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A Note on Tolerance Lattices of Algebras with Restricted Similarity Type

BEDŘICH PONDĚLÍČEK

Department of Mathematics, Faculty of Electrical Eng., Czech Techn. University, Technická 2, 166 27 Praha 6, Czech Republic

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Abstract

It is shown that for every finite algebra $A$ of a finite similarity type there exists a finite algebra $C$ of type $(2,1,1)$ such that the lattices of tolerances (congruences) on $A$ and $C$ are isomorphic.

Key words: Finite algebra of a finite similarity type, Lattice of congruences, Lattice of tolerances.

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In [1] I. Chajda proved that for every algebra $A$ of a finite similarity type there exists an algebra $B$ of type $(2,1,1)$ such that $\text{Tol } A \cong \text{Tol } B$.

Recall that $A = (A, F)$ is of a finite similarity type if $F$ is a finite set. By $\text{Tol } A$ we shall mean the lattice of all tolerances on an algebra $A$ with respect to set inclusion and a tolerance on $A$ is a binary reflexive and symmetric relation on $A$ which is a subalgebra of the direct product $A \times A$. See [2].

Unfortunately, if an algebra $A$ is finite and nontrivial, then Chajda’s algebra $B$ of type $(2,1,1)$ satisfying $\text{Tol } A \cong \text{Tol } B$ is not finite. In this note we shall show the following:

Theorem For every finite algebra $A$ of a finite similarity type there exists a finite algebra $C$ of type $(2,1,1)$ such that $\text{Tol } A \cong \text{Tol } C$.

Proof Let $A = (A, F)$ be a finite algebra of finite similarity type. Choose a positive integer $n \geq 2$ such that $\text{card } F \leq n$ and arity $f \leq n$ for all $f \in F$. 

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We can write a finite sequence \( f_1, f_2, \ldots, f_n \), where \( \{ f_1, f_2, \ldots, f_n \} = F \) and consider every \( f_i \) as an \( n \)-ary operation on \( A \).

Introduce one binary and two unary operations on \( C = A^n \) as follows:

\[
\begin{align*}
x \cdot y &= (x_1, y_1, y_2, \ldots, y_{n-1}), \\
g_i(x) &= (f_1(x), f_2(x), \ldots, f_n(x)), \\
h(x) &= (x_2, x_3, \ldots, x_n, x_1).
\end{align*}
\]

Then \( C = (C, \{., g, h\}) \) is a finite algebra of type \((2,1,1)\). For \( x^{(1)}, x^{(2)}, \ldots, x^{(k)} \in C \) \((k \geq 2)\) we can put inductively

\[
x^{(1)} \cdot x^{(2)} \cdot \ldots \cdot x^{(k)} = x^{(1)} \cdot (x^{(2)} \cdot x^{(3)} \cdot \ldots \cdot x^{(k)}).
\]

Define the map \( \varphi : \text{Tol} \ A \rightarrow \text{Tol} \ C \). Let \( T \in \text{Tol} \ A \). We put

\[
\langle (x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \rangle \in \varphi(T) \quad \text{if and only if} \quad \langle x_i, y_i \rangle \in T \quad \text{for} \quad i = 1, 2, \ldots, n.
\]

Clearly \( \varphi(T) \) is a reflexive and symmetric binary relation on \( C \) and from (1) it is easy to show that \( \varphi(T) \in \text{Tol} \ C \).

Evidently, for \( S, T \in \text{Tol} \ A \) we have \( S \subseteq T \) if and only if \( \varphi(S) \subseteq \varphi(T) \) and so \( \varphi \) is an injection. Now, it remains to prove that \( \varphi \) is a surjection. For every \( x = (x_1, x_2, \ldots, x_n) \in C \) we put \( I(x) = x_1 \). Suppose that \( R \in \text{Tol} \ C \). Let \( T \subseteq A \times A \) such that

\[
\langle u, v \rangle \in T \quad \text{if and only if} \quad \langle x, y \rangle \in R \quad \text{for} \quad I(x) = u, I(y) = v.
\]

Clearly \( T \) is a reflexive and symmetric binary relation on \( A \).

First we shall show that

\[
\langle (x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \rangle \in R
\]

whenever \( \langle x_i, y_i \rangle \in T \) for all \( i = 1, 2, \ldots, n \).

Assume that \( \langle x_i, y_i \rangle \in T \). Then there exist \( x^{(i)}, y^{(i)} \in C \) such that \( \langle x^{(i)}, y^{(i)} \rangle \in R \), \( I(x^{(i)}) = x_i \) and \( I(y^{(i)}) = y_i \). It follows from (1) and (2) that \( x = (x_1, x_2, \ldots, x_n) = x^{(1)} \cdot x^{(2)} \cdot \ldots \cdot x^{(n)} \), \( y = (y_1, y_2, \ldots, y_n) = y^{(1)} \cdot y^{(2)} \cdot \ldots \cdot y^{(n)} \) and so \( \langle x, y \rangle \in R \).

Now we shall show that \( T \in \text{Tol} \ A \). Let \( \langle x_i, y_i \rangle \in T \) for \( i = 1, 2, \ldots, n \). It follows from (5) that \( \langle x, y \rangle \in R \), where \( x = (x_1, x_2, \ldots, x_n) \), \( y = (y_1, y_2, \ldots, y_n) \).

According to (1), we obtain \( \langle g(x), g(y) \rangle \in R \) and so \( \langle f_1(x), f_1(y) \rangle \in T \) for \( k = 1, 2, \ldots, n - 1 \) we have \( \langle h^k g(x), h^k g(y) \rangle \in R \) and so \( \langle f_i(x), f_i(y) \rangle \in T \) for \( i = 2, 3, \ldots, n \). Thus \( T \in \text{Tol} \ A \).

Finally we shall show that \( R = \varphi(T) \). Let

\[
\langle x, y \rangle = \langle (x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \rangle \in R,
\]

then according to (1), we have \( \langle h^k(x), h^k(y) \rangle \in R \) for \( k = 1, 2, \ldots, n - 1 \) and so \( \langle x_i, y_i \rangle \in T \) for \( i = 1, 2, \ldots, n \). This means that \( \langle x, y \rangle \in \varphi(T) \). Therefore \( R \subseteq \varphi(T) \).
Assume that \((x, y) \in \varphi(T)\), then by (3) we have \((x_i, y_i) \in T\) for \(i = 1, 2, \ldots, n\), where \(x = (x_1, x_2, \ldots, x_n)\) and \(y = (y_1, y_2, \ldots, y_n)\). According to (5), we have \((x, y) \in R\). Consequently we obtain \(\varphi(T) \subseteq R\) and so \(R = \varphi(T)\).

**Note** It is well known that the set \(\text{Con } A\) of all congruences on an algebra \(A\) is a subset of \(\text{Tol } A\) (the set of all transitive tolerances on \(A\)). From (3) it follows that the map \(\varphi\) has the property:

\[\varphi(\text{Con } A) \subseteq \text{Con } C.\]

Now, we shall prove that

\[\varphi(\text{Con } A) = \text{Con } C.\]

It suffices to show that \(T\) defined by (4) is a congruence on \(A\), whenever \(R \in \text{Con } C\).

Suppose that \((u, v), (v, w) \in T\) and \(R \in \text{Con } C\). According to (4), there exist \(x, y^{(1)}, y^{(2)}, z \in C\) such that \((x, y^{(1)}), (y^{(2)}, z) \in R\) and \(I(x) = u, I(y^{(1)}) = v = I(y^{(2)}), I(z) = w\). It is easy to show that by (1) and (2) we have \(x^n = x \cdot x \ldots x = (u, u, \ldots, u)\). Analogously we can obtain that \((y^{(1)})^n = (v, v, \ldots, v) = (y^{(2)})^n\) and \(z^n = (w, w, \ldots, w)\). We have \((x^n, (y^{(1)})^n), ((y^{(2)})^n, z^n) \in R\) and so \((x^n, z^n) \in R\) and \(I(x^n) = u, I(z^n) = w\). Hence, by (4), we get \((u, w) \in T\). Consequently \(T \in \text{Con } A\).

**Corollary** For every finite algebra \(A\) of a finite similarity type there exists a finite algebra \(C\) of type \((2, 1, 1)\) such that \(\text{Con } A \cong \text{Con } C\).

**References**
