Tadeusz Dłotko

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PERIODIC SOLUTIONS OF SOME SECOND ORDER DIFERENTIAL SYSTEMS

TADEUSZ DŁOTKO

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Abstract

The question of existence of periodic solutions to the problem

$$(1)\qquad \begin{aligned} &x''(t) = g(t, x(t)) + f(t, x(t), x(r(t)), x'(t), x'(z(t))) \\ &x(o) = x(w), \quad x'(o) = x'(w), \quad w > o, \quad t \in [o, w] \end{aligned}$$

is considered. The proofs are based on topological degree methods, especially on the homotopy theory.

Key words: Existence in periodic BVPs, second order ODE with deviated argument, a priori bounds, perturbed systems.

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In this paper there are demonstrated some theorems for the existence of $AC^1[o, w]$ solutions to the problem (1). Assuming degree conditions a lot of papers was devoted to second order BVPs (see[1]–[4], [6]–[11]).

The concept of the a priori estimate and the extension to equations with deviated argument seems to be new in this paper.We assume that the function $g(t, x)$ is strong nonlinear in neighbourhood of $(t, x) = (o, o)$ and linear outside of it.

The formulations of degree theory here follows M.A.Krasnoselskij [4].

Notations and assumptions The symbols x, g, f denote $n \times 1$ vectors, in the following $A(t)$ is a $n \times n$ integrable matrix. (u, v) is the inner product of vectors $u, v \in \boldsymbol{R}^n$. $r(t), z(t)$ are continuous functions $r, z\colon [o, w] \to [o, w]$. We assume that f fulfils the Caratheodory conditions and it is bounded for arbitrary arguments. Further $(x, A(t)x) > o$ for $x \neq o$ and $t \in [o, w]$. A function $x \in AC^1[o, w]$ which fulfils (1) for a.e. $t \in [o, w]$ will be called a solution of (1).

Lemma 1 *The problem*

(2) $$x''(t) = A(t)x(t), \quad x(o) = x(w), \quad x'(o) = x'(w), \quad t \in [o, w]$$

has exactly the solution $x(t) \equiv o$ *for* $t \in [o, w]$.

In fact let us consider the inner product

(3) $$(x(t), x'(t)),$$

where $x(t)$ denote a solution of (2). Then

$$(x(t), x'(t))' = (x(t), A(t)x(t)) + (x'(t))^2 \geq o \quad \text{for} \quad t \in [o, w]$$

and the function $(x(t), x'(t))$ is nondecreasing in $[o, w]$.

When

$$(x(t), x'(t)) \not\equiv \text{const.},$$

then

$$(x(o), x'(o)) \neq (x(w), x'(w))$$

and (2) has only the zero solution.

When

$$(x(t), x'(t)) = c = \textit{const.} \neq o,$$

then

$$(x(t))^2 = 2ct + d, \quad c, d = \textit{const.}$$

The boundary conditions

$$x(o) = x(w), \qquad x'(o) = x'(w)$$

fulfils the system

$$(x(o))^2 = 2\,co + d, \qquad (x(w))^2 = 2\,cw + d,$$

which is contradictory.

When $c = o$, then

$$o \equiv (x(t), x'(t))' = (x(t), A(t)x(t)) + (x'(t))^2$$

and

$$x(t) \equiv o \quad \text{for} \quad t \in [o, w].$$

A consequence of the lemma is the existence of a Green function $G(t, s)$ to the problem

(4) $$(Lx)(t) = x''(t) - A(t)x(t) = o$$
$$x(o) = x(w), \quad x'(o) = x'(w), \quad t \in [o, w].$$

The solution of the problem

$$(1') \qquad \begin{aligned} &x''(t) = A(t)x(t) + f(t, x(t), x(r(t)), x'(t), x'(z(t))) \\ &x(o) = x(w), \quad x'(o) = x'(w), \quad w > o, \quad t \in [o, w] \end{aligned}$$

can be written in the form

$$(5) \qquad x(t) = \int_o^w G(t,s) f(s, x(s), x(r(s)), x'(s), x'(z(s))) ds.$$

It follows of [5] p.60 that the operator

$$(6) \qquad (Fx)(t) := \int_o^w G(t,s) f(s, x(s), x(r(s)), x'(s), x'(z(s))) ds$$

is completely continuous in $AC^1[o,w]$. The fixed point of (6) is a $AC^1[o,w]$ solution of (5).

Theorem 1 *Let us consider the problem (1') and assume that*

$$(A(t)x, x) > o \quad \text{for} \quad t \in [o,w] \quad \text{and} \quad x \neq o.$$

The functions $f(t,x,y,z,u)$, $r(t)$, $z(t)$ fulfil the above formulated assumptions, especially f is bounded for $t \in [o,w]$ and arbitrary (x,y,z,u).
Then (1') has at least one solution $x \in AC^1[o,w]$.

Proof For the following considerations let us introduce the trivial vector field

$$(7) \qquad (\phi x)(t) = x(t) - \int_o^w ox(s) ds,$$

it is completely continuous in $AC^1[o,w]$,

$$\phi : AC^1[o,w] \to AC^1[o,w]$$

and for $x \in AC^1[o,w]$, $\| x \| = R > o$ it is $\phi(x) \neq o$. The rotation $\gamma(\phi, S_R)$ of $\phi(x)$ on the sphere

$$S_R = \{x : x \in AC^1[o,w], \| x \| = R > o\}$$

may be considered. The vector field ϕ fulfils the condition

$$\phi(-x) = -\phi(x) \quad \text{hence} \quad \gamma(\phi, S_R) \neq o.$$

It is

$$\inf_{x \in S_R} \| \phi(x) \| = \alpha_R = R > o.$$

Therefore

$$(8) \qquad \inf_{x \in kS_R} \| \phi(x) \| = k\alpha_R = kR \to \infty, \quad \text{when} \quad R \to \infty.$$

Further let

$$(9) \qquad \begin{array}{c} (\psi x)(t) =: x(t) - \int_o^w G(t,s)f(s,x(s),x(r(s)),x'(s),x'(z(s)))ds, \\ x \in AC^1[o,w], \quad \| x \| = R > o. \end{array}$$

The vector field ψ is completly continuous,

$$\psi : AC^1[o,w] \to AC^1[o,w]$$

and

$$(10) \quad \| \phi x - \psi x \| = \| \int_o^w G(t,s)f(s,x(s)x(r(s)),x'(s),x'(z(s))ds \| \leq c = \text{const.}$$

The constant c in (10) is independent of the radius R of S_R. Conditions (8) and (10) enable for R_1 sufficiently large the inequality

$$(11) \qquad \| \phi x - \psi x \| < \| \phi x \|, \quad x \in S_{R_1}.$$

From the last inequality it follows that the vector fields ϕ and ψ on S_{R_1} are homotopic and therefore

$$(12) \qquad \gamma(\psi, S_{R_1}) \neq o.$$

The nonzero rotation (12) is sufficient for the existence of at least one solution $x \in AC^1[o,w]$ of the problem (1').

Let us now consider the problem (1) with the function $g(t,x)$ instead of $A(t)x$ in (1'). The behaviour of $g(t,x)$ in the neighbourhood of the origin $(t,x) = (o,o)$ is not important for the validity of theorem.

To examine this question we formulate the following lemma.

Lemma 2 *Let us consider the problem (1) and suppose that there exists $R_o > o$ such that for*

$$(t,x) \in D_1 =: \{(t,x) : t \in [o,w], \ \| x \| \geq R_o\}$$

$g(t,x) \equiv A(t)x$ and $(A(t)x,x) > o$. In the set

$$D_2 =: \{(t,x) : t \in [o,w], \ \| x \| < R_o\}$$

the function $g(t,x)$ in general is nonlinear.

Then there exists a linear function $\bar{A}(t)x$ for $(t,x) \in D_2$ and $(\bar{A}(t)x,x) > o$ such that

$$(13) \qquad \hat{A}(t)x =: \begin{cases} \bar{A}(t)x, & (t,x) \in D_2 \\ g(t,x) = A(t)x, & (t,x) \in D_1 \end{cases}$$

fulfils the Caratheodory conditions and

$$(14) \qquad \| A(t)x - g(t,x) \| \leq m = const. \quad for \quad (t,x) \in D_2.$$

To prove the lemma it is enough to take as $\bar{A}(t)x$, $(t,x) \in D_2$ any continuous prolongation of $A(t)x$, $(t,x) \in D_1$, such that

$$(\bar{A}(t)x, x) > o \quad \text{for} \quad (t,x) \in D_2.$$

Now we may demonstrate the following theorem.

Theorem 2 *Let us examine problem (1) and assume lemma 2 and that the functions f, r, z and g fulfil the conditions formulated at the beginning of the paper. Then (1) has at least one solution.*

Proof Let us rewrite (1) in the form

$$\begin{aligned} &x''(t) = \\ (15) \qquad &= \bar{A}(t)x(t) + f(t, x(t), x(r(t)), x'(t), x'(z(t))) + g(t, x(t)) - \bar{A}(t)x(t), \\ &x(o) = x(w), \quad x'(o) = x'(w), \quad x \in AC^1[o, w], \quad \| x \| = R > o. \end{aligned}$$

The difference

$$f(t, x(t), x(r(t)), x'(t), x'(z(t))) + g(t, x(t)) - \bar{A}(t)x(t) \quad (t,x) \in D_2$$

is bounded and $\hat{A}(t)x$ fulfils the condition $(\hat{A}(t)x, x) > o$ for $t \in [o, w]$ and $x \neq o$. Therefore theorem 1 may be applicated to (14) and this finished the proof.

In a well known manner it is possible to get w-periodic solutions of the problem (1).

Theorem 3 *If $A(t), r(t), z(t), f(t, x, y, z, u)$ are w-periodic in t and fulfil the assumptions of theorem 2, then problem (1) has at least one w-periodic solution $x \in AC^1[o, w]$.*

Examples The above theorems can be applied to the system

$$\begin{cases} x_1''(t) = x_1(t) + tx_2(t) + \arctan(x_2(\sin t)) \\ x_2''(t) = -tx_1(t) + 4x_2(t) + \exp(-[x_1^2(\sin t) + x_2^2(\sin t)]) \end{cases} \quad t \in [o, 2\pi]$$

but it is impossible to apply the existence theorems to the scalar problem

$$x''(t) = ox(t) + 2, \quad x(o) = x(w), \quad x'(o) = x'(w), \quad t \in [o, w], \quad w > o$$

arbitrary.

The condition $(A(t)x, x) > o$ for $t \in [o, w]$, $x \neq o$ is not valid.

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Author's address: Institute of Mathematics
Silesian University
Bankowa 14
40 007 Katowice
Poland