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CIRCULAR TOTALLY SEMI-ORDERED GROUPS

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Abstract

In the paper, circular totally semi-ordered groups are introduced and some properties of them, especially for the cases having least strictly positive elements, are studied.

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Let $T \neq \emptyset$ be a set. Then a binary relation "$\leq$" on A is called a *semi-order* if it is reflexive and antisymmetric. The pair $(T, \leq)$ is then said to be a *semi-ordered set* (a *so-set*).

If moreover

$$\forall a, b \in T; \quad a \leq b \text{ or } b \leq a,$$

then $(T, \leq)$ is called a *tournament*. Denote

$$a < b \Longleftrightarrow_{\mathrm{df}} a \leq b \text{ and } a \neq b.$$

A tournament $T = (T, \leq)$ is said to be *circular* (see [1]) if

(a) there exist $a, b, c \in T$ such that $a < b < c < a$,

and if

(b) whenever $x, y, z \in T$ satisfy $x < y < z < x$, then there exists no $w \in T$ such that $w < \{x, y, z\}$ or $\{x, y, z\} < w$.

If $(G,+)$ is a group and $(G,\leq)$ is a *so*-set, and if

$$a \leq b \Rightarrow c+a+d \leq c+b+d$$

for any $a,b,c,d \in G$, then $G=(G,+,\leq)$ is called a *semi-ordered group* (a *so-group*). If, moreover, $(G,\leq)$ is a tournament, then $G=(G,+,\leq)$ is called a *totally semi-ordered group* (a *to-group*). A *to*-group G is said to be *circular* if the tournament $(G,\leq)$ is circular.

We will denote by G^+ the *positive cone* of any *so*-group G (i.e. $G^+=\{x\in G;\ 0\leq x\}$).

Some properties of *so*-groups and *to*-groups were studied in [2], [3], [4] and [5].

The definition of a *to*-group of course admits essentially more possibilities of total semi-orders than total orders on a given group. For example, if G is an abelian group, then any subset P with 0 of G containing no non-zero element together with its opposite element such that $P\cup -P=G$ is the positive cone of a total semi-order on G.

Therefore, first it is important to study classes of *to*-groups which are "enough close" to totally ordered groups. Evidently the circular *to*-groups form such a class of *to*-groups. The study of properties of circular *to*-groups is the aim of the paper.

Proposition 1 *A to-group $G=(G,+,\leq)$ is circular if and only if there are $u,v\in G$ with $0<u<v<0$ and if $(G^+,\leq)$ satisfies the condition (b) from the definition of a tournament.*

Proof Let G be circular, $a,b,c\in G$, $a<b<c<a$. Then $0<b-a<c-a<0$. The condition (b) is satisfied trivially.

Conversely, let $x,y,z,w\in G$, $x<y<z<x$, $w<\{x,y,z\}$. Then $0<\{x-w,y-w,z-w\}$ and $x-w<y-w<z-w<x-w$, and so we get a contradiction with the hypothesis of the validity of (b) in G^+. Similarly for $\{x,y,z\}<w$. The condition (a) is for G valid trivially. □

Example 1 We will show that the *to*-group $G=(G,+,\leq)$, where $(G,+)=(\mathbb{Z},+)$ and

$$G^+=\{0,1,-2,3,4,-5,6,7,-8,9,10,-11,\dots,3n,3n+1,-(3n+2),\dots\}$$

is circular.

(a) We have e.g. $0<1<-1<0$.

(b) Let $x,y,z\in G^+\setminus\{0\}$, $x<y<z<x$. Then $y-x,z-y,x-z\in G^+\setminus\{0\}$.

1. Let $y-x=3a$, $z-y=3b$, $x-z=3c$, $a,b,c\in\mathbb{N}$. Then

$$3a=y-x=z-3b-z-3c=3(-b-c),$$

a contradiction, hence such elements x,y,z do not exist.

2. Let $y - x = 3a$, $z - y = 3b + 1$, $x - z = -(3c + 2)$, $a \in \mathbb{N}$, $b, c \geq 0$. Then

$$3a = y - x = z - 3b - 1 - z + 3c + 2 = 3(-b + c) + 1,$$

a contradiction.

3. Let $y - x = 3a$, $z - y = 3b$, $x - z = 3c + 1$. Then

$$3a = y - x = z - 3b - z - 3c - 1 = 3(-b - c) - 1,$$

a contradiction.

4. Let $y - x = 3a$, $z - y = 3b$, $x - z = -(3c + 2)$. Then

$$3a = y - x = z - 3b - z + 3c + 2 = 3(-b + c) + 2,$$

a contradiction.

5. Let $y - x = 3a + 1$, $z - y = 3b + 1$, $x - z = -(3c + 2)$. Then

$$3a + 1 = y - x = z - 3b - 1 - z + 3c + 2 = 3(-b + c) + 1,$$

hence $a = -b + c$.

Let $x = 3n$. Then

$$z = x + 3c + 2 = 3n + 3c + 2 = 3(n + c) + 2 \notin G^+,$$

a contradiction.

Let $x = 3n + 1$. Then

$$y = 3n + 1 + 3a + 1 = 3(n + a) + 2 \notin G^+,$$

a contradiction.

Let $x = -(3n + 2)$. Then

$$y = -3n - 2 + 3a + 1 = 3(-n + a) - 1 \notin G^+,$$

a contradiction.

6. Let $y - x = 3a + 1$, $z - y = -(3b + 2)$, $x - z = -(3c + 2)$. Then

$$3a + 1 = y - x = z + 3b + 2 - z + 3c + 2 = 3(b + c + 1) + 1,$$

hence $a = b + c + 1$.

Let $x = 3n$. Then

$$z = 3n + 3c + 2 = 3(n + c) + 2 \notin G^+,$$

a contradiction.

Let $x = 3n + 1$. Then

$$y = 3n + 1 + 3a + 1 = 3(n + a) + 2 \notin G^+,$$

a contradiction.

Let $x = -(3n+2)$. Then

$$y = -3n - 2 + 3a + 1 = 3(-n+a) - 1 \notin G^+,$$

a contradiction.

7. Let $y - x = 3a + 1$, $z - y = 3b + 1$, $x - z = 3c + 1$. Then

$$3a + 1 = y - x = z - 3b - 1 - z - 3c - 1 = 3(-b - c - 1) + 1,$$

hence $a = -b - c - 1$, a contradiction.

8. Let $y - x = -(3a+2)$, $z - y = -(3b+2)$, $x - z = -(3c+2)$. Then

$$-(3a+2) = y - x = z + 3b + 2 - z + 3c + 2 = 3(b + c + 1) + 1,$$

hence $3(-a-1)+1 = 3(b+c+1)+1$, therefore $a = -b-c-2$, a contradiction.

9. Let $y - x = 3a$, $z - y = 3b + 1$, $x - z = 3c + 1$. Then

$$3a = y - x = z - 3b - 1 - z - 3c - 1 = 3(-b - c) - 2,$$

a contradiction.

10. Let $y - x = 3a$, $z - y = -(3b+2)$, $x - z = -(3c+2)$. Then

$$3a = y - x = z + 3b + 2 - z + 3c + 2 = 3(b + c + 1) + 1,$$

a contradiction.

11. Let, for example, $y - x = 3a + 1$, $z - y = 3b + 1$, $x - z = 3c$. Then

$$3a + 1 = y - x = z - 3b - 1 - z - 3c = 3(-b - c) - 1,$$

a contradiction.

12. Let, for example, $y - x = -(3a+2)$, $z - y = 3b+1$, $x - z = 3c+1$. Then

$$3(-a) - 2 = y - x = z - 3b - 1 - z - 3c - 1 = 3(-b - c) - 2,$$

hence $a = b + c$.

Let $x = 3n$. Then

$$z = 3n - 3c - 1 = 3(n - c) - 1,$$

a contradiction.

Let $x = 3n + 1$. Then

$$y = 3n + 1 - 3a - 2 = 3(n - a) - 1,$$

a contradiction.

Let $x = -(3n+2)$. Then

$$y = -3n - 2 - 3a - 2 = 3(-n - a - 1) - 1,$$

a contradiction.

Therefore we can see that in all examined cases (and evidently also in all remaining ones) such elements x, y, z do not exist. Hence the condition (b) is for G^+ valid trivially.

Example 2 Denote $G = (\mathbb{Z}, +, \leq)$, where $G^+ = (\mathbb{Z}^+ \setminus \{4\}) \cup \{-4\}$. ($\mathbb{Z}^+$ is meant in the natural order of $(\mathbb{Z}, +)$.) Then G is a *to*-group, but it is not circular. Indeed, for example, $1 < 3 < 5 < 1$ and $0 < \{1, 3, 5\}$.

The positive cone G^+ of a *so*-group G need not be, in general, convex in G. (For instance, for $\mathbb{Z}_3$, where $\mathbb{Z}_3^+ = \{0, 1\}$, we have $1 < 2 < 0$, $1, 0 \in \mathbb{Z}_3^+$, but $2 \notin \mathbb{Z}_3^+$.

Lemma 2 *If G is a so-group such that G^+ is convex in G, then G satisfies one of the following conditions:*
a) *G is a po-group (i.e. "$\leq$" is transitive);*
b) $\exists a, b \in G;\ 0 < a < b,\ 0 \parallel b$.

Proof Let us suppose that $x, y, z \in G$ and $x < y < z$, that means $0 < -x+y < -x+z$. If in such a case always $0 < -x+z$, then G is a *po*-group.

Thus, let $0 \not< -x+z$. Suppose that $-x+z \leq 0$. Then $-x+y < -x+z \leq 0$, hence from the convexity of G^+ we have $-x+z \in G^+$. Therefore $-x+z \in G^+ \cap -G^+ = \{0\}$, i.e. $x = z$, a contradiction. Hence $0 \parallel -x+z$. □

Corollary *If G is a to-group, then the following conditions are equivalent:*
a) *G is an o-group (i.e. a totally ordered group).*
b) *G^+ is convex in G.*
c) *There are no elements $a, b \in G$ with $0 < a < b < 0$.*

Proof $a \iff b$: By Lemma 2.
$b \implies c$: Trivial.
$c \implies a$: Suppose that G is not an *o*-group. Then there exist elements $x, y, z \in G$ such that $x < y < z < x$, hence $0 < -x+y < -x+z < 0$, a contradiction. Therefore $x < z$, and thus "$\leq$" is transitive. □

Theorem 3 *Let G be a circular to-group which contains an element $a \in G^+ \setminus \{0\}$ such that $a \leq b$ for every $b \in G^+ \setminus \{0\}$ (i.e. a is the least element of $G^+ \setminus \{0\}$), and let a have infinite order. Then $[a] = \mathrm{grp}(a)$ is a subgroup of G that is an o-group and for which $[a]^+$ is convex in G^+.*

Proof a) Let a be the least element of $G^+ \setminus \{0\}$. Let us suppose that $x \in G$, $n \in \mathbb{N}$, and $0 < x \leq na$. Then $a \leq x$, and so $0 \leq x - a$. If $x - a = 0$, then $x \in [a]$. In the opposite case $0 < x - a$, hence $a \leq x - a$, that means $0 \leq x - 2a$. If $x - 2a = 0$, then $x \in [a]$, otherwise $0 < x - 2a$, etc. But because $x \leq na$, there exists $k \in \mathbb{N}$, $0 < k \leq n$, such that $x = ka$, therefore $x \in [a]$.

b) Let us show that the *to*-group $[a]$ is an *o*-group. First we will prove that $(-n)a < 0$ for any $n \in \mathbb{N}$. Let n be the least natural number with $0 < (-n)a$. (Clearly $n > 1$). Then we have:

$$\begin{aligned} (2n-1)a - (2n)a = -a < 0, &\quad \text{hence} \quad (2n-1)a < (2n)a; \\ (2n)a - na = na < 0, &\quad \text{hence} \quad (2n)a < na; \\ na - (2n-1)a = -(n-1)a < 0, &\quad \text{hence} \quad na < (2n-1)a. \end{aligned}$$

At the same time: Because $0 < (-n)a$, we have $a \leq (-n)a$, thus $0 \leq (-n-1)a$, and because a has infinite order, it must be $0 < (-n-1)a$. But this means that $a \leq (-n-1)a$, and so $0 < (-n-2)a$. By this method, we obtain $0 < (-2n+1)a$, $0 < (-2n)a$. Therefore we have

$$(-2n)a < (-2n+1)a < (-n)a < (-2n)a,$$
$$0 < (-2n+1)a, \quad 0 < (-2n)a, \quad 0 < (-n)a,$$

that contradicts the condition (b) from the circularity of G.

Hence $(-n)a < 0$, and therefore $0 < na$ for any $n \in \mathbb{N}$.

Now, if $m, n \in \mathbb{Z}$, $na \in [a]^+$, $0 \leq ma \leq na$, then $m, n \in \mathbb{Z}^+$, and thus $[a]^+$ is convex in $[a]$. But this means, by Corollary of Lemma 2, that $[a]$ is an *o*-group.

c) Now it is clear, by the preceding parts of the proof, that $[a]^+$ is convex in G^+. □

Theorem 4 *Let G be a circular to-group with the least strictly positive element a which has infinite order. Then $[a]$ is the least of all proper subgroups H of G such that H^+ is convex in G^+.*

Proof Let H be a subgroup of G and let H^+ be convex in G^+. If $0 < b \in H$, then $a \leq b$, and hence $0 < a \leq b$ implies $a \in H$. □

Theorem 5 *If G is a circular to-group with the least strictly positive element a, then there is no element x in G such that $0 < x$, $-x < x$ and $x < (-n)a$ for some $n \in \mathbb{N}$.*

Proof Suppose that for $0 < x$, $-x < x$, there exists $n \in \mathbb{N}$ such that $x < (-n)a$. Since $0 < x$ we have $a \leq x$, and since $x \neq a$ (it follows from the fact that $a \not< (-n)a$), $0 < -a + x$. From this $a \leq -a + x$, and because $x \neq 2a$, we obtain $0 < -2a + x$, i.e. $2a < x$, etc. Therefore $na < x$, that means $-x < (-n)a$. Hence $-x < \{0, x, (-n)a\}$, and at the same time $0 < x < (-n)a < 0$, a contradiction with the circularity of G. □

Example 3 Consider again the circular *to*-group G from Example 1. Let $n \in \mathbb{N}$.

Then

$$\begin{aligned} 3n-3 = 3(n-1) \in G^+, &\quad \text{hence} \quad 3n \geq 3, \\ (3n+1)-3 = 3(n-1)+1 \in G^+, &\quad \text{hence} \quad 3n+1 > 3, \\ -(3n+2)-3 = -3(n+2)+1 \in G^+, &\quad \text{hence} \quad -(3n+2) > 3, \end{aligned}$$

therefore 3 is the least element in $G^+ \setminus \{0\}$.

Hence the subgroup [3] is, by Theorem 3, an *o*-group and it is the least of all subgroups H of G such that H^+ is convex in G^+.

In this case, the subgroup $[3] = 3\mathbb{Z}$ has more properties. Consider the group $G' = (\mathbb{Z}_3, +)$ of numbers $\{0, 1, 2\}$ with the addition modulo 3 totally semi-ordered by $0 < 1 < 2 < 0$. Let f be the mapping of $\mathbb{Z}$ onto $\mathbb{Z}_3$ such that for $x \in 3\mathbb{Z} + i$, $f(x) = i$ $(i = 0, 1, 2)$. Clearly, f is a *wal*-homomorphism of G onto G' with the kernel $3\mathbb{Z}$, and hence $3\mathbb{Z}$ is a *wal*-ideal of G.

Let $n\mathbb{Z}$ $(n > 1)$ be a convex *wal*-ideal of G. If $n \in 3\mathbb{N}$, then $0 < 3$ and $3 \leq n$ imply $3 \in n\mathbb{Z}$. But this is possible only for $n = 3$.

If $n \in 3\mathbb{N} + 1$, then $0 < 1 < n$ imply $1 \in n\mathbb{Z}$, a contradiction.

If $n \in 3\mathbb{N} + 2$, then $0 < n - 1 < n$, hence $n - 1 \in n\mathbb{Z}$, a contradiction.

This means that $3\mathbb{Z}$ is the unique proper *wal*-ideal (and so also the unique convex *wal*-subgroup) of G.

Evidently $3\mathbb{Z}$ is also the only subgroup such that its positive cone is convex in G^+.

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