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ON THE APPROXIMATION OF THE BOOLEAN POWERS BY CARTESIAN POWERS

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Abstract

On the base of the Malcev — Cleave theorem on quasi-universal formulas it is proved, that a number of properties of congruences, tolerances, quasiorders can be transported from finite cartesian powers of algebraic systems on any boolean powers.

Key words: boolean power, congruence property, second order formula

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The concept of boolean power (the restricted boolean power) is introduced in the paper by Foster [1] and seemed to be useful enough in studying of some problems in the universal algebra and model theory. The survey of results, which are connected with the construction of boolean power, can be found in [2], [3]. This note is devoted to some observations, which are connected with the question of approximation of boolean power properties by cartesian power properties of algebraic systems.

Recal that if $\mathfrak{A}$ is some algebraic system, $\mathfrak{B}$ is a boolean algebra, then the boolean power $\mathfrak{A}^{\mathfrak{B}}$ is the subsystem of direct power $\mathfrak{A}^{\mathfrak{B}^*}$ with the underlying set $C(\mathfrak{B}^*,\mathfrak{A})$. Here $\mathfrak{B}^*$ is the Stone space of the boolean algebra $\mathfrak{B}$, and $C(\mathfrak{B}^*,\mathfrak{A})$ is a set of continuous maps of the space $\mathfrak{B}^*$ into the system $\mathfrak{A}$, equiped by the discrete topology. As the space $\mathfrak{A}$ is discrete and the space $\mathfrak{B}^*$ is compact, then the elements from $C(\mathfrak{B}^*,\mathfrak{A})$ are exactly functions $f \in \mathfrak{A}^{\mathfrak{B}^*}$. More precisely, for any finite subset $\{a_1, \dots, a_n\} \subseteq \mathfrak{A}$ and any finite subset $\{b_1, \dots, b_n\} \subseteq \mathfrak{B}$ which is the partition of unity of the boolean algebra $\mathfrak{B}$ the equality $f^{-1}(a_i) = b_i$ holds if elements b_i are identified with the corresponding clopen subsets of the space $\mathfrak{B}^*$. If is clear that if $\mathfrak{B}$ is a finite boolean algebra with n atoms, then

$\mathfrak{A}^{\mathfrak{B}} \simeq \mathfrak{A}^n$. A natural question arises, how far the properties of the cartesian powers $\mathfrak{A}^n$ influent the properties of arbitrary boolean powers $\mathfrak{A}^{\mathfrak{B}}$.

The concept of congruence–boolean power $\mathfrak{A}^{\mathfrak{B}}(\Theta)$, where Θ is a congruence of an algebraic system $\mathfrak{A}$, was introduced in [4]. Here $\mathfrak{A}^{\mathfrak{B}}(\Theta)$ is a subsystem of the system $\mathfrak{A}^{\mathfrak{B}}$ with the underlying set

$$\{f \in \mathfrak{A}^{\mathfrak{B}} \mid \langle f(i), f(j)\rangle \in \Theta \quad \text{for any} \quad i, j \in \mathfrak{B}^*\}.$$

By analogy, $\mathfrak{A}^n(\Theta)$ is a subsystem of the system $\mathfrak{A}^n$ with the underlying set

$$\{f \in \mathfrak{A}^n \mid \langle f(i), f(j)\rangle \in \Theta \quad \text{for any} \quad i, j \in n\},$$

moreover, if $\mathfrak{B}$ is a finite boolean algebra with n atoms, then $\mathfrak{A}^{\mathfrak{B}}(\Theta) \simeq \mathfrak{A}^n(\Theta)$.

In the papers by A. I. Malcev [5] and I. P. Cleave [6] there was introduced the notion of quasi-universal formula. A formula of the second order predicate calculus in prenex normal form is said to be object–universal if there is no occurences of the quantifier $\exists$ referring to an object variable. A formula of the second order predicate calculus is quasi–universal if it is obtained from object–universal formulas without free object variables, by first combining them using connectives only, and then using the universal quantifier $\forall$ to bind every predicate–variable.

A family $\mathfrak{A}_i$ $(i \in I)$ of subsystems of an algebraic system $\mathfrak{A}$ is said to form a local covering of $\mathfrak{A}$, if for any $a \in \mathfrak{A}$ there exists $i \in I$ such that $a \in \mathfrak{A}_i$, and if for each pair $\mathfrak{A}_i, \mathfrak{A}_j$ $(i, j \in I)$ there is a third $\mathfrak{A}_k$ $(k \in I)$ containing both of them. In [5], [6] (see also [7]) it is proved the following theorem.

Theorem 1 *If a quasi–universal formula is true on subsystems $\mathfrak{A}_i$ $(i \in I)$ locally covering an algebraic system $\mathfrak{A}$, then φ is also true on $\mathfrak{A}$.*

Proposition 1 *For every quasi–universal formula φ, and each algebraic system $\mathfrak{A}$, if for any $n \in \omega$ $\mathfrak{A}^n \models \varphi$, then for any boolean algebra $\mathfrak{B}$ $\mathfrak{A}^{\mathfrak{B}} \models \varphi$. Analogously, if Θ is a congruence on $\mathfrak{A}$ and $\mathfrak{A}^n(\Theta) \models \varphi$ for any $n \in \omega$, then $\mathfrak{A}^{\mathfrak{B}}(\Theta) \models \varphi$ for every boolean algebra $\mathfrak{B}$.*

Proof It is sufficient to show that for any boolean algebra $\mathfrak{B}$ there exists a local covering of the system $\mathfrak{A}^{\mathfrak{B}}$ by its subsystems which are isomorphic to the systems $\mathfrak{A}^n$ $(n \in \omega)$. For any partition $\{b_1, \ldots, b_n\}$ of the unit of the boolean algebra $\mathfrak{B}$ we put

$$\mathfrak{A}^{\mathfrak{B}}(b_1, \ldots, b_n) = \{f \in \mathfrak{A}^{\mathfrak{B}} \mid \text{for some } \ a_1, \ldots, a_n \in \mathfrak{A}, \ \ f^{-1}(a_i) \supseteq b_i \ \text{ for } i \leq n\}.$$

Obviously, the family of subsystems $\mathfrak{A}^{\mathfrak{B}}(b_1, \ldots, b_n)$, where $\{b_1, \ldots, b_n\}$ is any partition of the unit of the algebra $\mathfrak{B}$ is a local covering of the system $\mathfrak{A}^{\mathfrak{B}}$. Moreover, $\mathfrak{A}^{\mathfrak{B}}(b_1, \ldots, b_n) \cong \mathfrak{A}^n$.

Thus the proposition follows from the result of A. I. Malcev and I. P. Cleave.

The analogy can be proved for congruence–boolean powers.

Let $\mathrm{Con}'\mathfrak{A}$ be the lattice of congruences of an algebraic system $\mathfrak{A}$ which is equipped by the partial operation $\circ$ — the product of relations. By analogy the lattices $\mathrm{Tol}'\mathfrak{A}$, $\mathrm{Qord}'\mathfrak{A}$ of tolerances and quasiorders on system $\mathfrak{A}$ are defined, which are also enriched by the partial operation $\circ$ (on the $\mathrm{Tol}\,\mathfrak{A}$ and $\mathrm{Qord}\,\mathfrak{A}$ see, as example, [8], [9]).

Corollary 1 *If φ is a universal formula of the signature $\langle \vee, \wedge, \circ \rangle$ and if for some algebraic system $\mathfrak{A}$ of a finite signature $\mathrm{Con}'(\mathfrak{A}^n) \models \varphi$ ($\mathrm{Tol}'(\mathfrak{A}^n) \models \varphi$, $\mathrm{Qord}'(\mathfrak{A}^n) \models \varphi$) for any $n \in \omega$, then $\mathrm{Con}'(\mathfrak{A}^{\mathfrak{B}}) \models \varphi$ ($\mathrm{Tol}'(\mathfrak{A}^{\mathfrak{B}}) \models \varphi$, $\mathrm{Qord}'(\mathfrak{A}^{\mathfrak{B}}) \models \varphi$) for any boolean algebra $\mathfrak{B}$.*

Proof The proposition implies that it is sufficient to show that for any universal formula φ of the signature $\langle \vee, \wedge, \circ \rangle$ there exists a quasi-universal formula α_φ ($\beta_\varphi, \gamma_\varphi$) such that for any algebraic system $\mathfrak{A}$ of a finite signature $\mathrm{Con}'(\mathfrak{A}) \models \varphi$ ($\mathrm{Tol}'(\mathfrak{A}) \models \varphi$, $\mathrm{Qord}'(\mathfrak{A}) \models \varphi$) iff $\mathfrak{A} \models \alpha_\varphi$ ($\mathfrak{A} \models \beta_\varphi, \mathfrak{A} \models \gamma_\varphi$). Therefore, let us note that the property of a binary relation $\Theta \subseteq \mathfrak{A}^2$ to be a congruence (tolerance, quasiorder) on the algebraic system $\mathfrak{A}$ of some fixed finite signature is expressible in an object — universal formula, as well as in relations $\Theta_1 \wedge \Theta_2 = \Theta_3$, $\Theta_1 \vee \Theta_2 = \Theta_3$, $\Theta_1 \circ \Theta_2 = \Theta_3$ for congruences (tolerances, quasiorders) are expressable.

In particular, from Corollary 1 it follows that if for some algebraic system $\mathfrak{A}$ and any $n \in \omega$ the algebraic system $\mathfrak{A}^n$ is congruence–modular, congruence–distributive, (congruence–permutable, congruence–n-permutable, have Fraser–Horn property) then for any boolean algebra $\mathfrak{B}$ the system $\mathfrak{A}^{\mathfrak{B}}$ also has the corresponding property. An analogous statement remains true if we replace congruence–properties by corresponding tolerance and quasi–order–properties. Here the Fraser–Horn property for congruences (tolerances, quasiorders) on $\mathfrak{A}$ means, that for any systems $\mathfrak{A}_1$,$\mathfrak{A}_2$ such that $\mathfrak{A} \cong \mathfrak{A}_1 \times \mathfrak{A}_2$ and $\Theta \in \mathrm{Con}\,\mathfrak{A}$ there exist $\Theta_1 \in \mathrm{Con}\,\mathfrak{A}_1$, $\Theta_2 \in \mathrm{Con}\,\mathfrak{A}_2$ such that $\Theta = \Theta_1 \times \Theta_2$ (identifying $\mathfrak{A}$ and $\mathfrak{A}_1 \times \mathfrak{A}_2$).

In the same way it can be shown that if for any $n \in \omega$ the system $\mathfrak{A}^n$ is a-congruence regular (tolerance–trivial, q-trivial), then for any boolean algebra $\mathfrak{B}$ the system $\mathfrak{A}^{\mathfrak{B}}$ is a-congruence regular (tolerance–trivial, q-trivial), where a is some constant of an algebraic system $\mathfrak{A}$.

The variety $\mathfrak{M}$ has the Fraser–Horn congruence–property if any $\mathfrak{M}$-algebra has this property. It is known that any congruence–distributive variety, any variety of rings with a unit and so on, has the Fraser–Horn congruence–property (on the varieties with Fraser–Horn congruence–property, see e.g. [10]).

Corollary 2 *Let φ be a universal Horn formula, let $\mathfrak{A}$ be an algebraic system from a variety with the Fraser–Horn congruence–property and let $\mathrm{Con}(\mathfrak{A}) \models \varphi$, then $\mathrm{Con}(\mathfrak{A}^{\mathfrak{B}}) \models \varphi$ for any boolean algebra $\mathfrak{B}$.*

Proof This statement follows from Corollary 1 and the fact that the Fraser–Horn congruence–property implies that $\mathrm{Con}(\mathfrak{A}^n) \cong (\mathrm{Con}\,\mathfrak{A})^n$.

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