Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica

Miloslav Fialka

On the bundle of solutions of a nonhomogeneous second order linear differential equation

Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica, Vol. 34 (1995), No. 1, 15--20

Persistent URL: `http://dml.cz/dmlcz/120328`

Acta Univ. Palacki. Olomuc., Fac. rer. nat.,
Mathematica **34** (1995) 15–20

On the Bundle of Solutions of a Nonhomogenous Second Order Linear Differential Equation

Miloslav FIALKA

Department of Mathematics, Faculty of Technology, Technical University,
Nám. TGM 275, 762 72 Zlín, Czech Republic
E-mail: fialka@zlin.vutbr.cz

(Received January 13, 1995)

Dedicated to Prof. O. Borůvka to his 96 birthday

Abstract

The aim of author's contribution is
a) to recall concept of the node system which is sometimes called the system of knots of the first kind corresponding to a nonhomogeneous second order linear differential equation
b) to give a physical interpretation for a bundle of solutions and a node system of the first kind
c) to show graphs of a bundle and a node system in special cases of nonhomogeneous differential equations.

Key words: N-th central dispersion of the 1st kind, node system, bundle of solutions.

MS Classification: 34C10

1 Introduction

We consider a nonhomogeneous differential equation of the second order

$$y'' - q(t)\,y = r(t), \qquad q, r \in C^0(j),\ j = (a,b),\ -\infty \le a < b \le +\infty. \qquad (r)$$

The respective homogeneous differential equation of Jacobian form

$$y'' - q(t)\,y = 0\,, \tag{q}$$

will be always understood to be *oscillatory* on j, i.e. both end points of the interval j are cluster points of any solutions of equation (q).

In this paper we denote (r), (q) either the given differential equation or the set of all solutions of the differential equation in question, R the set of all real numbers, $C^0(j)$ the set of all continuous functions defined in the interval j.

Trivial solutions of (q) will not be considered. Concurrently with O. Borůvka [1] we make use of the following concepts:

Dispersion Let n be a positive integer, $t \in j$ and u be a solution of (q) such that $u(t) = 0$. If $\phi_n(t)$ $(\phi_{-n}(t))$ denotes the n-th zero of the u lying to the right (left) of t, then the function ϕ_n (ϕ_{-n}) is called the n-th $(-n$-th$)$ *central dispersion of the first kind* or the *1st kind central dispersion with the index* $n\,(-n)$ *of* (q). Instead of ϕ_1 we write ϕ which is called the *fundamental dispersion of the first kind.*

It is convenient also to introduce the *zero-th central dispersion of the first kind* by setting

$$\phi_0(t) := t \qquad \text{for all } t \in j.$$

By the above definitions, the values of the central dispersions at an arbitrary point $t \in j$ represent the zeros of integrals of the equation (q) which vanish at the point t. For more details see [1].

2 Auxiliary results

Convention 1 Let $t_0 \in j$, $z_0, z_0' \in R$ be arbitrary numbers and $z \in (r)$ throughout the paper be a solution, for which

$$z(t_0) = z_0, \quad z'(t_0) = z_0'.$$

Let $\phi_n(t)$ denote the n-th central dispersion of the 1st kind of (q), where $n = 0, \pm1, \pm2, \ldots$. In accordance with M. Laitoch [2], let $\mathscr{S}(r; t_0, z(t_0))$ or briefly $\mathscr{S}$ always denote a node system of the 1st kind corresponding to the equation (r) and to the initial condition $[t_0, z_0]$, i.e. the set of all points $[\phi_n(t_0), z(\phi_n(t_0))]$ for n being an integer.

Remark 1 We known from [2] that the node system of the 1st kind is uniquely determined by any of its points. Now it is suitable to recall the definition of the bundle of solutions of the 1st kind.

Definition 1 By a bundle of solutions of the 1st kind appropriate to the equation (r) and to the initial condition $[t_0, z_0]$ we mean all solutions $y \in (r)$ satisfying the condition $y(t_0) = z_0$ which we write as $\mathscr{S}(r; t_0, z(t_0))$, i.e. like the node system all solutions of which are passing through. We write $y \in \mathscr{S}$.

For some properties of node system and bundles the following theorem is important.

Theorem 1 *Given a bundle of solutions* $\mathscr{S}(r; t_0, z(t_0))$. *If* $\widetilde{y} \in (r)$ *and* $\widetilde{y} \notin \mathscr{S}$ *then exactly one bundle of solutions* $\widetilde{\mathscr{S}}(r; x_0, z(x_0))$ *exists, where* $x_0 \in (t_0, \phi(t_0))$, *so that*

$$\widetilde{y} \in \widetilde{\mathscr{S}}(r; x_0, z(x_0)).$$

Proof This would be similar to the proof of theorem 1 from [3, p. 109]. We use the method of [2] by means of Sturm separation theorem [4, p. 224].

3 Physical interpretation

Now we can give a physical interpretation for a bundle of solutions and a node system of the 1st kind

$$\mathscr{S} := \mathscr{S}(r; t_0, z(t_0)).$$

Let equation (q) be a *right-oscillatory* equation, which describes the time behaviour of vibration y in a certain physical process.

Assume the vibrations run from initial time $t = t_0$ to a sufficiently large time

$$t \in [t_0, T], \quad T < b.$$

Suppose the function $y = z(t)$, $t \in j$ describes the first time behaviour of variable y in a following physical process. Besides coefficient $q(t)$ this process is influenced by another physical variable, defined by function $r(t)$ in equation (r).

We denote the initial values of solution $z(t)$ of equation (r) by symbols

$$z_0 := z(t_0), \qquad z_0' := z'(t_0).$$

Now let us assume that the solutions

$$y = z_k(t), \quad z_k \in (r), \qquad k = 1, 2, 3, \ldots$$

describe the successive time values of variable y of the physical process. At the instant t_0 we assume

$$z_k(t_0) = z_0 \quad \text{and} \quad z_k'(t_0) \neq z_0',$$

i.e. we assume initial velocities of oscillations z_k and z are mutually different. Therefore z_k belongs to the bundle $\mathscr{S}$: $z_k \in \mathscr{S}$.

Any node of the 1st kind

$$[\phi_n(t_0),\, z(\phi_n(t_0))], \qquad n = 0, 1, 2, \ldots$$

for equation (r) and initial condition $[t_0, z_0]$ gives an instant $t_n := \phi_n(t_0)$ at which the following equality

$$z_k(\phi_n(t_0)) = z(\phi_n(t_0)), \qquad k = 1, 2, 3, \ldots$$

holds. This means,

$*$ *the physical variable y (vibration y) takes the same value at successive time behaviours of the physical process which is described by equation (r)*

$*$ *the exact instants* $t_n = \phi_n(t_0)$ *are described by the 1st kind central dispersion* $\phi_n(t)$ *with the non-negative integer index n of equation (q).*

We say (see [5, p. 78]), the solutions (vibrations) $z_k \in \mathscr{S}$ are right-oscillatory on the interval j with respects to the solution $z(t)$.

4 Applications

The following examples and figures show some posibilities for equation (r) and various initial values of the first derivatives of solutions from the set (r).

Example 1 The bundle of solutions and the respective node system we can demonstrate for nonhomogeneous Bessel differential equation (Fig. 1)

$$y'' - \left(\frac{63}{4t^2} - 1\right) y = e^{-t}$$

and the following initial values

$$z(0.5) = 0, \quad z'(0.5) = -0.18, \quad z_1'(0.5) = -0.20, \quad z_2'(0.5) = 0.05\,.$$

The function $r(t) = e^{-t}$ is the damping term of forced vibrations.

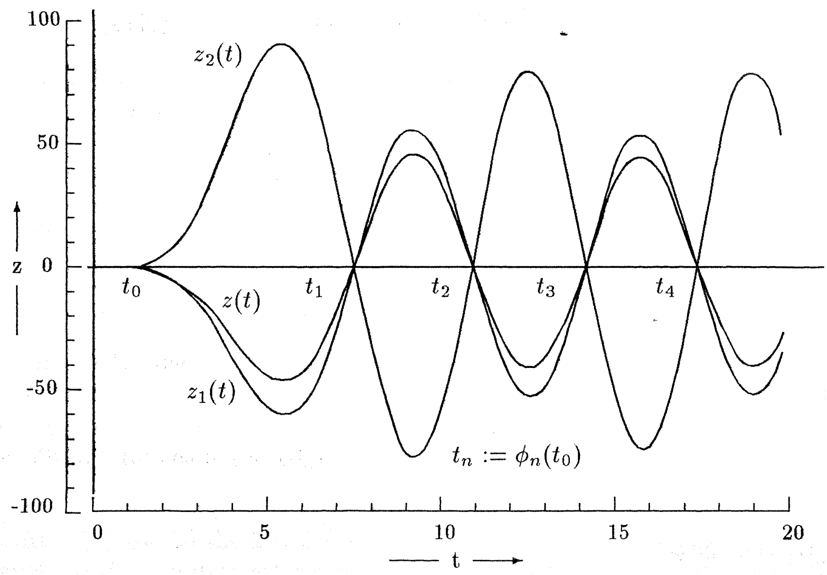

Fig. 1

Example 2 Quite different situation occures for nonhomogeneous Euler differential equation (Fig. 2)

$$y'' + \frac{10}{t^2}y = \frac{3}{2t}$$

and the following initial values

$$z(2) = 1, \quad z'(2) = 0.2, \quad z_1'(2) = -0.2, \quad z_2'(2) = 0.4\,.$$

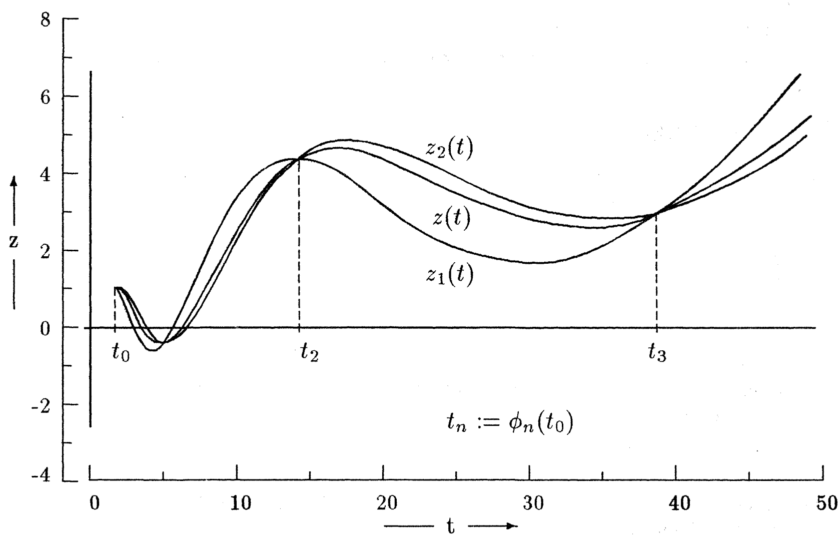

Fig. 2

From the course of all three graphs on Fig. 1–2 which are starting out from the same initial nodal point we can see, that the graphs have the same points of intersection. This confirms the computing accuracy. Runge–Kutta method of the 4th order was used for our computation.

5 Conclusion

The concept of the node system and bundle of solutions introduced by M. Laitoch in [2] was applied in papers of another authors. M. Laitoch generalized Sturm separation theorem on zeros of solutions or its derivatives of homogeneous equation for the case of nonhomogeneous equation. The Sturm comparison theorem is used in [6].

In [7] S. Staněk used the properties of nodes of the 1st and 2nd kind for existence and uniqueness of the solution of Two-point boundary value problem in a 2nd order equation (r), where the carrier q comprisses one parameter and the right hand side of equation (r) the other one.

In [8] J. Kojecká generalized the results from the paper [2] for the functions relative to the classes of the factor spaces.

In the above mentioned paper [5] J. Palát used the concept oscillatory solution of equation (r) with respect to a certain function and modified Sturm comparison theorem.

In [3] there are derived theorems on distribution of nodes of the 1st kind of two associated node systems $\mathscr{S}_1$ and $\mathscr{S}_2$ with the real constant bases (α, β) and (γ, δ) respectively, corresponding to some nonhomogeneous equation and to the initial condition.

For this contribution author was motivated by collaboration with his colleagues from the department of physics and material engineering [9]–[11].

References

[1] Borůvka, O.: *Linear Differential Transformations of the Second Order.* The English University Press, London, 1971.

[2] Laitoch, M.: *A Modification of the Sturm's Theorem on Separating Zeros of Solutions of a Linear Differential Equation of the 2nd Order.* Acta Univ. Palacki. Olomuc., Fac. rer. nat. **57**, Math. 17 (1978), 27–33.

[3] Fialka, M.: *On the Position of Nodes of Associated Equations to the Differential Equation $y'' - q(t)y = r(t)$.* Acta Univ. Palacki. Olomuc., Fac. rer. nat. **100**, Math. 30 (1991), 107–120.

[4] Neuman, F.: *Global Properties of Linear Ordinary Differential Equations.* Academia, Praha, 1991.

[5] Palát, J.: *Prilozhenie Teoremy Shturma.* Acta Univ. Palacki. Olomuc., Fac. rer. nat. **69**, Math. 20 (1981), 77–83

[6] Laitochová, J.: *The Sturm Comparison Theorem for i-Conjugate Numbers.* Acta Univ. Palacki. Olomuc., Fac. rer. nat. **97**, Math. 29 (1990), 197–206

[7] Staněk, S.: *Two Point Boundary Problem in a Second Order Nonhomogeneous Linear Differential Equation.* Acta Univ. Palacki. Olomuc., Fac. rer. nat. **61**, Math. 18 (1979), 59–71.

[8] Kojecká, J.: *On Some Properties of Functions Relative to the Classes of Certain Factor Spaces.* Acta Univ. Palacki. Olomuc., Fac. rer. nat. **61**, Math. 18 (1979), 29–34.

[9] Ponížil, P., Vašut, S., Novák, P.: *Sandvich Structures for Damping of Vibrations.* Polymer Processing Society, European Regional Meeting, Sept. 21–24, 1992, Praha, Convention and Culture Centre, (1992), 139–140.

[10] Briš, P., Ponížil, P., Vašut, S.: *The Damping of the Vertical Vibrations with the Aid of the Bottom Parts of Shoe.* Kožařství, 42 (1992), Odborná příloha, 3–4.

[11] Vašut, S., Ponížil, P., Voříšek, M., Novák, P.: *Vibroisolating Properties of Structures from Polymeric Materials.* Makrotest 1992, X. celost. konference, 2.–5. listopadu 1992, Seč u Chrudimi, Pobočka ČSPCH a. s. SYNPO Pardubice, (1992), 83–84.