a neighborhood U of $x\in M$ with $\varphi(x)=0$. Then the diagram

$$\begin{array}{ccc} TT_xM & \xrightarrow{\;\;TT_x\varphi\;\;} & TT_0\mathfrak{R}^n \\ \psi_xM \big\downarrow & & \big\downarrow \psi_0\mathfrak{R}^n \\ TT_xM & \xrightarrow{\;\;TT_x\varphi\;\;} & TT_0\mathfrak{R}^n \end{array}$$

commutes which proves our assertion.

Finally, if $f\in Z_{ss}(TT_0\mathfrak{R}^n)$ there exists a natural transformation Ψ: $TT \longrightarrow TT$ such that $\psi_0\mathfrak{R}^n = f$. We define $\psi_xM=(TT_x\varphi)^{-1}.f.(TT_x\varphi)$ where φ is a map chosen as above. The map ψM:: $TTM \longrightarrow TTM$ coinciding with ψ_xM on the fibre over $x\in M$ is differentiable, independent of the choice of φ, and satisfies $\psi_0\mathfrak{R}^n=f$. So we have proved:

Proposition 5. *There exists a bijective correspondence between all natural transformations of the functor TT into TT and the set* $Z_{ss}(TT_0\mathfrak{R}^n)$.

Proposition 6. *Using the operation of composition and the operation + (the action of the vector fibration $V=TM$ on the affine fibration TTM, [5]), the set of all natural transformations of TT into itself is generated by the following natural transformations:*

(16) $X\in T_y(TM) \longrightarrow \lambda._1(\lambda'._2X)\in T_{\lambda._2y}(TM)$, $\lambda,\lambda'\in\mathfrak{R}$,

(17) $X\in T_y(TM) \longrightarrow 0\in T_{y+Tp(X)}(TM)$ *where* 0 *is a zero vector,* $Tp: TTM \longrightarrow TM$ *is a tangent map of the natural projection* $p: TM \longrightarrow M$,

(18) $X\in T_y(TM) \longrightarrow (To_M)_x(y+Tp(X))$ *where* $x=p(y)$, *and* o_M *denotes a zero section of the vector fibration TM* ,

(19) $X\in T_y(TM) \longrightarrow e_M(y+Tp(X))\in T_0(T_xM)$ *where* $x\in p(y)$, *and*

$e_M: T_x M \longrightarrow T_0(T_x M)$ *is a canonical isomorphism,*

(20) $X \in T_y(TM) \longrightarrow i_M X \in T_{Tp(X)}(TM)$ *where* i_M *denotes a canonical involution on* TTM.

Proof. By **Prop. 5.**, the set of all natural transformations of TT into itself is generated by the natural transformations corresponding to the generators of $Z_{ss}(TT_0\mathbb{R}^n)$ described in **Prop. 4.** An evaluation in local coordinates shows that the transformations from (16)-(20) correspond respectively to the maps given by the formulas (10)-(14).

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