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Solvability of Nonlinear Functional Boundary Value Problems

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Abstract

New boundary value problems for the functional differential equation $x'' = f(t, x, x', x_t, x'_t)$ are considered. By the Leray–Schauder degree method, existence results are proved under assumption that $f$ is the Carathéodory operator.

Key words: Existence, Carathéodory solution, functional boundary conditions, functional differential equation, Leray–Schauder degree, Borsuk theorem.

1991 Mathematics Subject Classification: 34B15, 34K10

1 Introduction

Let $C_r$ ($r > 0$) be the Banach space of $C^0$-functions on $[-r, 0]$ with the norm $\|x\|_* = \max\{|x(t)| : t \in [-r, 0]\}$. For any continuous function $x : [-r, 1] \to \mathbb{R}$ and each $t \in [0, 1] =: J$ denote by $x_t$ the element of $C_r$ defined by

$$x_t(s) = x(t + s) \quad \text{for } s \in [-r, 0].$$

Let $X$ be the Banach space of $C^0$-functions on $J$ with the norm $\|x\|_\infty = \max\{|x(t)| : t \in J\}$ and $L_k(J)$ ($k \in \mathbb{N}$) be the Banach space of measurable functions $x : J \to \mathbb{R}$ such that

$$\|x\|_k = \left[ \int_0^1 |x(t)|^k \, dt \right]^{\frac{1}{k}} < \infty.$$
For each interval $I \subset J$ denote by $D_I$ the set of surjective functionals $\gamma : X \to \mathbb{R}$ which are

(i) continuous, $\gamma(0) = 0$, and

(ii) increasing (i.e. $x, y \in X$, $x(t) < y(t)$ for $t \in I \Rightarrow \gamma(x) < \gamma(y)$)

and set $D_J^* = \{ \gamma : \gamma \in D_I, \lim_{n \to \infty} \gamma(\varepsilon x_n) = \varepsilon \infty$ for each $\varepsilon \in \{-1, 1\}$ and any $\{x_n\} \subset X$, $\lim_{n \to \infty} x_n(t) = \infty$ locally uniformly on $I \}$ (see [11] and [12] where also some examples of functionals belonging to $D_I$ are given).

From the following Example 1 follows that $D_J \neq D_J^*[0,1]$. 

**Example 1** Consider the functional $\gamma : X \to \mathbb{R}$ defined by

$\gamma(x) = x(1) + \arctan x \left( \frac{1}{2} \right) .

Obviously, $\gamma(0) = 0$, $\gamma(\mathbb{R}) = \mathbb{R}$ and $\gamma$ is continuous increasing; hence $\gamma \in D_J$. Set $x_n(t) = n \cos \left( \frac{t\pi}{2} \right)$ for $t \in J$ and $n \in \mathbb{N}$. Then $\lim_{n \to \infty} x_n(t) = \infty$ locally uniformly on $[0,1]$ and

$$\begin{align*}
\lim_{n \to \infty} \gamma(\varepsilon x_n) &= \lim_{n \to \infty} \left( \varepsilon x_n(1) + \arctan \left( \varepsilon x_n \left( \frac{1}{2} \right) \right) \right) \\
&= \lim_{n \to \infty} \arctan \left( \varepsilon n \cos \left( \frac{\pi}{4} \right) \right) = \frac{\varepsilon \pi}{2}
\end{align*}$$

for $\varepsilon \in \{-1, 1\}$. Thus $\gamma \not\in D_J^*[0,1]$.

We say that $f : J \times \mathbb{R}^2 \times C_r \times C_r \to \mathbb{R}$ satisfies assumption $(H)$ if

$(H)$: (a) $f(\cdot, x, y, g, \psi)$ is measurable on $J$ for each $(x, y, g, \psi) \in \mathbb{R}^2 \times C_r \times C_r$,

(b) $f(t, \cdot, \cdot, \cdot, \cdot)$ is continuous on $\mathbb{R}^2 \times C_r \times C_r$ for a.e. $t \in J$, and

(c) there exist $k, l, p, q, r \in L_1(J)$ such that

$$|f(t, x, y, g, \psi)| \leq k(t)|x| + l(t)|y| + p(t)||g||_{\infty} + q(t)||\psi||_{\infty} + r(t) \quad (1)$$

for a.e. $t \in J$ and each $(x, y, g, \psi) \in \mathbb{R}^2 \times C_r \times C_r$.

Let $f$ satisfy assumption $(H)$. In the paper we consider the functional differential equation

$$x'' = f(t, x, x', x_t, x'_t) \quad (2)$$

together with the functional boundary conditions

$$(x_0, x'_0) \in \{(\varphi + c_1, \chi + c_2) : c_1, c_2 \in \mathbb{R}\}, \quad \alpha(x|_{J}) = A, \quad \beta(x'|_{J}) = B \quad (3)$$

or

$$(x_0, x'_0) \in \{(\varphi + c_1, \chi + c_2) : c_1, c_2 \in \mathbb{R}\}, \quad \alpha(x|_{J}) = A, \quad \beta_1(x(1) - x|_{J}) = B. \quad (4)$$
Here $\varphi, \chi \in C_r$, $\alpha, \beta \in D_J$, $\beta_1 \in D_{[0,1]}$, $A, B \in \mathbb{R}$ and $x|_J$ is the restriction of $x$ to $J$.

By a solution of BVP $(2), (i)$ $(i = 3, 4)$ we mean a continuous function $x : [-r, 1] \rightarrow \mathbb{R}$ having the absolutely continuous first derivative on $J$ (i.e. $x|_J \in AC^1(J)$), $(x_0, x'_0) = (\varphi - \varphi(0) + x(0), \chi - \chi(0) + x'(0))$ and satisfying the last two boundary conditions of (i) (see [11]).

This definition of a solution of BVP $(2), (i)$ $(i = 3, 4)$ is motivated by the papers of Haščák ([5]-[7]) where some formulations of BVPs for the $n$-th order linear differential equations with delays were given. We observe that for any solution $x$ of BVP $(2), (i)$ $(i = 3, 4)$ the functions $x_1, x_2$ defined by

$$
x_1(t) = \begin{cases} 
\varphi(t) + c_1 & \text{for } t \in [-r, 0] \\
x(t) & \text{for } t \in (0, 1], 
\end{cases}
$$

$$
x_2(t) = \begin{cases} 
\chi(t) + c_2 & \text{for } t \in [-r, 0] \\
x'(t) & \text{for } t \in (0, 1], 
\end{cases}
$$

with $c_1 = -\varphi(0) + x(0)$, $c_2 = -\chi(0) + x'(0)$ are continuous on $[-r, 1]$.

**Remark 1** If $f(t, x, y, \varphi, \psi) = f_1(t, x, y)$ is independent of $\varphi$, $\psi$ and if we set $\alpha(x) = x(0)$, $\beta(x) = x(1)$, $\beta_1(x) = x(\eta)$ for $x \in X$ with an $\eta \in (0, 1)$, then $(2)-(4)$ (with $A = B = 0$) imply

$$
x'' = f_1(t, x, y),
$$

$$
x(0) = 0, \ x'(1) = 0
$$

and

$$
x(0) = 0, \ x(1) - x(\eta) = 0.
$$

This paper was motivated by the recently papers of Marano [8] and Gupta [3] where sufficient conditions for the existence of BVP $(\tau)$, $(i)$ $(i = 6, 7)$ where given. In [8] the results are proved by an existence theorem for operator inclusions by O. N. Ricceri and B. Ricceri [10]. In [3] it is given a simple proof of Theorem 1 of [8] using a Leray–Schauder continuation theorem by Mawhin [9] and the author also obtained a better analogue of Theorem 3 of [8]. The results of [3] and [8] improve those of [2].

In this paper we generalize results of [3] and [8] especially in the following directions:

(i) there are considered functional differential equations, and

(ii) boundary conditions have a nonlinear functional form.

The existence theorems are proved by the Leray–Schauder degree method and by the Borsuk theorem (see e.g. [1], [9]).
2 Lemmas, notation

Lemma 1 Let $I \subset J$ be an interval, $u \in X$, $a \in D_I$ and $c \in [0, 1]$. Let the equality
\[ a(x + u) + (c - 1)a(-x + u) = ca(u) \]
be satisfied for an $x \in X$. Then there exists $a \xi \in I$ such that
\[ x(\xi) = 0 . \]

Proof Set $\gamma(z) = a(z + u) + (c - 1)a(-z + u) - ca(u)$ for $z \in X$. Then $\gamma \in D_I$ and $\gamma(x) = 0$. If $x(t) \neq 0$ on $I$ we obtain $\gamma(x) \neq 0$, a contradiction. □

Lemma 2 Let $\alpha, \beta \in D_J$ and $A, B \in \mathbb{R}$. Then the system
\[ \alpha(a + bt) = A, \quad \beta(b) = B \quad (8) \]
has a unique solution $(a_0, b_0) \in \mathbb{R}^2$.

Proof Define the continuous functions $p : \mathbb{R}^2 \rightarrow \mathbb{R}$, $q : \mathbb{R} \rightarrow \mathbb{R}$ by
\[ p(a, b) = \alpha(a + bt), \quad q(b) = \beta(b) . \]
Since $q$ is increasing on $\mathbb{R}$ and $\lim_{b \to \pm \infty} q(b) = \pm \infty$, there exists a unique $b_0 \in \mathbb{R}$ such that $q(b_0) = B$. The function $p(\cdot, b_0)$ is increasing on $\mathbb{R}$ and $\lim_{a \to \pm \infty} p(a, b_0) = \pm \infty$, and consequently $p(a_0, b_0) = A$ for a unique $a_0 \in \mathbb{R}$. We see that $(a_0, b_0) \in \mathbb{R}^2$ is the unique solution of (8). □

Lemma 3 Let $\alpha \in D_J$, $\beta_1 \in D^*_{(0,1)}$ and $A, B \in \mathbb{R}$. Then the system
\[ \alpha(a + bt) = A, \quad \beta_1(b(1 - t)) = B \quad (9) \]
has a unique solution $(a_0, b_0) \in \mathbb{R}^2$.

Proof Since the proof is very similar to that of Lemma 2, it is omitted. □

Let $u, v \in X$, $\alpha, \beta \in D_J$, $\beta_1 \in D^*_{(0,1)}$, $\varphi, \chi \in C_r$ and let $h$ satisfy assumption (H) (with $f = h$). To prove the main existence results we consider the auxiliary BVPs (10), (i) $(i = 11, 12)$ where
\[ x'' = h(t, x, x', x_t, x'_t), \quad (10) \]
\[ (x_0, x'_0) \in \{(\varphi + c_1, \chi + c_2) : c_1, c_2 \in \mathbb{R}\}, \]
\[ a(x|J + u) = a(u), \quad \beta_1(x''|J + v) = \beta_1(v), \quad (11) \]
\[ (x_0, x'_0) \in \{(\varphi + c_1, \chi + c_2) : c_1, c_2 \in \mathbb{R}\}, \]
\[ a(x|J + u) = a(u), \quad \beta_1(x(1) - x|J + v) = \beta_1(v). \quad (12) \]

Let $Y$ be the Banach space of $AC^1$-functions on $J$ endowed with the norm $\|x\|_A = \max\{\|x\|, \|x'\|, \|x''\|\}$. For each $c \in [0, 1]$ define the operators $H_c, V_c : Y \times \mathbb{R}^2 \rightarrow Y \times \mathbb{R}^2$.
by

\[ H_c(x, A, B) = \left( A + Bt + c \int_0^t \int_0^s h(\tau, x(\tau), x'(\tau), x''(\tau), x'''(\tau), x''') \, d\tau \, ds, \right. \]

\[ A + \alpha(x + u) + (c - 1)\alpha(-x + u) - c\alpha(u), \]

\[ B + \beta(x' + v) + (c - 1)\beta(-x' + v) - c\beta(v), \]

\[ V_c(x, A, B) = \left( A + Bt + c \int_0^t \int_0^s h(\tau, x(\tau), x'(\tau), x''(\tau), x'''(\tau), x''') \, d\tau \, ds, \right. \]

\[ A + \alpha(x + u) + (c - 1)\alpha(-x + u) - c\alpha(u), \]

\[ B + \beta(x(1) + x + v) + (c - 1)\beta(-x(1) + x + v) - c\beta(v), \]

where

\[ x_t(s) = \begin{cases} \varphi(t + s) - \varphi(0) + x(0) & \text{for } t + s \in [-r, 0] \\ x(t + s) & \text{for } t + s \in (0, 1], \end{cases} \] (13')

\[ x'_t(s) = \begin{cases} \chi(t + s) - \chi(0) + x'(0) & \text{for } t + s \in [-r, 0] \\ x'(t + s) & \text{for } t + s \in (0, 1]. \end{cases} \] (13'')

Consider the operator equations

\[ H_c(x, A, B) = (x, A, B), \quad c \in [0, 1] \] (14c)

and

\[ V_c(x, A, B) = (x, A, B), \quad c \in [0, 1]. \] (15c)

**Remark 2** We see that \( x \) is a solution of BVP (10), (11) (resp. (10), (12)) if \( (x, x(0), x'(0)) \) is a solution of (14) (resp. (15)). And conversely, let \( (x, A, B) \) be a solution of (14) (resp. (15)) and \( \tilde{x} : [-r, 1] \to \mathbb{R} \) be given by \( \tilde{x}(t) = \varphi(t) - \varphi(0) + x(0) \) for \( t \in [-r, 0] \) and \( \tilde{x} = x \). Then \( (\tilde{x}, A, B) \) is a solution of BVP (10), (11) (resp. (10), (12)). So to prove existence results for BVP (10), (11) and BVP (10), (12) it is enough to show ones for functional equations (14) and (15), respectively.

**Lemma 4** Let \( h \) satisfy assumption \( (H) \) (with \( f = h \)). Let

\[ \lambda := ||k||_1 + ||l||_1 + ||p||_1 + ||q||_1 < 1 \] (16)

and set

\[ \Lambda = \frac{1}{1 - \lambda} \left[ 2(||p||_1||\varphi||_1 + ||q||_1||x||_1) + ||r||_1 \right] + 1, \]

\[ \Omega = \left\{ (x, A, B) : (x, A, B) \in Y \times \mathbb{R}^2, ||x||_A < \Lambda, |A| < \Lambda, |B| < \Lambda \right\}. \]

If \( (x, A, B) \) is a solution of (14) or (15) for a \( c \in [0, 1] \), then \( (x, A, B) \in \Omega. \)
Proof. Let \((x,A,B)\) be a solution of \((14_\omega)\) for \(c \in [0,1]\). Then the following equalities

\[
x(t) = A + Bt + c \int_0^t \int_0^s h(\tau, x(\tau), x'(\tau), x', x'_\tau) \, d\tau \, ds, \quad t \in [0,1],
\]

\[
\alpha(x + u) + (c-1)\alpha(-x + u) = c\alpha(u),
\]

\[
\beta(x' + v) + (c-1)\beta(-x' + v) = c\beta(v)
\]

hold, where \(x_t\) and \(x'_t\) are defined by \((13')\) and \((13'')\), respectively. By \((18), (19)\) and Lemma 1, there exist some \(\xi, \eta \in J\) such that \(x(\xi) = 0, x'(\eta) = 0\); hence (cf. \((17))

\[
x(t) = c \int_\xi^t \int_\eta^s h(\tau, x(\tau), x'(\tau), x, x'_\tau) \, d\tau \, ds,
\]

\[
x'(t) = c \int_\eta^t h(s, x(s), x'(s), x, x'_s) \, ds
\]

for \(t \in J\). Thus (cf. \((1)\) with \(f = h\))

\[
|x'(t)| \leq c \left( ||k||_1 ||x||_\infty + ||l||_1 ||x'||_\infty + ||p||_1 \max\{||x_t|| : t \in J\} 
+ ||q||_1 \max\{||x'_t|| : t \in J\} + ||r||_1 \right)
\]

\[
\leq (||k||_1 + ||p||_1) ||x||_\infty + (||l||_1 + ||q||_1) ||x'||_\infty 
+ 2(||p||_1 ||x||_\infty + ||q||_1 ||x'||_\infty) + ||r||_1, \quad t \in J
\]

since \(||x||_\infty \leq 2||x'||_\infty, ||x'_t|| \leq ||x'||_\infty + 2||x||_\infty\) for \(t \in J\). Consequently,

\[
||x'||_\infty \leq \left( ||k||_1 + ||p||_1 \right) ||x||_\infty + \left( ||l||_1 + ||q||_1 \right) ||x'||_\infty 
+ 2(||p||_1 ||x||_\infty + ||q||_1 ||x'||_\infty) + ||r||_1. \tag{20}
\]

We next have \(x(t) = \left| \int_\xi^t x'(s) \, ds \right| \leq ||x'||_\infty\) for \(t \in J\) and therefore

\[
||x||_\infty \leq ||x'||_\infty \tag{21}
\]

which implies (cf. \((20))

\[
||x'||_\infty \leq \lambda ||x'||_\infty + 2(||p||_1 ||x||_\infty + ||q||_1 ||x'||_\infty) + ||r||_1
\]

and \(||x'||_\infty < \Lambda\). Then \(||x||_\infty < \Lambda\) and since \(A = x(0), B = x'(0)\) we obtain \(|A| < \Lambda, |B| < \Lambda\). Finally,

\[
||x'||_1 = c \int_0^t |h(s, x(s), x'(s), x, x'_s)| \, ds
\]

\[
\leq \lambda ||x'||_\infty + 2(||p||_1 ||x||_\infty + ||q||_1 ||x'||_\infty) + ||r||_1
\]

\[
\leq \lambda \Lambda + (\Lambda - 1)(1 - \lambda) < \Lambda.
\]

Hence \((x, A, B) \in \Omega\).
Let \((x, A, B)\) be a solution of (15c) for a \(c \in [0,1]\). Then the equalities (17), (18) and
\[
\beta_1(x(1) - x + v) + (c - 1)\beta_1(-x(1) + x + v) = c\beta(v)
\] (22)
are satisfied. By (18), (22) and Lemma 1, there exist a \(\xi \in J\) and an \(\varepsilon \in [0,1]\) such that \(x(\xi) = 0, x(1) - x(\varepsilon) = 0\). Thus \(x'(\eta) = 0\) for an \(\eta \in (\varepsilon, 1)\) and, in the same manner as in the first part of our proof, we obtain \((x, A, B) \in \Omega\). \(\square\)

**Lemma 5** Let \(h\) satisfy assumption \((H)\) (with \(f = h\)). Assume \(k \in L_2(J), l \in L_i(J), p \in L_j(J)\) and \(q \in L_m(J)\) where \(i, j, m \in \{1, 2\}\) and
\[
\lambda^* := \frac{2}{\pi}||k||_2 + ||l||_i + ||p||_j + ||q||_m < 1.
\]
Let \((x, A, B)\) be a solution of (14c) or (15c) for a \(c \in [0,1]\). Then
\[
||x||_\infty < \Lambda_1, \quad ||x'||_\infty < \Lambda_1, \quad ||x''||_1 < \Lambda_1, \quad |A| < \Lambda_1, \quad |B| < \Lambda_1,
\] (23)
where
\[
\Lambda_1 = \frac{1}{1 - \lambda^*} \left[2(||p||_j||\varphi||_* + ||q||_m||\chi||_*) + ||r||_1\right] + 1.
\]

**Proof** By the proof of Lemma 4, \(A = x(0), B = x'(0)\) and there exist some \(\xi, \eta \in J\) such that \(x(\xi) = 0, x'(\eta) = 0\). Hence
\[
||x||_\infty \leq ||x'||_\infty \leq ||x''||_1, \quad ||x'||_2 \leq ||x'||_\infty
\]
and
\[
||x||_2 \leq \frac{2}{\pi}||x'||_2
\]
by the Wintinger inequality (see e.g. [4], Theorem 256). Using (1) (with \(f = h\)) we get
\[
||x''||_1 = c \int_0^1 \left|h(t, x(t), x'(t), x_1(t), x'_1(t))\right| dt
\]
\[
< \left(||k||_2||x||_2 + ||l||_i||x'||_\infty + ||p||_j(||x||_\infty + 2||\varphi||_*\right)
\]
\[
+ ||q||_m(||x'||_\infty + 2||\chi||_* + ||r||_1
\]
\[
\leq \left(\frac{2}{\pi}||k||_2 + ||l||_i + ||p||_j + ||q||_m\right)||x'||_\infty
\]
\[
+ 2(||p||_j||\varphi||_* + ||q||_m||\chi||_* + ||r||_1
\]
\[
\leq \lambda^*||x''||_1 + 2(||p||_j||\varphi||_* + ||q||_m||\chi||_* + ||r||_1
\]
\[
and consequently \(||x''||_1 < \Lambda_1\) which implies that (23) holds. \(\square\)
3 Existence theorems

Proposition 1 Let $h$ satisfy assumption (H) (with $f = h$) and $\Omega \subset \mathbf{Y} \times \mathbf{R}^2$ be open bounded and symmetric with respect to $0 \in \Omega$. Then operator equation (14.1) and (15.1) has a solution in $\Omega$ provided $H_c(x, A, B) \neq (x, A, B)$ and $V_c(x, A, B) \neq (x, A, B)$ on $\partial \Omega$ for any $c \in [0, 1]$, respectively.

Proof Assume $(x, A, B) \neq \partial \Omega$ for any solution $(x, A, B)$ of the family of equations (14.1) (resp. (15.1)) with $c \in [0, 1]$. Set $W(c, x, A, B) = H_c(x, A, B)$ (resp. $W(c, x, A, B) = V_c(x, A, B)$) for $(c, x, A, B) \in [0, 1] \times \mathbf{Y} \times \mathbf{R}^2$. Then $W$ is a compact operator on the closure $\bar{\Omega}$ of $\Omega$ by the Arzelà-Ascoli theorem, the Bolzano-Weierstrass theorem and the Lebesgue theorem, and $W(c, x, A, B) \neq (x, A, B)$ for any $(x, A, B) \in \partial \Omega$ and each $c \in [0, 1]$ by our assumption. Thus

$$D(I - W(1, \cdot, \cdot, \cdot), \Omega, 0) = D(I - W(0, \cdot, \cdot, \cdot), \Omega, 0)$$

where "D" denotes the Leray-Schauder degree (see e.g. [1]). To prove the existence of a solution for equation $W(1, x, A, B) = (x, A, B)$ (that is (14.1) resp. (15.1)) we have to show that

$$D(I - W(0, \cdot, \cdot, \cdot), \Omega, 0) \neq 0.$$ 

Since

$$H_0(-x, -A, -B) =$$

$$= (-A - Bt, -A + \alpha(-x + u) - \alpha(x + u), -B + \beta(-x' + u) - \beta(x' + u))$$

$$= -H_0(x, A, B)$$

and

$$V_0(-x, -A, -B) =$$

$$= (-A - Bt, -A + \alpha(-x + u) - \alpha(x + u), -B + \beta_1(-x(1) + x + v) - \beta_1(x(1) - x + v))$$

$$= -V_0(x, A, B)$$

for $(x, A, B) \in \mathbf{Y} \times \mathbf{R}^2$, $W(0, \cdot, \cdot, \cdot)$ is an odd operator and then

$$D(I - W(0, \cdot, \cdot, \cdot), \Omega, 0) \neq 0$$

by the Borsuk Theorem (see [1], Theorem 8.3).

$$\Box$$

Theorem 1 Let $h$ satisfy assumption (H) (with $f = h$). Then BVP (10). (i) $(i = 11, 12)$ has at least one solution for each $u, v \in \mathbf{X}$, $\alpha, \beta \in \mathbf{D}_j$, $\beta_1 \in \mathbf{D}_0^{*}$ and $\varphi, \chi \in \mathbf{C}_j$ provided

$$||k||_1 + ||l||_1 + ||p||_1 + ||q||_1 < 1. \quad (24)$$
Proof Let $u, v \in X$, $\alpha, \beta \in D_J$, $\beta_1 \in D^*_{[0,1]}$ and $\varphi, \chi \in C_r$ and let (24) be satisfied. By Remark 2, it is sufficient to show that operator equations (14) and (15) have solutions. By Lemma 4, there exists an open bounded subset $\Omega$ of $Y \times \mathbb{R}^2$ which is symmetric with respect to $0 \in \Omega$ such that $(x, A, B) \notin \partial \Omega$ for any solution $(x, A, B)$ of the family of equations (14) and (15) with $c \in [0, 1]$. The conclusion of Theorem 1 follows immediately from Proposition 1.

Using Proposition 1 and Lemma 5 we can prove the following theorem.

**Theorem 2** Let $h$ satisfy assumption (H) (with $f = h$). Assume $k \in L_2(J)$, $l \in L_1(J)$, $p \in L_2(J)$ and $q \in L_2(J)$ where $i, j, m \in \{1, 2\}$. Then BVP (10), (i) $(i = 11, 12)$ has at least one solution for each $u, v \in X$, $\alpha, \beta \in D_J$, $\beta_1 \in D^*_{[0,1]}$ and $\varphi, \chi \in C_r$ provided

$$\frac{2}{\pi} ||k||_2 + ||l||_i + ||p||_j + ||q||_m < 1.$$  

(25)

The main existence results for BVP (2), (i) $(i = 3, 4)$ are given in the following two theorems.

**Theorem 3** Let $f$ satisfy assumption (H). Assume that (24) is satisfied. Then BVP (2), (i) $(i = 3, 4)$ has at least one solution for each $\alpha, \beta \in D_J$, $\beta_1 \in D^*_{[0,1]}$, $\varphi, \chi \in C_r$ and $A, B \in \mathbb{R}$.

Proof Fix $\alpha, \beta \in D_J$, $\beta_1 \in D^*_{[0,1]}$, $\varphi, \chi \in C_r$ and $A, B \in \mathbb{R}$. By Lemma 2 (resp. Lemma 3) there exist (unique) $a_0, b_0 \in \mathbb{R}$ such that $\alpha(a_0 + b_0 t) = A$, $\beta(b_0) = B$ (resp. $\alpha(a_0 + b_0 t) = A$, $\beta(b_0(1 - t)) = B$). Set

$$h(t, x, y, \varphi, \psi) = f(t, x + a_0 + b_0 t, y + b_0, \varphi + \omega_t, \psi + b_0)$$

for $(t, x, y, \varphi, \psi) \in J \times \mathbb{R}^2 \times C_r \times C_r$ where

$$\omega_t(s) = \begin{cases} a_0 & \text{for } t + s \in [-r, 0] \\ a_0 + b_0(t + s) & \text{for } t + s \in (0, 1] \end{cases}$$

We see that $x$ is a solution of BVP (10), (11) with $u = a_0 + b_0 t$ and $v = b_0$ if and only if $x + a_0 + b_0 t$ is a solution of BVP (2), (3) and $x$ is a solution of BVP (10), (12) with $u = a_0 + b_0 t$ and $v = b_0(1 - t)$ if and only if $x + a_0 + b_0 t$ is a solution of BVP (2), (4). Since (cf. (1))

$$|h(t, x, y, \varphi, \psi)| = k(t)|x + a_0 + b_0 t| + l(t)|y + b_0| + p(t)||\varphi + \omega_t|| + q(t)||\psi + b_0|| + r(t)$$

$$\leq k(t)|x| + l(t)|y| + p(t)||\varphi|| + q(t)||\psi|| + r_1(t)$$

for $(t, x, y, \varphi, \psi) \in J \times \mathbb{R}^2 \times C_r \times C_r$, where

$$r_1(t) = (k(t) + p(t))(|a_0| + |b_0|) + (l(t) + q(t))|b_0| + r(t),$$

there exists a solution of BVP (10), (i) $(i = 11, 12)$ by Theorem 1. This completes the proof. □
Theorem 4 Let f satisfy assumption (H). Assume \( k \in L^2(J) \), \( l \in L^4(J) \), \( p \in L^2(J) \) and \( q \in L^m(J) \) where \( i, j, m \in \{1, 2\} \) and (25) is satisfied. Then BVP (2), (i) \((i = 3, 4)\) has at least one solution for each \( \alpha, \beta \in \mathcal{D}J \), \( \beta_1 \in \mathcal{D}_{[0,1)} \), \( \varphi, \chi \in C_r \) and \( A, B \in \mathbb{R} \).

Proof We proceed exactly as in the proof of Theorem 3 but instead of Theorem 1 we now use Theorem 2. \(\Box\)

Remark 3 Note that analogously existence results as above can be shown for the functional differential equation of the form

\[ x''(t) = f(t, x(t), x'(t), x(a(t)), x'(b(t)), x_t, x'_t) \]

with \( a : J \to J, \; b : J \to J \) continuous.

References


[10] Ricceri, O. N., Ricceri, B.: *An existence theorem for inclusions of the type \( \Psi(u)(t) \in F(t, \Phi(u)(t)) \) and application to a multivalued boundary value problem*. Appl. Anal. 38 (1990), 259–270.
