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The Semigroup of Varieties of Weakly Associative Lattice Groups

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Abstract

In the paper it is proved that the varieties of weakly associative lattice groups form an ordered semigroup with one of distributive laws.

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A *semi-order* of a non-void set A is a binary reflexive and transitive relation "$\leq$" on A and $(A, \leq)$ is then called a *semi-ordered set.* If for each $a, b \in A$ there exist their join $a \vee b$ and meet $a \wedge b$ then $(A, \leq)$ is called a *weakly associative lattice (wa-lattice).* Similarly as lattices, *wa*-lattices can be equivalently defined as algebras $(A, \vee, \wedge)$ with two binary operations satisfying the identities

(I)	$a \vee a = a;$	$a \wedge a = a$
(C)	$a \vee b = b \vee a;$	$a \wedge b = b \wedge a$
(Abs)	$a \vee (a \wedge b) = a;$	$a \wedge (a \vee b) = a$
(WA)	$((a \wedge c) \vee (b \wedge c)) \vee c = c;$	$((a \vee c) \wedge (b \vee c)) \wedge c = c.$

(See [2] and [10].)

If $(G, +)$ is a group and $(G, \vee, \wedge)$ is a *wa*-lattice and if for each elements $a, b, c, d \in A$

$$(D_\vee) \qquad a + (b \vee c) + d = (a + b + d) \vee (a + c + d),$$

then $G = (G, +, \vee, \wedge)$ is called a *weakly associative lattice group (wal-group)*. (For basic properties of *wal*-groups see [6] and [7].) The notion of a *wal*-group is an essential generalization of that of lattice ordered group (l-group) because, in contrast to l-groups, there exist many non-trivial finite *wal*-groups.

Let G be a *wal*-group and A a *wal*-subgroup of G (i.e. a subgroup of G closed under $\vee$ and $\wedge$). Then A is called a *wal-ideal* of G if it is normal in G and if it satisfied the following condition:

$$\forall a, b, c \in A,\ x, y \in G; \qquad x \le a,\ y \le b \implies (x \vee y) \vee c \in A.$$

By [6] and [7], the kernels of *wal*-homomorphisms of *wal*-groups are exactly all *wal*-ideals. (In the paper, the fact that A is a *wal*-ideal of an *wal*-group G will be denoted by $A \trianglelefteq G$).

The class of all *wal*-groups is a variety of type $\mathscr{L} = (+, 0, -(.), \vee, \wedge)$ of signature $< 2, 0, 1, 2, 2 >$. Next we will consider all *wal*-groups in the language $\mathscr{L}$. The varieties of *wal*-groups form by [8] a complete lattice **WAL** which is distributive and contains the lattice **L** of the varieties of l-groups as a complete $\wedge$-subsemilattice. (Infima in both lattices coincide with intersections.) The structure of the lattice **WAL** differs from that of the lattice **L**. For instance, the variety $\mathscr{A}_l$ of abelian l-groups is an atom in **WAL** but, contrary to **L**, it is not the smallest non-trivial variety of *wal*-groups. Further, the variety $\mathscr{R}_{wal}$ of representable *wal*-groups (i.e. the variety generated by all totally semi-ordered groups) is not comparable to the variety $\mathscr{A}_{wal}$ of abelian *wal*-groups. However, finite joins in **WAL** can be characterized similarly as in **L**.

Proposition 1 *Let $\mathscr{U}$ and $\mathscr{V}$ be varieties of wal-groups and G be a wal-group. Then $G \in \mathscr{U} \vee \mathscr{V}$ if and only if there are wal-ideals M and N of G such that $M \cap N = \{0\}$, $G/M \in \mathscr{U}$, and $G/N \in \mathscr{V}$.*

Proof The lattice $\mathscr{L}(G)$ of *wal*-ideals of G is, by [8, Theorem 4], distributive and hence the proposition can be proved in the same way as for the analogical proposition in [4] for varieties of l-groups. □

Now we define similarly as for groups (see [5]) and for l-groups (see [4] or [9]) the product of varieties of *wal*-groups.

Definition If $\mathscr{U}$ and $\mathscr{V}$ are varieties of *wal*-groups then their *product* $\mathscr{U}\mathscr{V}$ will be the class of *wal*-groups such that $G \in \mathscr{U}\mathscr{V}$ if and only if there exists a *wal*-ideal A of G with $A \in \mathscr{U}$ and $G/A \in \mathscr{V}$.

Theorem 2 *If $\mathscr{U}$ and $\mathscr{V}$ are arbitrary varieties of wal-groups then their product $\mathscr{U}\mathscr{V}$ is a variety of wal-groups too.*

Proof a) Let $G \in \mathscr{U}\mathscr{V}$, $A \trianglelefteq G$, $A \in \mathscr{U}$, $G/A \in \mathscr{V}$ and let H be a *wal*-subgroup of G. Denote $H_1 = H \cap A$.

If $a \in H_1$, $x \in H$, $0 \le x \le a$, then $x \in H \cap A = H_1$, hence H_1 is convex in H. Consider $a, b, c \in H_1$ and $x, y \in H$ such that $x \le a$ and $y \le b$. Clearly

$(x \vee y) \vee c \in H$. At the same time, A is a *wal*-ideal of G, hence $(x \vee y) \vee c \in A$, and so $(x \vee y) \vee c \in H_1$. That means H_1 is a *wal*-ideal of H. Moreover, H_1 is a *wal*-subgroup of A, therefore $H_1 \in \mathcal{U}$.

Consider now the factor *wal*-group $H/H_1 = H/(H \cap A)$. By [6, Theorem 13], $(H + A)/A \cong H/(H \cap A)$ (as *wal*-groups), and since $(H + A)/A$ is by [3, III.2.12] a *wal*-subgroup of G/A, $H/H_1 \in \mathcal{V}$. That means $H \in \mathcal{U}\mathcal{V}$.

b) Let G and G' be *wal*-groups, $G \in \mathcal{U}\mathcal{V}$, $A \trianglelefteq G$, $A \in \mathcal{U}$, $G/A \in \mathcal{V}$, and let $\varphi : G \to G'$ be a surjective *wal*-homomorphism. Denote $A' = \varphi[A]$. Since kernels of *wal*-homomorphisms and *wal*-ideals coincide, we have, by [3, III.2.13], that A' is a *wal*-ideal of G'. By the assumption $A \in \mathcal{U}$, hence $A' \in \mathcal{U}$ too.

We will show that $G'/A' \in \mathcal{V}$. If $K = \operatorname{Ker} \varphi$, then the following possibilities can come:

α) $K \subseteq A$: Then by [3, III.2.13] $G'/A' \cong G/A$, and thus $G'/A' \in \mathcal{V}$.

β) $A \subseteq K$: Then $A' = \varphi[A] = \{0'\}$ and $G' \cong G/K$. Moreover, by [6, Theorem 12], $G/K \cong (G/A)/(K/A)$, and because of $G/A \in \mathcal{V}$ we have $G/K \in \mathcal{V}$ too, and so $G'/A' \in \mathcal{V}$.

γ) $K \parallel A$: By [6, Theorem 12], $G/(K+A) \cong (G/A)/\big((K+A)/A\big)$. Moreover $\varphi[K + A] = A'$, thus, by [6, Theorem 12], we have $G/(K + A) \cong G'/A'$. By the assumption, $G/A \in \mathcal{V}$, hence also $G/(K + A) \in \mathcal{V}$, that means $G'/A' \in \mathcal{V}$.

Therefore we get $G' \in \mathcal{U}\mathcal{V}$, hence the class $\mathcal{U}\mathcal{V}$ is closed under *wal*-homomorphic images.

c) Let G_i $(i \in I)$ be *wal*-groups such that $G_i \in \mathcal{U}\mathcal{V}$ for each $i \in I$. Then for each $i \in I$ there is a *wal*-ideal $A_i \trianglelefteq G_i$ such that $A_i \in \mathcal{U}_i$ and $G_i/A_i \in \mathcal{V}$. Denote $G = \prod_{i \in I} G_i$ and $A = \prod_{i \in I} A_i$. It is easy to verify that A is *wal*-ideal of G. Moreover $A \in \mathcal{U}$.

Further, the mapping $\varphi : G/A \to \prod_{i \in I} G_i/A_i$ that for each element $g = (g_i)_{i \in I} \in G$ assigns to $g + A$ in G/A the element $(g_i + A_i)_{i \in I}$ in $\prod_{i \in I} G_i/A_i$, is a *wal*-isomorphism. Hence $G/A \in \mathcal{V}$, that means $\prod_{i \in I} G_i \in \mathcal{U}\mathcal{V}$. □

Theorem 3 WAL *is an ordered semigroup with respect to the multiplication of varieties and to the order by inclusion.*

Proof The associativity of the multiplication of varieties of *wal*-groups can be proved likewise as the associativity of varieties of groups in [5, Theorem 21.51]. The validity of the implications

$$\mathcal{U} \subseteq \mathcal{V} \implies \mathcal{U}\mathcal{W} \subseteq \mathcal{V}\mathcal{W} \text{ and } \mathcal{W}\mathcal{U} \subseteq \mathcal{W}\mathcal{V}$$

is obvious. □

Theorem 4 *If $\mathcal{U}$ and $\mathcal{V}_i$ $(i \in I)$ are varieties of wal-groups then*

$$\Big(\bigcap_{i \in I} \mathcal{V}_i\Big)\mathcal{U} = \bigcap_{i \in I} \mathcal{V}_i\mathcal{U} .$$

Proof It is evident that the left side of the equality is contained in the right one. Conversely, let $G \in \bigcap_{i\in I} \mathscr{V}_i \mathscr{U}$. Then for each $i \in I$ there exists a *wal*-ideal $H_i \trianglelefteq G$ such that $H_i \in \mathscr{V}_i$ and $G/H_i \in \mathscr{U}$. Set $H = \bigcap_{i\in I} H_i$. Then H is a *wal*-ideal of G and $H \in \bigcap_{i\in I} \mathscr{V}_i$.

Denote $\varphi : G/H \to \prod_{i\in I} G/H_i$ the mapping such that for each $x \in G$, $\varphi(x + H) = (x + H_i)_{i\in I}$. It is clear that φ is an isomorphic embedding and that $\varphi[G/H]$ is a *wal*-subgroup of $\prod_{i\in I} G/H_i$. And because of $\prod_{i\in I} G/H_i \in \mathscr{U}$ we have $G \in \left(\bigcap_{i\in I} \mathscr{V}_i\right) \mathscr{U}$. □

Remark For the varieties of l-groups, also other distributivity laws (distributivity of multiplication over lattice operations from the left or the right) in the ordered semigroup **L** of l-varieties are valid too. (See [1, Theorem 6.1], [9, Theorem 10.9.7].) It remains an open question which of those laws are valid in **WAL**.

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