Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica

Irena Rachůnková

A transmission problem

Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica, Vol. 31 (1992), No. 1, 45--59

Persistent URL: http://dml.cz/dmlcz/120280

ACTA UNIVERSITATIS PALACKIANAE OLOMUCENSIS FACULTAS RERUM NATURALIUM

1992 Mathematica XXXI Vol.105

A TRANSMISSION PROBLEM

Irena Rachůnková

(Received February 25,1991)

Abstract. Let I, I_i $(i=1,2,3)$ be compact intervals and $I=I_1\cup I_2\cup I_3$. We consider the equation (D_i) $u_i''=f_i(t,u_i,u_i')$ subject to the condition P_i on I_i for $i=1,2,3$, where P_1 and P_3 are boundary or initial conditions and P_2 is a transmission condition. We prove the existence of a Car-solution to the transmission problem $(D_i,P_i;i=1,2,3)$ on I. Our method of proofs is based on the topological degree theory. We obtain the existence results without growth conditions of Nagumo-Bernstein type.

Key words: transmission condition, four-point, Dirichlet and mixed problems, a priori estimates, Brouwer degree, the Mawhin Continuation Theorem.

MS Classification: 34B10, 34B15

INTRODUCTION

Notations. Let $I\subset\mathbb{R}$ be a compact interval. We write $C^k(I)$ for the space of C^k functions $u:I\to\mathbb{R}$ with the norm $\|u\|_k=\sum_{i=0}^{k}\max\{|u^{(i)}(t)|:t\in I\}$, $AC^k(I)$ denotes the set of real functions having absolutely continuous k-derivatives on I, for $p\geq 1$, $L^p(I)$ is the space of functions $u:I\to\mathbb{R}$ such that $|u|^p$ is Lebesgue integrable on I with the norm $\|u\|_{L^p}=(\int_I |u(t)|^p dt)^{1/p}$, $Car(I\times\mathbb{R}^2)$ signifies the set of functions $f:I\times\mathbb{R}^2\to\mathbb{R}$ satisfying the local Caratheodory conditions on $I\times\mathbb{R}^2$, i.e. the map $t\to f(t,x,y)$ is Lebesgue measurable on I for each $x,y\in\mathbb{R}$, the map $(x,y)\to f(t,x,y)$ is continuous on $\mathbb{R}^2$ for almost each (a.e.) $t\in I$, for each $\rho>0$ there exists $h_\rho\in L^1(I)$ such that $|x|+|y|<\rho\Rightarrow|f(t,x,y)|\leq h_\rho(t)$ for a.e. $t\in I$.

Formulation of Problem. Let $a,b,c,d\in\mathbb{R}$, $a<c\leq d<b$, $I_1=[a,c]$, $I_2=[c,d]$, $I_3=[d,b]$, and $f_i\in Car(I_i\times\mathbb{R}^2)$, $i=1,2,3$. We consider the equation

$$(D_i)\qquad u_i''=f_i(t,u_i,u_i')$$

subject to the condition P_i on $I_i (i=1,2,3)$, where P_1 and P_3 are boundary or initial conditions and P_2 is a transmission condition.

We shall find conditions for the existence of a function $u\in AC^1(I)$, which is a Car-solution to the transmission problem $(D_i,P_i;i=1,2,3)$, i.e. $u=u_i$ verifies P_i and fulfils (D_i) for a.e. $t\in I_i$, $i=1,2,3$.

Let us suppose that P_1 has one of the three following forms

$$(P.1.1)\qquad u_1(a)=0,$$

$$(P1.2)\qquad u_1'(a)=0,$$

$$(P1.3)\qquad u_1(c)-u_1(a)=0.$$

Similarly for P_3 we will choose one of the forms

$$(P3.1)\qquad u_3(b)=0,$$

$$(P3.2)\qquad u_3'(b)=0,$$

$$(P3.3)\qquad u_3(b)-u_3(d)=0.$$

Then, for $c<d$, P_2 has the form

$$(P2.1)\qquad \begin{cases} u_1(c)=u_2(c), & u_2(d)=u_3(d), \\ u_1'(c)=u_2'(c), & u_2'(d)=u_3'(d), \end{cases}$$

while, for $c=d$, it is

$$(P2.2)\qquad \begin{cases} u_1(c)=u_3(c), \quad u'_1(c)=u'_3(c) \\ (D_2) \text{ is omitted.} \end{cases}$$

Let us put $f(t,x,y)=f_i(t,x,y)$ for a.e. $t\in I_i$ and each $x,y\in\mathbb{R}$, $i=1,2,3$, and consider the equation

$$(D)\qquad u''=f(t,u,u') \quad \text{on } I.$$

Clearly $f\in Car(I\times\mathbb{R}^2)$ and $u\in AC^1(I)$ is a Car-solution to the transmission problem $(D_i,P_i;i=1,2,3)$, iff u is a Car-solution to the boundary value problem $(D),P_1,P_3$.

1. AUXILIARY RESULTS

Problem $(D),P_1,P_3$ will be studied by means of topological degree arguments and therefore we remind some notions and results (see [1]).

Let X,Y be real vector normed spaces and $domL\subset X$ a vector subspace. A linear map $L:domL\to Y$ will be called a Fredholm map of index zero, iff $dim\ kerL = codim\ imL < \infty$ and imL is closed in Y. If L is a Fredholm map of index zero, then there exist continuous projectors $P:X\to X$ and $Q:Y\to Y$ such that

$$(1.1)\qquad imP=kerL \quad\text{and}\quad kerQ=imL$$

and $X=KerL\oplus KerP$, $Y=ImL\oplus ImQ$ as topological direct sums. Consequently, the restriction L_p of L to $domL\cap KerP$ is one-to-one and onto ImL, so that its (algebraic) inverse

$$(1.2)\qquad K_p:\ ImL \to domL\cap KerP$$

is defined.

Let $L:domL\to Y$ be a Fredholm map of index zero and let $\Omega\subset X$ be an open bounded set. A continuous (not necessarily linear) map $N:X\to Y$ will be called L-compact on $\overline{\Omega}$ iff the maps $QN:\overline{\Omega}\to X$ and $K_p(I-Q)N:\overline{\Omega}\to X$ are compact.

Note.

1. $\overline{\Omega}$ and $\partial\Omega$ will denote the closure and the boundary of $\Omega\subset X$, respectively.

2. One can show that L-compactness of N does not depend upon the choice of P,Q.

3. Since $\dim \ker L = \dim \operatorname{im} Q < \infty$, there exists an isomorphism

(1.3) $$J: \operatorname{im} Q \to \ker L .$$

Let us consider the maps

$$N^{*}: \bar{\Omega}\times[0,1]\to Y, \quad (x,\lambda)\to N^{*}(x,\lambda)$$

with $N^{*}(.,1)=N$, and

(1.4) $$N_0 = JQN^{*}(.,0): \ker L \to \ker L.$$

Theorem 1 *(Mawhin Continuation Theorem). Let $L: \operatorname{dom} L \to Y$ be a Fredholm map of index zero and let $\Omega \subset X$ be an open bounded set. Let N^{*} be L-compact on $\bar{\Omega}\times[0,1]$. Suppose*

a) for each $\lambda \in (0,1)$, every solution x of $Lx=\lambda N^{*}(x,\lambda)$ is such that $x \notin \partial\Omega$,

b) $QN^{*}(x,\lambda)\neq 0$ for each $x \in \ker L \cap \partial\Omega$,

c) the Brouwer degree $d[N_0, \Omega\cap \ker L, 0]\neq 0$.

Then the equation $Lx=Nx$ has at least one solution in $\operatorname{dom} L\cap\bar{\Omega}$.

Proof. See [1,p.29].

Corollary. Let $\ker L=\{0\}$, let $\Omega\subset X$ be an open bounded set with $0\in\Omega$ and such that $Lx\neq\lambda N^{*}(x,\lambda)$ for each $x\in \operatorname{dom} L\cap\partial\Omega$ and each $\lambda\in(0,1)$. Then the equation $Lx=Nx$ has at least one solution in $\operatorname{dom} L\cap\bar{\Omega}$.

2. A Fredholm map L

In what follows let $X=C^{1}(I)$, $Y=L^{1}(I)$, and

$$\operatorname{dom} L=\{x\in AC^{1}(I): x \text{ satisfies } P_1, P_3\}$$

(2.1) $$L: \operatorname{dom} L\to Y, \quad x\to x''.$$

Lemma 1. *Let $i,j\in\{1,2,3\}$ and $P_1=(P1.i)$, $P_3=(P3.j)$. Then L is a Fredholm map of index zero.*

Proof. a) If $i=1$ or $j=1$, then $\ker L=\{0\}$, L is one-to-one and onto Y, so that L is a Fredholm map of index zero.

b) Now, let $i,j\in\{2,3\}$. Then $\ker L$ consists of all constant functions and therefore

(2.2) $$\dim \ker L = 1$$

and $\operatorname{im} L$ is the set of all functions $y\in Y$ for which there exist functions $x\in \operatorname{dom} L$ verifying the equation $x''(t)=y(t)$ for a.e. $t\in I$.

Let us put for $y \in Y$

(2.3) $$\overline{y}_{2,2} = \frac{1}{b-a} \int_a^b y(t)dt ,$$

(2.4) $$\overline{y}_{2,3} = \frac{1}{(b+d)/2-a} \left[\frac{1}{b-d} \int_d^b \int_a^s y(t)dtds \right] ,$$

(2.5) $$\overline{y}_{3,2} = \frac{1}{b-(c+a)/2} \left[\frac{1}{c-a} \int_a^c \int_s^b y(t)dtds \right] ,$$

(2.6) $$\overline{y}_{3,3} = \frac{1}{c_0} \left[\frac{1}{b-d} \int_d^b \int_a^s y(t)dtds - \frac{1}{c-a} \int_a^c \int_a^s y(t)dtds \right],$$

where $c_0=(b+d)/2 - (c+a)/2$.

Then, for $i,j \in \{2,3\}$, $imL=\{y \in Y: \overline{y}_{i,j}=0\}$. In all the cases we have

(2.7) $$\dim Y/imL = 1$$

An application of the Lebesgue convergence theorem will prove that imL is closed in Y for $i,j \in \{2,3\}$. Lemma is proved.■

3. Projectors P and Q

Let $P_1=(P1.i)$, $P_3=(P3.j)$, $i,j \in \{1,2,3\}$. Then, by Lemma 1, there exist continuous projectors satisfying (1.1). If $i=1$ or $j=1$, then $P=Q=0$, where 0 is a zero mapping. Let $i=2$, $j=\{2,3\}$ or $i=3, j=3$. Then we can put

(3.1) $$P:X \to X,\ x \to x(a);\ Q:Y \to Y,\ y \to \overline{y}_{i,j} ,$$

For $i=3, j=2$ we can put

(3.2) $$P:X \to X,\ x \to x(b);\ Q:Y \to Y,\ y \to \overline{y}_{3,2} ,$$

We can easily prove the following

Lemma 2. *The maps P,Q defined by (3.1) or (3.2) are continuous projectors satisfying (1.1).*

Now, let us consider the Nemyckii operator

(3.3) $$N:X \to Y,\ x \to f(\cdot, x(\cdot), x'(\cdot))$$

Lemma 3. *Let $\Omega \subset X$ be an open bounded set. Let N and Q be the maps (3.3) and (3.i), $i \in \{1,2\}$, respectively.*

Then the map $QN: \overline{\Omega} \to Y$ is compact.

Proof. Since Ω is bounded and $f\in Car(I\times\mathbb{R}^2)$, there exists $h\in L^1(I)$ such that $|f(t,x(t),x'(t)|\le f(t)$ for a.e. $t\in I$. Then, using the Lebesgue convergence theorem we get that N is continuous. Moreover, since $QN(\bar{\Omega})$ is bounded in Y and $\dim imQ = 1$ (see (2.7)), $QN(\bar{\Omega})$ is relatively compact, which completes the proof. ∎

4. An inverse map K_P

We shall study map (1.2) in the cases $P_1=(P1.i)$, $P_3=(P3.j)$, $i,j\in\{1,2,3\}$. If $i=1$ or $j=1$, then

$$K_p = L^{-1}: Y\to domL,\ y\to \int_a^b G(t,s)y(s)ds, \tag{4.1}$$

where G is the Green function of the problem

$x''= 0$,(P1.i),(P3.j), $i=1, j\in\{1,2,3\}$ or $j=1, i\in\{2,3\}$.

Let $i=2, j=\{2,3\}$. Then

$$K_p : imL\to domL\cap kerP,\ y\to \int_a^t \int_a^s y(\tau)d\tau ds, \tag{4.2}$$

For $i=3, j=2$ we get

$$K_p : imL\to domL\cap kerP,\ y\to \int_t^b \int_s^b y(\tau)d\tau ds, \tag{4.3}$$

Finally, for $i=3, j=3$ we have

$$K_p : imL\to domL\cap kerP,\ y\to -\frac{t-a}{c-a}\int_a^c \int_a^s y(\tau)d\tau ds+\int_a^t \int_a^s y(\tau)d\tau ds, \tag{4.4}$$

Lemma 4. *Let $i,j\in\{1,2,3\}$ and $P_1=(P1.i)$, $P_3=(P3.j)$. Let $\Omega\subset X$ be an open bounded set and let L and N be the maps (2.1) and (3.3), respectively. Then N is L-compact on $\bar{\Omega}$.*

Proof. According to Lemma 3 it is sufficient to prove that $K_p(I-Q)N:\bar{\Omega}\to X$ is compact. This assertion can be proved by standard arguments using the Lebesgue Convergence Theorem and the Arzela-Ascoli Theorem in all the cases $i,j\in\{1,2,3\}$. ∎

Lemma 5. *Let $\Omega\subset X$ be an open bounded set and let $f^*\in Car(I\times(\mathbb{R}^2\times[0,1]))$. Then the assertion of Lemma 4 is valid for the map*

$$N^* : \bar{\Omega}\times[0,1] \to Y,\ (x,\lambda)\to f^*(\cdot,x(\cdot),x'(\cdot),\lambda). \tag{4.5}$$

Proof. Lemma 5 can be proved in a similar way as Lemma 4. In the space $X\times[0,1]$ we work with the norm $\|(x,\lambda)\| = \|x\|_{C^1} + |\lambda|$. ∎

5. Auxiliary theorems of the Leray-Schauder type

Let us choose a function $f^*\in Car(I\times(\mathbb{R}^2\times[0,1]))$ such that $f^*(t,x,y,1) = f(t,x,y)$ for a.e. $t\in I$ and for each $x,y\in\mathbb{R}$, and consider the set of the equations

$$u'' = \lambda f^*(t,u,u',\lambda), \quad \lambda\in[0,1]. \tag{5.1λ}$$

I. A case of non-resonance.

Theorem 2. *Let $i=1$, $j\in\{1,2,3\}$ or $j=1$, $i\in\{2,3\}$ and let $P_1=(P1.i)$, $P_3=(P3.j)$. Let there exist an open bounded set $\Omega\subset X$ containing the zero-function and let for each $\lambda\in(0,1)$, every Car-solution u of the problem (5.1λ),P_1,P_3 fulfil $u\notin\partial\Omega$.*

Then the problem (D),P_1,P_3 has at least one Car-solution in $\bar{\Omega}$.

Proof. According to Lemmas 1-5, the assertion of Theorem 2 follows from Corollary of Part 1, where L and N^* are given by (2.1) and (4.5), respectively. ∎

II. A case of resonance.

Let us put $\phi(t,x)=f^*(t,x,0,0)$ on $I\times\mathbb{R}$ and, for $i,j\in\{2,3\}$

$$g_{i,j}(x)=\overline{\phi(x)}_{i,j}, \quad \text{for } x\in\mathbb{R}. \tag{5.2}$$

(See (2.3)-(2.6).)

Theorem 3. *Let $P_1=(P1.i)$, $P_3=(P3.j)$ where $i,j\in\{2,3\}$. Let there exist an open bounded set $\Omega\subset X$ such that*

(a) for any $\lambda\in(0,1)$, every Car-solution u of the problem (5.1λ), P_1,P_3 satisfies $u\notin\partial\Omega$,

(b) for any root $x_0\in\mathbb{R}$ of the equation $g_{i,j}(x)=0$, the condition $x_0\notin\partial\Omega$ is fulfilled, where x_0 is considered as a constant function on I,

(c) the Brouwer degree $d[g_{i,j},\Delta,0]\neq0$, where $\Delta\subset\mathbb{R}$ is the set of such constants c that the constant functions $u(t)=c$ belong to Ω.

Then the problem $(D),P_1,P_3$ *has at least one Car-solution in* $\bar{\Omega}$.

Proof. According to (3.1),(3.2),(4.5) and (5.2) we have $QN^*(x,0)=g_{i,j}(x)$, and in view of (1.3),(1.4) and (2.2), $N_0=kg_{i,j}$, where $k\in\mathbb{R}$, $k\neq 0$. Therefore, by Lemmas 1-5, the conditions of Theorem 1 are satisfied, which completes the proof.∎

In the next parts, using Theorem 2 or 3, we shall prove existence theorems for the boundary value problems $(D),(P1.j),(P3.j)$, where $i=j=1$ or $i=1,j=2$ or $i=j=3$. (The other possibilities for i,j could be solved similarly.)

6. DIRICHLET PROBLEM

We shall investigate the case of $i=j=1$, i.e. the Dirichlet problem

(6.1) $$u''=f(t,u,u'),\quad u(a)=u(b)=0.$$

Lemma 6. *Let* $g\in Car(I\times\mathbb{R}^2)$ *and* $r,k\in(0,\infty)$ *be such that*

(6.2) $$\int_a^b |g(t,x,y)|dt \le K \quad \text{for each } x\in[-r,r],\ y\in\mathbb{R}$$

and

(6.3) $$g(t,-r,0)\le 0,\quad g(t,r,0)\ge 0 \quad \text{for a.e. } t\in I.$$

Then the problem

(6.4) $$u''=g(t,u,u'),\quad u(a)=u(b)=0$$

has at least one Car-solution u_0 *with*

(6.5) $$\|u_0\|\le r.$$

Proof. For $m\in\mathbb{N}$ let us put

$$g_m(t,x,y)=\begin{cases} g(t,r,0) & \text{for } x>r+1/m\\ g(t,r,y)+[g(t,r,0)-g(t,r,y)]m(x-r) & \text{for } r<x\le r+1/m\\ g(t,x,y) & \text{for } -r\le x\le r\\ g(t,-r,y)-[g(t,-r,0)-g(t,-r,y)]m(x+r) & \text{for } -r-1/m\le x<-r\\ g(t,-r,0) & \text{for } x<-r-1/m \end{cases}$$

and consider the auxiliary problem

(6.6m) $$u''=g_m(t,u,u'),\quad u(a)=u(b)=0.$$

Now choose an arbitrary but fixed $m\in\mathbb{N}$. We shall prove the existence of a solution of (6.6m) by means of Theorem 2 and therefore we need to study the parameter-set of equations

$$(6.7\lambda)\qquad u''=\lambda g^*_m(t,u,u',\lambda)\ ,\ u(a)=u(b)=0,$$

where

$$g^*_m(t,x,y,\lambda)=\lambda g_m(t,x,y)+(1-\lambda)x \quad\text{and}\quad \lambda\in[0,1].$$

Suppose that u is a Car-solution to (6.7λ) for some $\lambda\in(0,1)$. First, we shall show that

$$(6.8)\qquad \|u\|\le r+1/m.$$

Put $v(t)=u(t)-r-1/m$. Then $v(a)=v(b)=-r-1/m<0$ and $u'=v'$ on I. Let us suppose that there exists $t_0\in(a,b)$ such that $v(t_0)>0$. Then there exists $\bar{t}\in(a,b)$ such that $0<v(\bar{t})=\max\{v(t):t\in I\}$ and $v'(\bar{t})=0$. Therefore we can find $\delta>0$ and the interval $I_\delta=(\bar{t}-\delta,\bar{t}+\delta)\subset(a,b)$ such that $v'(\bar{t}-\delta)\ge0$, $v'(\bar{t}+\delta)\le0$ and $v(t)\ge0$ for each $t\in I_\delta$. From this it follows

$$v''(t)=u''(t)=\lambda g_m(t,u,u')+(1-\lambda)u=\lambda g(t,r,0)+(1-\lambda)u>0$$

for a.e. $t\in I_\delta$. Integrating the last inequality, we get

$$0\ \ge\ v'(\bar{t}+\delta)-v'(\bar{t}-\delta)\ =\ \int_{I_\delta} v''(t)dt\ >\ 0,$$

a contradiction.

So, we have proved $v(t)\le0$ on I, which means that $u(t)\le r+1/m$ on I. Similarly, putting $v(t)=-r-1/m-u(t)$, we can prove $v(t)\le0$ on I, which means $u(t)\ge-r-1/m$ on I (see proof of Lemma 7). Hence u satisfies (6.8).

Further we shall estimate u'. Since $u(a)=u(b)$, there exists $a_0\in(a,b)$ such that $u'(a_0)=0$. Integrating (6.7λ) from a_0 to t, we have

$$(6.9)\qquad \|u'\|<K_0,$$

where $K_0 = K+(b-a)(r+1)$.

Finally, define

$$\Omega=\{x\in X:\ \|x\|<r+2/m\ ,\ \|x'\|<K_0\}.$$

Then, by (6.8),(6.9), $u\notin\partial\Omega$ and according to Theorem 2, problem (6.6m) has at least one solution $u_m\in\bar{\Omega}$.

In this way, we can get the sequence of solutions $(u_m)_1^\infty$ which is for $m=1,2,\dots,$ bounded in $C^1(I)$ and hence also equicontinuous in $C^1(I)$ by the equation. By the Arzela-Ascoli Theorem and the integrated forms of the equations (6.6m) one gets the existence of a converging subsequence whose limit is a solution u_0 of problem (6.4) satisfying (6.5).∎

Theorem 4. *Let $f\in Car(I\times\mathbb{R}^2)$ and $R\in(0,\infty)$, $\varepsilon\in(0,b-a]$, $r\in(0,R\varepsilon/2]$ be such that*

(6.10) $f(t,-r,0)\le 0$, $f(t,r,0)\ge 0$ *for a.e.* $t\in I$

(6.11) $f(t,x,-R)\le 0$, $f(t,x,R)\ge 0$ *for a.e.* $t\in I$, *each* $x\in[-r,r]$

and

(6.12) $\int_{b-\varepsilon}^{b} |f(t,x,(-1)^i R)|dt < R/2$ *for* $x\in[-r,r], i\in\{-1,1\}$.

Then problem (6.1′) has at least one Car-solution u such that

(6.13) $\|u\|\le r$, $\|u'\|\le R$.

Proof. According to (6.12) we can find such a small positive number ε_0 that

(6.14) $\int_{b-\varepsilon}^{b} |f(t,x,(-1)^i R)|dt+\varepsilon\varepsilon_0<R/2$ for $i\in\{-1,1\}$.

Let us put

$$g(t,x,y) = \begin{cases} f(t,x,R) + \frac{y-R}{y-R+1}\varepsilon_0 & \text{for } y>R \\ f(t,x,y) & \text{for } -R\le y\le R \\ f(t,x,-R) + \frac{y+R}{|y+R|+1}\varepsilon_0 & \text{for } y<-R \end{cases}$$

and consider the auxiliary problem

(6.15) $u''=g(t,u,u')$, $u(a)=u(b)=0$.

We shall show that g satisfies the conditions of Lemma 6. Since $f\in Car(I\times\mathbb{R}^2)$, there exists $h\in L^1(I)$ such that $|f(t,x,y)|\le h(t)$ for a.e. $t\in I$ and for each $x\in[-r,r]$, $y\in[-R,R]$. Therefore

$\int_a^b |g(t,x,y)dt\le\int_a^b h(t)dt+\varepsilon_0(b-a)=K,$ for each $x\in[-r,r]$, $y\in\mathbb{R}$.

Further $g(t,-r,0)=f(t,-r,0\le 0$ and $g(t,r,0)=f(t,r,0)\ge 0$ for a.e. $t\in I$. Hence, by Lemma 6, problem (6.15) has at least one Car-solution u satisfying (6.5).

Now, we shall prove that

(6.16) $\|u'\|\le R$.

Let us suppose on the contrary that

$$\max\{u'(t):t\in I\}=u'(t_0)>R.$$

a) Let $t_0\in[a,b)$. Then there exist $\delta>0$ and $I_\delta=[t_0,t_0+\delta]\subset[a,b)$ such that $u'(t)>R$ for each $t\in I_\delta$ and $u'(t_0+\delta)\le u'(t_0)$.

Then for a.e. $t\in I_\delta$ we have $u''(t)=g(t,u,u')=f(t,u,R)+\frac{u'-R}{u'-R+1}\varepsilon_0>0$. Thus $0<u'(t_0+\delta)-u'(t_0)\le 0$, a contradiction.

b) Let $t_0=b$. Then $u'(b)>R$ and by (6.14) we get for any $t\in[b-\varepsilon,b)$

$$u'(b)-u'(t)=\int_t^b u''(s)ds\le\int_{b-\varepsilon}^b|u''(t)|dt\le\int_{b-\varepsilon}^b|f(t,u,R)|dt+\varepsilon\varepsilon_0<R/2$$

which implies $u'(t)>R/2$ on $[b-\varepsilon,b]$.

Hence $r\le R\varepsilon/2<\int_{b-\varepsilon}^b u'(t)dt=u(b)-u(b-\varepsilon)=-u(b-\varepsilon)$, which contradicts (6.5).

Supposing $\min\{u'(t):t\in I\}<-R$, we get a contradiction in a similar way. Therefore u fulfils (6.16). This implies that u is also a solution of (6.1). Theorem is proved. ∎

7. Mixed problem

Now we consider the case $i=1$, $j=2$, i.e. the mixed problem

(7.1) $$u''=f(t,u,u'),\ u(a)=0,\ u'(b)=0.$$

Lemma 7. *Let $g\in Car(I\times\mathbb{R}^2)$ and $r\in(0,\infty)$ be such that (6.3) is fulfilled.*

Then the problem

(7.2) $$u''=g(t,u,u'),\ u(a)=0,\ u'(b)=0$$

has at least one Car-solution u satisfying (6.5).

Proof. For $m\in\mathbb{N}$ define the function g_m in the same way as in the proof of Lemma 6 and consider the problem

(7.3m) $$u''=g_m(t,u,u'),\ u(a)=0,\ u'(b)=0$$

and the parameter-set of problems

(7.4λ) $$u''=\lambda g_m^*(t,u,u',\lambda),\ u(a)=0,\ u'(b)=0,$$

where g_m^* and λ are also the same as in the proof of Lemma 6.

Let us suppose that u is a Car-solution to (7.4λ) for some $\lambda\in(0,1)$ and let us show that u fulfils (6.8).

Put $v(t)=-r-1/m-u(t)$. Then $v(a)<0$, $v'(b)=-u'(b)=0$ and $v'(t)=-u'(t)$ on I. Suppose that there exists $t_0\in(a,b]$ such that $v(t_0)>0$. Then there exists $\bar{t}\in(a,b]$ such that $0<v(\bar{t})=\max\{v(t):t\in I\}$ and $v'(\bar{t})=0$. Therefore we can find $\delta>0$ and the interval $I_\delta=(\bar{t}-\delta,\bar{t}]\subset(a,b]$ such that $v'(\bar{t}-\delta)\ge0$ and $|v'(t)|=|u'(t)|<R$, $v(t)\ge0$ for each $t\in I_\delta$. From this it follows $v''(t)=-u''(t)=-\lambda g_m(t,u,u')-(1-\lambda)u=-\lambda g(t,-r.0)-(1-\lambda)u>0$ for a.e. $t\in I_\delta$. Integrating the last inequality from $\bar{t}-\delta$

to $\overline{t}$, we get $0\geq v'(\overline{t})-v'(\overline{t}-\delta)=\int_{I_\delta} v''(t)dt>0$, a contradiction. Therefore $v(t)\leq 0$ on I, i.e. $u(t)\geq -r-1/m$ on I. Similarly (see proof of Lemma 6) we can prove $u(t)\leq r+1/m$ on I. Hence u satisfies (6.8).

Further we can follow the proof of Lemma 6, where $a_0=b$.

Theorem 5. *Let $f\in Car(I\times\mathbb{R}^2)$ and $r\in(0,\infty)$ be such that (6.10) and (6.11) are fulfilled.*

Then problem (7.1) has at least one Car-solution u with the property (6.13).

Proof. Theorem 5 can be proved in the same way as Theorem 4, only instead of Lemma 6 we use Lemma 7. ∎

8. Four-point problem

Finally, we shall study the case $i=j=3$, i.e. the four-point problem

(8.1) $$u''=f(t,u,u'),\ u(c)=u(a),\ u(b)=u(d).$$

Lemma 8. *Let $g\in Car(I\times\mathbb{R}^2)$ and $r,K\in(0,\infty)$ be such that (6.2) and (6.3) are satisfied.*

Then the problem

(8.2) $$u''=g(t,u,u'),\ u(a)=u(c),\ u(b)=u(d)$$

has at least one Car-solution u_0 satisfying (6.5).

Proof. For $m\in\mathbb{N}$ define the function g_m in the same way as in the proof of Lemma 6 and consider the problem

(8.3m) $$u''=g_m(t,u,u'),\ u(a)=u(c),\ u(b)=u(d).$$

For a fixed m we shall use Theorem 3 to prove the existence of a solution to (8.3m). Therefore we need the parameter-set of problems

(8.4λ) $$u''=\lambda g^*_m(t,u,u',\lambda),\ u(a)=u(c),\ u(b)=u(d),$$

where $g^*_m(t,x,y,\lambda)=\lambda g_m(t,x,y)+(1-\lambda)(x-r/2)$, $\lambda\in[0,1]$.

(a) If we define a set Ω in the same way as in the proof of Lemma 6 with $K_0=K+2(b-a)(r+1)$, we can get by the same arguments like there that for any $\lambda\in(0,1)$ every Car-solution u of (8.4λ) does not belong to $\partial\Omega$.

(b) $g_{3,3}(x)=\frac{1}{c_0}\left[\frac{1}{b-a}\int_d^b\int_a^s g_m^*(t,x,0,0)dtds -\right.$

$$\left. -\frac{1}{c-a}\int_a^c\int_a^s g_m^*(t,x,0,0)dtds\right] =$$

$$=\frac{1}{c_0}\left[\frac{1}{b-d}\int_d^b\int_a^s(x-r/2)dtds-\frac{1}{c-a}\int_a^c\int_a^s(x-r/2)dtds\right]=x-r/2.$$

So the equation $g_{3,3}(x)=0$ has just one root $x_0=r/2$ and the constant function $u_0(t)=r/2$ on I belongs to Ω. Thus $u_0\notin\partial\Omega$.

(c) Finally $\Delta=(r-2/m,r+2/m)$ and $d[g_{3,3},\Delta,0]\neq 0$.

We have shown that all conditions of Theorem 3 are fulfilled which implies that problem (8.3m) has at least one solution $u_m\in\overline{\Omega}$. Further we can follow the proof of Lemma 6.■

Theorem 6. *Let all conditions of Theorem 4 are satisfied. Then problem (8.1) has at least one Car-solution u with the property (6.13).*

Proof. Theorem 6 can be proved in the same way as Theorem 4, only instead of Lemma 6 we use Lemma 8.■

9. Examples

Example 1. Let us consider three equations

(9.1) $$u''=e^{uu'}(u^5+(u')^3+3t^2-1),$$

(9.2) $$u''=e^{uu'}(u^7+(u')^5+3t^2+5),$$

and

(9.3) $$u''=e^{u}(u^5+(u')^3+3t^2-1).$$

Further let us put $I=[0,1]$, $\varepsilon=10^{-4}$, $I_1=[0,\varepsilon]$, $I_2=[\varepsilon,1-\varepsilon]$, $I_3=[1-\varepsilon,1]$. We want to prove the existence of a function $u\in AC^1(I)$ which satisfies equation (9.1) on the initial part of I (i.e. for a.e. $t\in I_1$), equation (9.2) on the middle part of I (i.e. for a.e. $t\in I_2$) and equation (9.3) on the end part of I (i.e. for a.e. $t\in I_3$). Moreover u has to satisfy on I_1 the condition

(9.4) $$u(0)=u(\varepsilon)$$

and on I_3 the condition

(9.5) $$u(1-\varepsilon)=u(1).$$

We shall use Theorem 6. Let us put

$$f(t,x,y)=\begin{cases} e^{xy}(x^5+y^3+3t^2-1) & \text{for a.e. } t\in I_1 \\ e^{xy}(x^7+y^5+3t^2+5) & \text{for a.e. } t\in I_2 \\ e^{x}(x^5+y^3+3t^2-1) & \text{for a.e. } t\in I_3 \end{cases}$$

where $x,y\in\mathbb{R}$. Then $f\in Car(I\times\mathbb{R}^2)$ and for $r=2$, $R=20$ f satisfies conditions (6.10), (6.11) and (6.12) which implies the existence of a solution u of our problem (9.1)-(9.5).

In the same way we could prove the existence of a solution u of equations (9.1)-(9.3) satisfying

(9.6) $$u(0)=u(1)=0$$

or

(9.7) $$u(0)=u'(1)=0.$$

Example 2. Let us consider the equations (9.1) and (9.2) and let us put $I=[0,1]$, $\varepsilon=10^{-1}$, $I_1=[0,\varepsilon]$, $I_2=[\varepsilon,1-\varepsilon]$, $I_3=[1-\varepsilon,1]$. We want to prove the existence of a function $u\in AC^1(I)$ which satisfies equation (9.1) on I_1 and I_3 and equation (9.2) on I_2 and moreover it fulfils the condition (9.7). We can put

$$f(t,x,y)=\begin{cases} e^{xy}(x^5+y^3+3t^2-1) & \text{for a.e. } t\in I_1\cup I_3 \\ e^{xy}(x^7+y^5+3t^2+5) & \text{for a.e. } t\in I_2 \end{cases} .$$

Then similarly as in **Example** 1 we can show that for $r=2$, $R=20$, $f\in Car(I\times\mathbb{R}^2)$ fulfils (6.10) and (6.11). So the existence of a solution to the transmission problem (9.1),(9.2),(9.8) follows from Theorem 5.

Notice that in this case f does not fulfil (6.12) and so we can not get the existence of solutions of problem (9.1),(9.2), (9.6) or problem (9.1),(9.2),(9.4),(9.5).

Example 3. Let $I=[0,10]$, $\varepsilon=10^{-2}$. Let us consider two equations

(9.8) $$u''=h_1(t)(u^{2k+1}+(u')^{2n+1}+2\pi)$$

(9.9) $$u''=h_2(t)(u^{2k+1}+(u')^{2n+1})$$

where $k,n\in\mathbb{N}$, $k<n$ and $h_1,h_2\in L^1(I)$ are positive with

$$\int_{10-\varepsilon}^{10} h_2(t)dt < \frac{1}{4\cdot 10^{2n}}$$

Let us put

$$f(t,x,y)=\begin{cases} h_1(t)(x^{2k+1}+y^{2n+1}+2\pi) & \text{for a.e. } t\in[0,1-\varepsilon) \\ h_2(t)(x^{2k+1}+y^{2n+1}) & \text{for a.e. } t\in[1-\varepsilon,1] \end{cases}$$

and for each $x,y\in\mathbb{R}$. Then $f\in Car(I\times\mathbb{R}^2)$ satisfies for $r=R=10$ the condition (6.10)-(6.12). Therefore there exists a function $u\in AC^1(I)$ which fulfils (9.8) on $[0,10-\varepsilon]$ and (9.7) on $[10-\varepsilon,10]$ and moreover it satisfies (9.4),(9.5) (or (9.6) or (9.7)).

References

[1] J.L.Mawhin: Topological Degree Methods in Nonlinear Boundary Value problems, Providence, R.I.,1979.

[2] I.Rachůnková: An existence theorem of the Leray-Schauder type for four-point boundary value problems, Acta UPO,Fac.rer.nat. 100(1991).

[3] I.Rachůnková: On a certain four-point problem, Radovi Matematički 8(1992).

Author's address: Department of Mathematical Analysis
Faculty of Sciences, Palacký University
Vídeňská 15, 77146 Olomouc
Czechoslovakia