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SUBDIRECTLY IRREDUCIBLE ALGEBRAS OF QUASIORDERED LOGICS

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Abstract

An algebra of quasiordered logic is a generalization of Boolean algebra such that the induced relation is not an order but only a quasiorder in the general case. We give a list of all subdirectly irreducible algebras of quasiordered logic which are not degenerated.

Key words: q-lattice, q-algebra, quasiorder, lattice, Boolean algebra, subdirectly irreducible algebra.

MS Classification: 06E05, 08A05

The concept of a q-lattice which generalizes lattices for quasiordered sets was introduced in [2]:

Definition 1 By a q-lattice is meant an algebra $(A; \vee, \wedge)$ with two binary operations satisfying the following axioms:

- (associativity) $a \vee (b \vee c) = (a \vee b) \vee c \quad a \wedge (b \wedge c) = (a \wedge b) \wedge c$
- (commutativity) $a \vee b = b \vee a \quad a \wedge b = b \wedge a$
- (weak absorption) $a \vee (a \wedge b) = a \vee a \quad a \wedge (a \vee b) = a \wedge a$
- (weak idempotence) $a \vee (b \vee a) = a \vee b \quad a \wedge (b \wedge a) = a \wedge b$
- (equalization) $a \vee a = a \wedge a$

It was proven in [2] that the binary relation defined on $A$ by

$\langle a, b \rangle \in Q$ if and only if $a \vee b = b \vee b$
(or, equivalently, if $a \land b = b \land a$) is a quasiorder on $A$; the so called \textit{induced quasiorder}.

A $q$-lattice $(A; \lor, \land)$ is \textit{distributive} if it satisfies the distributive identity:

$$a \lor (b \land c) = (a \lor b) \land (a \lor c)$$

for each $a, b, c$ of $A$. Note that this identity is equivalent to its dual.

A $q$-lattice $(A; \lor, \land)$ is \textit{bounded} if there exist elements 0 and 1 of $A$, the so called \textit{zero} and \textit{unit}, such that

$$0 \land a = 0 \quad \text{and} \quad 1 \lor a = 1$$

for every element $a$ of $A$.

Let us remark that

(i) such elements are unique in $A$;

(ii) it can happen that $0 \lor a \neq a$ and $1 \land a \neq a$, however $0 \lor a = a \lor a$ and $1 \land a = a \land a$ for each $a \in A$;

(iii) $(0, a) \in Q$ and $(a, 1) \in Q$ for the induced quasiorder $Q$; by (i) and (ii), it can also happen $(b, 0) \in Q$ and/or $(1, c) \in Q$ for some $b, c \in A$.

For some examples, see [2] and [3].

A $q$-lattice $(A; \lor, \land)$ is \textit{complementary} if it is bounded and for each $a \in A$ there exists $b \in A$ with $a \lor b = 1$ and $a \land b = 0$; such element $b$ is called a \textit{complement} of $a$.

Let $(A; \lor, \land)$ be a bounded distributive $q$-lattice, let $a, b, c \in A$ and $b, c$ be complements of $a$. It was proven in [3] that in such a case $b \lor b = c \lor c$. Henceforth, we can introduce the unary operation $'$ in a complementary distributive $q$-lattice defined as follows:

$$a' = b \lor b,$$

where $b$ is a complement of $a$.

\textbf{Definition 2} An algebra $A = (A; \lor, \land, \land', 0, 1)$ with two binary operations $\lor, \land$, with one unary operation $'$ and two nullary operations $0, 1$ is called an \textit{algebra of quasiordered logic} if $(A; \lor, \land)$ is a complementary distributive $q$-lattice where 0 and 1 satisfy $(\ast)$ and $'$ is defined in $(A; \lor, \land)$ by $(\ast\ast)$.

An algebra $A$ of quasiordered logic is called \textit{trivial} if $\text{card } A = 1$;

$A$ is \textit{nondegenerated} if $A$ is trivial whenever $0 = 1$.

We can visualize $q$-lattices in diagrams as follows:

if $a, b \in A$ and $(a, b) \in Q$, where $Q$ is the induced quasiorder, then $a$ is connected with $b$ by a path consisting of arrows oriented in the same direction. An example of a nine-element algebra of quasiordered logic is shown in Fig.1:
Although this $q$-lattice is distributive and complementary, it is not uniquelly complementary since 0 has two complements 1 and $q$; 1 has two complements 0 and $p$; $w$ has four complements $x, y, z, v$.

An element $a$ of a $q$-lattice $(A; \lor, \land)$ is called the idempotent if $a \lor a = a$ (or, equivalently, $a \land a = a$). If $a, b \in A$, then clearly $a \lor b$ is the idempotent as it follows from weak idempotence. If $\text{card} \ C > 1$ and $C$ is a maximal subset of $A$ such that $C \times C \subseteq Q$ for the induced quasiorder $Q$, then $C$ is called a cell of $A$. It is easy to see that every cell has just one idempotent.

In the foregoing example, $\{0, p\}$, $\{1, q\}$ and $\{x, y, z, v\}$ are cells of $A$. If $x$ is the idempotent, then

$$x = x \lor x = y \lor y = z \lor z = v \lor v.$$  

Since $0 \land a = 0$ and $1 \lor a = 1$ for each $a \in A$, the zero 0 and the unit 1 are idempotents. Also $w \in A$ is the idempotent because it is not contained in any cell of $A$.

The connection between algebras of quasiordered logic and propositional calculus is shown in [3]. The aim of this paper is to list all subdirectly irreducible algebras of quasiordered logic. It was shown in [3] that the algebra of quasiordered logic is a Boolean algebra if and only if it has no cell. By [1], the variety of all Boolean algebras has just one subdirectly irreducible member, namely the two-element algebra. We are going to show that the situation is different in our case:

**Theorem 1** Let $\mathcal{V}$ be the variety of all algebras of quasiordered logic. A nondegenerated algebra $A \in \mathcal{V}$ is subdirectly irreducible if and only if it has either two or three elements, i.e. if $A$ is isomorphic to one of the three algebras $B, C_1, C_2$ in Fig. 2.
Proof Trivially, $B$ is subdirectly irreducible since it has two elements only.
Denote by $\omega$ the identity relation, i.e. the least congruence, and by $\varepsilon$ the greatest congruence (i.e. the full relation). Thus $C_1, C_2$ has the following lattices of congruences $\Theta$ for which $(0, 1) \notin \Theta$ (see Fig. 3):

![Fig. 3](image)

Hence $C_1$ and $C_2$ are also subdirectly irreducible since their congruence lattices have only one atom.

Now, let $A$ be an algebra of quasiordered logic different from $B, C_1, C_2$. We have the following possibilities:

(a) $A$ has no cell. Then, by [3], $A$ is a Boolean algebra. Since $A$ is not isomorphic to $B$, it is subdirectly reducible by [1].

(b) Let $A$ has at least two different cells, say $D_1, D_2$. Then, evidently, $D_1 \cap D_2 = \emptyset$. We can put $\Theta_1 = D_1 \times D_1 \cup \omega, \Theta_2 = D_2 \times D_2 \cup \omega$ where $\omega$ is the identity relation. It is easy to see that $\Theta_1, \Theta_2$ are congruences on $A$ with $\Theta_1 \cap \Theta_2 = \omega$, thus $A$ is subdirectly reducible, see e.g. [1].

(c) It remains the possibility when $A$ has just one cell $D$.

(i) Suppose that $A$ has only two idempotents, namely 0 and 1. Since $A$ is not isomorphic to $C_1$ or $C_2$, it means that $D$ contains at least two non-idempotent elements, say $a$ and $b$.

Suppose $0 \in D$ and put $A_1 = \{0, 1, a\}, A_2 = A - \{a\}$. Clearly both $A_1, A_2$ are algebras of quasiordered logic ($A_1 \cong C_2$). Introduce $\alpha : A \rightarrow A_1 \times A_2$ as follows:

$$\alpha(0) = (0, 0) \quad \alpha(1) = (1, 1) \quad \alpha(a) = (a, 0) \quad \alpha(x) = (0, x) \quad \text{for} \ x \in D, \ x \neq a.$$
We can see that $\alpha$ is an injection and $pr_1\alpha(A) = A_1$, $pr_2\alpha(A) = A_2$. If $z, y \in A_1$ or $z, y \in A_2$, we can easily testify

$$
\alpha(z \lor y) = \alpha(z) \lor \alpha(y), \quad \alpha(z \land y) = \alpha(z) \land \alpha(y).
$$

If $z \in A_1 - A_2$, $y \in A_2 - A_1$, then $z = a$ and $y \in D$, and we have

$$
\begin{align*}
\alpha(z \lor y) &= \alpha(a \lor y) = \alpha(0) = \langle 0, 0 \rangle \\
\alpha(z) \lor \alpha(y) &= \alpha(a) \lor \alpha(y) = \langle a, 0 \rangle \lor \langle 0, x \rangle = \langle 0, 0 \rangle
\end{align*}
$$

and

$$
\alpha(z) \land \alpha(y) = \langle a, 0 \rangle \land \langle 0, x \rangle = \langle 0, 0 \rangle = \alpha(0) = \alpha(z \land y).
$$

It is evident that $\langle 0, 0 \rangle$ is the zero and $\langle 1, 1 \rangle$ the unit in $A_1 \times A_2$, thus $\alpha$ preserves both nullary operations.

\begin{align*}
\alpha(0') &= \alpha(1) = \langle 1, 1 \rangle = \langle 0, 0 \rangle' \\
\alpha(1') &= \alpha(0) = \langle 0, 0 \rangle = \langle 1, 1 \rangle' \\
\alpha(a') &= \alpha(1) = \langle 1, 1 \rangle = \langle a, 0 \rangle' = \alpha(a)' \\
\alpha(x') &= \alpha(1) = \langle 1, 1 \rangle = \langle 0, x \rangle' = \alpha(x)' & \text{for } x \in D, x \neq a,
\end{align*}

thus $\alpha$ is an injective homomorphism. In the summary, $A$ is isomorphic to a subdirect product of $A_1, A_2$.

If we suppose $1 \in D$, the proof is dual to the previous one for $0 \in D$.

(ii) Suppose that $A$ contains an idempotent $d$ such that $0 \neq d \neq 1$. Put

$$
A_1 = \{ x; \langle x, d \rangle \in Q \}, \quad A_2 = \{ x; \langle d, x \rangle \in Q \},
$$

where $Q$ is the induced quasiorder.

(a) If $d \in D$ (the unique cell of $A$), define

$$
\begin{align*}
\alpha(x) &= \langle x \land d, x \lor d \rangle & \text{for } x \notin D & \text{and} \\
\alpha(x) &= \langle x, x \rangle & \text{for } x \in D.
\end{align*}
$$

Since every $x \notin D$ is an idempotent of $A$, it is easy to check that $\alpha$ is an injective homomorphism of $A$ into the direct product $A_1 \times A_2$ and $pr_1\alpha(A) = A_1$, $pr_2\alpha(A) = A_2$, i.e. $A$ is isomorphic to a subdirect product of $A_1, A_2$.

(b) If $0 \notin D$, then $d \notin D$ and we can define

$$
\begin{align*}
\alpha(x) &= \langle x \land d, x \lor d \rangle & \text{for } x \notin D & \text{and} \\
\alpha(x) &= \langle x, d \rangle & \text{for } x \in D.
\end{align*}
$$

Analogously as in the case (a), we can prove that $A$ is isomorphic to a subdirect product of $A_1, A_2$.

(c) If $1 \in D$, define

$$
\begin{align*}
\alpha(x) &= \langle x \land d, x \lor d \rangle & \text{for } x \notin D & \text{and} \\
\alpha(x) &= \langle d, x \rangle & \text{for } x \in D.
\end{align*}
$$

The proof is dual to that of (b).
Corollary 1 Every algebra of quasiordered logic is isomorphic to a subdirect product of algebras $B, C_1, C_2$ (see Fig.2).

Example The algebra $A$ in Fig.1 is isomorphic to $C_1 \times C_2$.

References


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