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SOLID SUBGROUPS OF WEAKLY ASSOCIATIVE LATTICE GROUPS

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Abstract. In the paper some properties of weakly associative lattice ordered groups (*wal*-groups) are shown, solid, prime and straightening subgroups of *wal*-groups are studied and transitive *wal*-groups are characterized.

Key words: Weakly associative lattice group, solid subgroup, transitive *wal*-group.

MS Classification: 06F15, 06F99

The notion of a weakly associative lattice group (*wal*-group) is a generalization of that of a lattice ordered group (*l*-group) in which a weakly associative lattice is used instead of a lattice. In the paper some properties of *wal*-groups are shown, solid, prime, and straightening subgroups of *wal*-groups are studied, and transitive *wal*-groups are characterized.

1. Basic properties

A *weakly associative lattice* (*wa-lattice*) is an algebra

$A = (A, \wedge, \vee)$ with two binary operations such that

1. $\forall a \in A;\ a \wedge a = a\ ,\ a \vee a = a\ ;$
2. $\forall a,b \in A;\ a \wedge b = b \wedge a\ ,\ a \vee b = b \vee a\ ;$
3. $\forall a,b \in A;\ a \wedge (a \vee b) = a\ ,\ a \vee (a \wedge b) = a\ ;$
4. $\forall a,b,c \in A;\ ((a \wedge c) \vee (b \wedge c)) \vee c = c\ ,\ ((a \vee c) \wedge (b \vee c)) \wedge c = c\ .$

Define a binary relation " $\leq$ " on A as follows:

$\forall a,b \in A;\ a \leq b \Leftrightarrow_{df} a \wedge b = a$ (or equivalently $a \leq b \Leftrightarrow_{df} a \vee b = b$).

Then it holds:

5. $\forall a \in A;\ a \leq a\ ;$
6. $\forall a,b \in A;\ a \leq b\ \&\ b \leq a \Rightarrow a = b\ ;$
7. $\forall a,b \in A \exists d \in A;\ (a \leq d\ \&\ b \leq d)\ \&\ (\forall u \in A;\ (a \leq u\ \&\ b \leq u) \Rightarrow d \leq u)\ ;$
8. $\forall a,b \in A \exists e \in A;\ (e \leq a\ \&\ e \leq b)\ \&\ (\forall v \in A;\ (v \leq a\ \&\ v \leq b) \Rightarrow v \leq e)\ .$

It is also true that if a relation "$\leq$ " satisfies the conditions 5 - 8 and if we denote d by $a \vee b$ and e by $a \wedge b$, then the algebra $(A, \wedge, \vee)$ satisfies the conditions 1 - 4. (See [3].)

If a binary relation " $\leq$ " on A satisfies the conditions 5 and 6, then " $\leq$ " is called a *semi-order* on A and $(A, \leq)$ is called a *semi-ordered set* (*so-set*). If a semi-ordered set $(A, \leq)$ satisfies the condition 7, then it is called a $\vee$-*semilattice-ordered set* ($\vee$-*wa-semilattice*). A semi-ordered set $(A, \leq)$ is said to be a *tournament* if any elements $a,b \in A$ are comparable.

A system $G=(G,+,\leq)$ is called a *semi-ordered group* (*so-group*) if

a) $(G,+)$ is a group;

b) $(G,\leq)$ is a so-set;

c) $\forall a,b,c,d \in G;\ a \leq b \Rightarrow c+a+d \leq c+b+d\ .$

If $(G,\leq)$ is a *wa*-lattice, then we say that $(G,+,\leq)$ is a *weakly associative lattice group* (*wal-group*). If $(G,\leq)$ is a lattice, then $(G,+,\leq)$ is said to be a *lattice ordered group* (*l-group*).

(For necessary results concerning ordered groups and *l*-groups see e.g. [1], for some properties of *so*-groups and *wal*-groups see [2].)

Let G be a *so*-group. Denote $G^+ = \{x \in G;\ 0 \leq x\}$. Then G^+ will be called the *positive cone* of G. Evidently we have

Proposition 1.1. *If G is a wal-group, then G is an l-group if and only if G^+ is a subsemigroup of G.* □

The proofs of the following propositions are (formally) the same as the proofs of the analogical propositions for l-groups in [1], and then they are omitted.

Proposition 1.2. *Let G be a so-group. Then for any $a,b,c,d\in G$ it holds:*

(a) *If $b\vee c$ exists, then $(a+b+d)\vee(a+c+d)$ exists and $a+(b\vee c)+d = (a+b+d)\vee(a+c+d)$.*

(b) *If $b\wedge c$ exists, then $(a+b+d)\wedge(a+c+d)$ exists and $a+(b\wedge c)+d = (a+b+d)\wedge(a+c+d)$.*

(c) *If $a\wedge b$ exists, then $-a\vee -b$ exists and $-a\vee -b = -(a\wedge b)$.* □

Proposition 1.3. *If G is a so-group, $a,b\in G$, and if $a\vee b$ exists, then $a\wedge b$ exists, too, and $a\wedge b = b+(-(a\vee b))+a$.* □

Corollary 1.4. *If $(G,+,\le)$ is a $\vee$-semilattice semi-order, then the following conditions are equivalent.*

(a) *G is a wal-group.*

(b) *$\forall a,b,c,d\in G;\ a+(b\vee c)+d = (a+b+d)\vee(a+c+d)$.* □

Proposition 1.5. *Let for elements a,b in a so-group G $a\wedge b$ exist. Let $a = x+(a\wedge b)$, $b=y+(a\wedge b)$, $c = a-b$. Then*

$$x\wedge y = 0\ ,\ x-y = c\ ,\ x = c\vee 0\ ,\ y = -c\vee 0\ .$$ □

Proposition 1.6. *If G is a so-group, then the following conditions are equivalent:*

(a) *G is a wal-group.*

(b) *$\forall a\in G\exists x,y\in G;\ a = x-y\ ,\ x\wedge y = 0$.*

(c) *$\forall a\in G;\ a\vee 0$ exists.* □

Example 1.1. Denote $(G,+) = (\mathbb{Z},+)$, $G^+ = \{0,1,2,4,\dots\}$. It is evident that G^+ is the positive cone of a semi-order of the group G. If $x\in G$, then it holds:

a) $x\in G^+ \Rightarrow x\vee 0 = x$;

b) $-x\in G^+ \Rightarrow x\vee 0 = 0$;

c) $x\notin G^+, -x\notin G^+ \Rightarrow x\vee 0 = \max\{x,0\}+1$, where $\max\{x,0\}$ is meant in the natural ordering of $\mathbb{Z}$.

Denote " $\le$ " the semi-order defined by G^+. Then, by Proposition 1.6, $(G,+,\le)$ is a *wal*-group. Note that G is neither an *l*-group nor a *to*-group.

Proposition 1.7. *A wal-group G is a to-group if and only if*

$$\forall a,b\in G;\ a\wedge b = 0 \Rightarrow a=0 \text{ or } b=0. \qquad \square$$

Proposition 1.8. *For any so-group G, the following conditions are equivalent:*

(a) *G is a wal-group.*

(b) *G is directed (i.e. for each $x,y\in G$ there exists $z\in G$ such that $x,y\le z$) and for each $a,b\in G^+$ there exists their infimum in G (that belongs to G^+).*

Proof. a⇒b: Evident.

b⇒a: Let $a,b\in G$. Then there exists $c\in G$ such that $c\le a,b$, i.e. $0\le -c+a,\ -c+b$. Hence there exists $(-c+a)\wedge(-c+b)=d$, too. Therefore $c+d\le a,b$. Let $h\le a,b$. Then $-c+h\le -c+a,\ -c+b$, and thus $-c+h\le d$. That means $h\le c+d$, and so $c+d=a\wedge b$. □

Remark 1.1. If G is a *wal*-group, then G^+ need not be a $\vee$-*wa*-subsemilattice. For instance, if G is the *wal*-group in Example 1.1, then $1\vee 4=5$ in G, but $5\notin G^+$.

Remark 1.2. In a *wal*-group G the identity

$$a\wedge(b\vee c) = (a\wedge b)\vee(a\wedge c)$$

need not be satisfied in general. For example, let us consider the group $\mathbb{Z}_3 = \{0,1,2\}$ with the addition mod 3. Let $\mathbb{Z}_3$ be semi-ordered as a tournament such that $0<1$, $1<2$, $2<0$. Then

$$0\wedge(1\vee 2) = 0\wedge 2 = 2,\ (0\wedge 1)\vee(0\wedge 2) = 0\vee 2 = 0 .$$

Nevertheless we have the following proposition.

Proposition 1.9. *If G is a wal-group, then*

$$\forall a,b,c\in G;\ (a\vee c=b\vee c \ \&\ a\wedge c=b\wedge c) \Rightarrow a=b .$$

Proof. $a = 0+a = [(a\vee c)-(a\vee c)]+a = [(a\vee c)+(-c+(a\wedge c)-a)]+a = (a\vee c)-c+(a\wedge c) = (b\vee c)-c+(b\wedge c) = [(b\vee c)-(b\vee c)]+b = b$. □

Proposition 1.10. *A wal-group G is an l-group if and only if*

$$\forall a,b,c\in G;\ a\wedge(b\vee c) = (a\wedge b)\vee(a\wedge c) .$$

Proof. Let $a,b,c\in G$, $a\le b$, $b\le c$. Then

$$a\wedge(b\vee c) = a\wedge c , \quad (a\wedge b)\vee(a\wedge c) = a\vee(a\wedge c) = a ,$$

and if the considered condition is satisfied, we have $a\wedge c = a$, and so $a\le c$.

The converse implication is trivial. □

Theorem 1.11. *Let G be a wal-group, let $a_1,\dots a_m,b_1,\dots b_n\in G^+$ and let $a_1+\dots+a_m = b_1+\dots+b_n$. Then there exist elements $c_{ij}\in G^+$, $i=1,\dots,m$, $j=1,\dots,n$, such that*

$$a_i = \sum_{j=1}^{n} c_{ij} \quad , \quad b_j = \sum_{i=1}^{m} c_{ij} .$$ □

Now we get the following proposition as a consequence.

Proposition 1.12. *If $a,b_1,\dots,b_n\in G^+$ are such that $a\le b_1+\dots+b_n$, then there exist $a_1,\dots,a_n\in G^+$ such that $a_i\le b_i$ $(i=1,\dots n)$ and $a=a_1+\dots+a_n$.*

□

We say that elements $a,b\in G^+$ are *orthogonal* (denote: $a\perp b$) if $a\wedge b = 0$.

Proposition 1.13. *If $a,b\in G^+$, then $a\perp b$ if and only if*

$$a+b = a\vee b .$$ □

Proposition 1.14. *If $a,b\in G^+$ and $a\perp b$, then $a+b = b+a$.* □

Proposition 1.15. *If G is a wal-group, then the following conditions are equivalent:*

(a) *G is an l-group.*

(b) $\forall a,b,c\in G;\ a\perp b\ \&\ c\ge 0 \Rightarrow a\wedge c = a\wedge(b+c)$.

(c) $\forall a,b,c\in G;\ a\perp b\ \&\ a\perp c \Rightarrow a\perp(b+c)$.

Proof. The conditions (b) and (c) are satisfied in any l-group.

b⇒a, c⇒a: It is evident that for $x \in G$, $0 \perp x$ if and only if $0 \le x$. Hence, if (b) or (c) is true, then G^+ is a subsemigroup of G, and so G is an l-group.

2. PRIME SUBGROUPS AND STRAIGHTENING SUBGROUPS

Let $(G,+,\le)$ and $(G',+,\le)$ be *so*-groups. A mapping $\varphi: G \to G'$ is called a *homomorphism of so-groups* (*so-homomorphism*), if φ is simultaneously a group homomorphism of $(G,+)$ into $(G',+)$ and a *so*-homomorphism of $(G,\le)$ into $(G'\le)$ (i.e. $a \le b$ implies $\varphi(a) \le \varphi(b)$ for any $a,b \in G$).

If $(G,+,\le)$ and $(G',+,\le)$ are *wal*-groups and if φ is a *so*-homomorphism of $(G,+,\le)$ into $(G',+,\le)$ which is also a *wa*-lattice homomorphism, then φ is called a *homomorphism of wal-groups* (*wal-homomorphism*).

Let $(G,+,\le)$ be a *wal*-group and A a subgroup of G. Then A is said to be a *wal-subgroup* of G, if A is a *wa*-sublattice of $(G,\le)$. If a normal convex *wal*-subgroup A satisfies the condition:

(*) *For any $a,b \in A$, $x,y \in G$ such that $x \le a$, $y \le b$, there exists $c \in A$ such that $x \vee y \le c$,*

then A is called a *wal-ideal* of G.

It is proved (in [2]) that exactly all normal convex subgroups are kernels of *so*-homomorphisms and exactly all *wal*-ideals are kernels of *wal*-homomorphisms.

Lemma 2.1. *A normal convex wal-subgroup A of a wal-group G is a wal-ideal of G if and only if*

(**) $\qquad \forall a,b,c \in A, x,y \in G;\ x \le a,\ y \le b \Rightarrow (x \vee y) \vee c \in A.$

Proof. Let A be a *wal*-ideal, $x,y \in G$, $a,b,c \in A$, $x \le a$, $y \le b$. Then A is the kernel of some *wal*-homomorphism $\varphi: G \to G'$, and it holds

$$\varphi((x \vee y) \vee c) = \varphi(x \vee y) \vee \varphi(c) = \varphi(x \vee y) \vee 0',$$

where $0'$ is the zero-element in G'. But $\varphi(x) \le \varphi(a) = 0'$, $\varphi(y) \le \varphi(b) = 0'$, hence $\varphi(x \vee y) = \varphi(x) \vee \varphi(y) \le 0'$, and thus $\varphi((x \vee y) \vee c) = 0'$. Therefore $(x \vee y) \vee c \in A$.

Conversely, let a normal convex *wal*-subgroup A of G satisfy the condition (**) and let $a,b,c\in A$, $x,y\in G$, $x\le a$, $y\le b$. Then there exists $d\in A$ such that $(x\vee y)\vee c=d$, and so $x\vee y\le d$. Therefore A is a *val*-ideal of G. □

If A is a convex *val*-subgroup of G satisfying the condition (**), then A will be called a *solid subgroup* of G.

Denote by $\mathcal{L}(G)$ the set of all *val*-ideals and by $\mathcal{C}(G)$ the set of all solid subgroups of a *val*-group G. It is evident that, by means of set inclusion, $\mathcal{L}(G)$ and $\mathcal{C}(G)$ form complete lattices with the least element $\{0\}$ and the greatest element G and that infima are formed, in both cases, by set intersections.

Remark 2.1. Let G be a *so*-group, A a convex subgroup of G and $G/_1A$ the set of all left cosets modulo A. Put

$$x+A \le y+A \ \Leftrightarrow_{df}\ \exists a\in A;\ x+a \le y ,$$

for any $x,y\in G$. Then " $\le$ " is a semi-order on $G/_1A$.

Let G be a *val*-group and $H\in\mathcal{C}(G)$. Consider the following conditions for H.

(1) *If $x,y\in G$ and $0\le x\wedge y\in H$, then $x\in H$ or $y\in H$.*

(2) *If $x,y\in G$ and $x\wedge y=0$, then $x\in H$ or $y\in H$.*

(3) *$G/_1H$ is a tournament semi-ordered set.*

(4) *$\{A\in\mathcal{C}(G);\ H\subseteq A\}$ is a linearly ordered set.*

(5) *If $A,B\in\mathcal{C}(G)$ and $A\cap B=H$, then $A=H$ or $B=H$.*

Theorem 2.2. *If H is a solid subgroup of a val-group G, then* (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) .

Proof. (1) $\Rightarrow$ (2): Trivial.

(2) $\Rightarrow$ (3): Let $a+H, b+H \in G/_1H$. By Proposition 1.5, there exist $x,y\in G$ such that $a=(a\wedge b)+x$, $b=(a\wedge b)+y$, $x\wedge y=0$. If $x\in H$, then $a+H = ((a\wedge b)+x)+H = (a\wedge b)+H \le b+H$. If $y\in H$, then $b+H\le a+H$. Thus $G/_1H$ is a tournament.

(3) $\Rightarrow$ (1): Let $G/_1H$ be a tournament, $a,b\in G\setminus H$, $0\le a\wedge b$. By the assumption, $a+H$ and $b+H$ are comparable. If, for example, $a+H\le b+H$, then $(a\wedge b)+H = (a+H)\wedge(b+H) = a+H$, and hence $a\wedge b\notin H$.

(3) $\Rightarrow$ (4): Let $A,B\in\mathcal{C}(G)$, $H\subseteq A$, $H\subseteq B$ and $A\not\subseteq B$. Since (by [2, Theorem 3]) every *val*-subgroup of G is generated by its positive elements, there exists $0\le x\in A\setminus B$. Let $0\le b\in B$. If $x+H\le b+H$, then

there exists $h\in H$ such that $x+h\leq b$, i.e. $x\leq b-h$. Since $0\leq x\leq b-h\in B$, we get $x\in B$, a contradiction. Hence $b+H\leq x+H$, that means there exists $k\in H$ such that $b+k\leq x$. Then $0\leq b\leq x-k\in A$. Therefore $B^{+}\subseteq A$, and because A and B are *val*-subgroups, we have $B\subseteq A$.

(4) $\Rightarrow$ (5): Trivial. □

A solid subgroup H of a *val*-group G satisfying the conditions (1), (2) and (3) will be called a *straightening subgroup* of G.

If a solid subgroup H of a *val*-group G satisfies the condition (5), then H is said to be a *prime subgroup* of G.

Remark 2.2. It is well known (see e.g. [1, Théorème 2.4.1]) that for solid subgroups of an *l*-group all conditions (1)-(5) are equivalent.

But for *val*-groups this equivalence generally is not true, because there exist prime subgroups of *val*-groups not being straightening.

For example, let G be the direct product $\mathbb{Z}\times\mathbb{Z}$, where $(\mathbb{Z},+)$ is semi-ordered by the same semi-order as in Example 1.1, i.e. $(\mathbb{Z},+) = \{0,1,2,4,6,\dots\}$. G is, as a direct product of *val*-groups, a *val*-group. Denote $H = \{(x,0);\ x\in\mathbb{Z}\}$. Evidently, H is a *val*-ideal of G.

H is not a straightening subgroup, because, for example, $(1,4)\wedge(4,1) = (0,0)$ but neither $(1,4)$ nor $(4,1)$ belongs to H.

Let $A\in\mathcal{C}(G)$, let H be a proper subgroup of A and let $(a_1,a_2)\in A\setminus H$. Then $a_2\neq 0$ and $(0,a_2)=(a_1,a_2)-(a_1,0)\in A$. Since the convex subgroup of $\mathbb{Z}$ generated by a_2 is equal to $\mathbb{Z}$, we get $(x_1,x_2)=(x_1,0)+(0,x_2)\in A$ for any element (x_1,x_2) in G, hence $A=G$.

Therefore A is a prime subgroup of G that is not straightening.

Let G be a *val*-group, $0\neq a\in G$, $H\in\mathcal{C}(G)$. We say that H is a ***value of a*** if H is a maximal solid subgroup of G not containing **a**. (The set of all values of an element **a** will be denoted by **val(a)**.)

A solid subgroup H **of** G **is said to be** *regular* **if** $H = \cap(A_i;\ i\in I)$ $(A_i\in\mathcal{C}(G))$ **implies the existence of an** $i_0\in I$ **such that** $H=A_{i_0}$. **(Evidently every regular subgroup is prime.)**

Proposition 2.3. *$H\in\mathcal{C}(G)$ is regular if and only if there exists $a\in G$ such that $H\in val(a)$.*

Proof is the same as in [1, Propositon 2.5.3] and then it is omitted. □

Theorem 2.4. *If $H\in\mathcal{C}(G)$ and $a\in G\setminus H$, then there exists $C\in val(a)$ such that $H\subseteq C$.*

Proof. Let $(A_i;\ i\in I)$ be a linearly ordered system of solid subgroups such that $H\subseteq A_i$ and $a\notin A_i$ for each $i\in I$. Evidently, $A = \cup(A_i;\ i\in I)$ is a *wal*-subgroup of G. Let $a,b,c\in A$, $x,y\in G$, $x\leq a$, $y\leq b$. Then there exist $i_0, i_1, i_2\in I$ such that $a\in A_{i_0}$, $b\in A_{i_1}$, $c\in A_{i_2}$. Let e.g. $A_{i_1}\subseteq A_{i_0}$, $A_{i_2}\subseteq A_{i_0}$. Then $(x\vee y)\vee c\in A_{i_0}\subseteq A$, hence $A\in\mathcal{C}(G)$. That means (by the Zorn's lemma) the set of all $B\in\mathcal{C}(G)$ with $H\subseteq B$, $a\notin B$ contains a maximal element which is a value of a. □

Corollary 2.5. a) *Every solid subgroup of a wal-group is an intersection of regular subgroups.*

b) *Every prime subgroup is an intersection of a linearly ordered system of regular subgroups.* □

If G is a *wal*-group, then G is called *representable* if it is isomorphic to a subdirect sum of *to*-groups. It is clear that we have:

Theorem 2.6. *A wal-group is representable if and only if the intersection of all its straightening ideals is equal to* $\{0\}$. □

Corollary 2.7. *If a wal-group G is representable, then G contains a system of prime ideals such that the intersection of that system is equal to* $\{0\}$. □

3. Transitive WAL-groups

Let T be a tournament and $\mathcal{A}ut T$ be the set of all automorphisms of T. It is evident that $\mathcal{A}ut T$ forms a group with respect to the composition of mappings. For $f,g\in\mathcal{A}ut T$ we put

$$f\leq g \Leftrightarrow_{df} \forall t\in T;\ f(t)\leq g(t)\ .$$

Evidently "$\leq$" is a *va*-lattice semi-order on *AutT* and *AutT* with this semi-order is a *val*-group.

Suppose that G is a *val*-group of *AutT*. If t is an element in T, then the set $G_t = \{g\in G;\ g(t)=t\}$ will be called the *stabilizer of t*.

Proposition 3.1. *G_t is a straightening subgroup of G for any $t\in T$.*

Proof. Obviously, G_t is a convex *val*-subgroup of G.

Let $x,y\in G, f,g,h\in G_t$, $x\leq f$, $y\leq g$. Then $x(t)\leq y(t)\leq t$, and hence

$$[(x\vee y)\vee h](t) = (x\vee y)(t)\vee h(t) = [x(t)\vee y(t)]\vee t = t\ ,$$

so $(x\vee y)\vee h\in G_t$. Therefore G_t is a solid subgroup of G.

Let $x,y\in G$, $\mathrm{id}_T\leq x\wedge y\in G_t$. Then $x(t)=t$ or $y(t)=t$, thus $x\in G_t$ or $y\in G_t$, and so G_t is straightening. □

Theorem 3.2. *If G is a val-group, A a straightening subgroup of G, and u the canonical mapping of G into $Aut(G/_1A)$, then it holds:*

a) *u is a val-homomorphism;*

b) *$u(G)$ acts transitively on $G/_1A$;*

c) Ker *u is equal to the intersection of all the conjugates of A.* □

A *val*-group G is called *transitive* if there exists a tournament T and an injective *val*-homomorphism $u: G\to AutT$ such that $u(G)$ acts transitively on T.

Theorem 3.3. *A val-group G is transitive if and only if it contains a straightening subgroup A such that the intersection of all conjugates of A is equal to $\{0\}$.*

Proof. Let G be a transitive *val*-group. Consider G as a *val*-subgroup of *AutT*, where T is a tournament. Let $t,t'\in T$, $x\in G$, $x(t')=t$, $g\in G_t$. Then

$$(x^{-1}gx)(t') = (x^{-1}g)(x(t')) = (x^{-1}g)(t) = x^{-1}(t) = t',$$

hence $x^{-1}G_t x \subseteq G_{t'}$.

Let $g'\in G_{t'}$. Then

$$(xg'x^{-1})(t) = (xg')(x^{-1}(t)) = x(g'(t')) = x(t') = t \ ,$$

thus $xg'x^{-1}\in G_t$, and we have $g' = x^{-1}(xg'x^{-1})x$, so $G_{t'} \subseteq x^{-1}G_t x$.

Therefore from the transitivity of G we get for a fixed $t\in T$

$$\cap \{x^{-1}G\ x;\ x\in G\} = \cap \{G \quad t'\in T\} = \{0\}.$$

Conversely, let G be a *wal*-group and A a straightening subgroup of G such that the intersection of all the conjugates of A is equal to $\{0\}$. Then $G/_1 A$ is a tournament and the natural mapping $u: G\to Aut(G/_1 A)$ fulfils, by the preceding theorem, the condition of a transitive *wal*-group, □

Corollary 3.4. *A commutative wal-group is transitive if and only if it is a to-group.*

Proof. If G is a commutative *wal*-group, then for every its subgroup A and every $x\in G$ it holds $x^{-1}Ax = A$. Then, by the preceding theorem, G is transitive if and only if $\{0\}$ is a straightening subgroup, i.e. if G is a *to*-group. □

The following theorem could be proved by a similar way as Théorème 4.1.7 in [1].

Theorem 3.5. *If a wal-group G contains a system of straightening subgroups $(G_i;\ i\in I)$ such that $\cap(G_i; i\in I) = \{0\}$, then G is isomorphic to a subdirect sum of transitive wal-groups.* □

Corollary 3.6. *If a commutative wal-group G contains a system of straightening subgroups with the zero intersection, then G is a subdirect sum of to-groups.* □

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