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Congruence Semimodularity of Conservative Groupoids

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Abstract

A groupoid $\mathcal{G} = (G, \cdot)$ is conservative if $a \cdot b = a$ or $a \cdot b = b$ for each $a, b \in G$. We prove that for any conservative groupoid $\mathcal{G}$, the congruence lattice Con $\mathcal{G}$ is semimodular but it is not modular in a general case.

Key words: Groupoid, conservative groupoid, semimodular congruence lattice.

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The concept of conservative groupoid was introduced by B. Zelinka [3], [4] as an algebraic tool for some graph theoretical treatment. Recall that a groupoid $\mathcal{G} = (G, \cdot)$ is *conservative* if $a \cdot b = a$ or $a \cdot b = b$ for any two elements $a, b \in G$. It is almost trivial to show that the class $\mathcal{C}$ of all conservative groupoids is closed under homomorphic images and subalgebras. Call a groupoid $\mathcal{G} = (G, \cdot)$ *trivial* if card $G = 1$. Clearly every trivial groupoid is conservative and every conservative groupoid is idempotent.

A *tolerance* on a groupoid $\mathcal{G} = (G, \cdot)$ is a reflexive and symmetric binary relation T on G such that $\langle a, b\rangle \in T$ and $\langle c, d\rangle \in T$ imply $\langle a \cdot c, b \cdot d\rangle \in T$. Denote by Tol $\mathcal{G}$ the lattice of all tolerances on $\mathcal{G}$. Applying well-known facts on tolerances, see e. g. [2], and Theorem 1 of [5], we conclude:

Proposition 1 *For any conservative groupoid $\mathcal{G}$, the tolerance lattice* Tol $\mathcal{G}$ *is distributive.*

Denote by Con $\mathcal{G}$ the congruence lattice of a groupoid $\mathcal{G}$. A class $\mathcal{K}$ of groupoids is said to be *congruence modular* if Con $\mathcal{G}$ is a modular lattice for each $\mathcal{G} \in \mathcal{K}$.

Proposition 2 *The class $\mathcal{C}$ of all conservative groupoids is not congruence modular.*

Proof Evidently every semigroup $\mathcal{L}$ of left-zeros (i.e. $x \cdot y = x$ for any x, y of $\mathcal{L}$) is a conservative groupoid. Further, every equivalence on the support of $\mathcal{L}$ is a congruence on $\mathcal{L}$, thus for an n-element $\mathcal{L}$, Con $\mathcal{L} \cong \Pi_n$, where Π_n is the partition lattice. However, Π_n is not modular for $n > 3$ see e.g. [1]. □

Remark 1 For other types of algebras, the situation is different. E.g. for lattices, their congruence lattices are distributive but tolerance lattices are distributive only for a variety of distributive lattices, see [2]. Hence, conservative groupoid can serve as an example of algebras where tolerance lattices satisfy more strong condition than congruence lattices.

In what follows we are going to show that congruence lattices of conservative groupoids satisfy a weaker of condition, namely semimodularity.

Recall (see e.g. [1]) that a lattice L is *semimodular* if it satisfies the so called *covering condition* (cc) for each $x, y \in L$:

$$x \wedge y \prec x \qquad \text{implies} \qquad y \prec x \vee y\,. \tag{cc}$$

If L satisfies also the dual of (cc), it is modular.

A class $\mathcal{K}$ of algebras is *congruence semimodular* if Con A is semimodular for each $A \in \mathcal{K}$.

We accept the following notation. If $\mathcal{G} = (G, \cdot)$ is a groupoid and $a, b \in G$, denote by $\Theta(a, b)$ the least congruence on $\mathcal{G}$ containing the pair $\langle a, b\rangle$. $\Theta(a, b)$ is called the *principal congruence* (generated by $\langle a, b\rangle$). Further, denote by ω_G is the identity relation (the diagonal) of G.

The following result was proven in [5]:

Proposition 3 *Let $\mathcal{G} = (G, \cdot)$ be a conservative groupoid and $a, b \in G$, $a \neq b$. Then $\Theta(a, b)$ has exactly one non-singleton congruence class.*

We are ready to prove our main result:

Theorem 1 *The class $\mathcal{C}$ of all conservative groupoid is congruence semimodular.*

Proof (1) Suppose the existence of $\mathcal{G} \in \mathcal{C}$ such that Con $\mathcal{G}$ is not semimodular. Then there exist $\Theta, \Phi \in$ Con $\mathcal{G}$ which fail the covering condition

$$\Theta \cap \Phi \prec \Theta \Longrightarrow \Phi \prec \Theta \vee \Phi\,. \tag{1}$$

Since $\mathcal{C}$ is closed under homomorphic images, also $\mathcal{G}/\Theta \cap \Phi \in \mathcal{C}$. However,

$$\text{Con } \mathcal{G}/\Theta \cap \Phi \cong [\Theta \wedge \Phi, G^2]\,.$$

the interval in Con $\mathcal{G}$. Hence also the lattice $L = \text{Con}\,\mathcal{G}/\Theta \cap \Phi$ is not semimodular and there exist elements $\bar{\Theta}, \bar{\Phi}$ which fail semimodularity at the bottom of L, i.e. $\bar{\Theta} \cap \bar{\Phi}$ is the least element of L and $\bar{\Theta}$ is an atom of L. Hence, if there exists a conservative groupoid which is not congruence semimodular then there exists a groupoid of the same property and, moreover, Θ of (1) is an atom of the congruence lattice.

(2) With respect to (1), suppose $\mathcal{G} \in \mathcal{C}$ with $\Theta, \Phi \in \text{Con}\,\mathcal{G}$ such that

$$\Theta \cap \Phi = \omega_G \prec \Theta \qquad \text{but not} \qquad \Phi \prec \Theta \vee \Phi \tag{2}$$

Hence, there exists $\Psi \in \text{Con}\,\mathcal{G}$ with

$$\Phi \subset \Psi \subset \Theta \vee \Phi, \qquad \Phi \neq \Psi \neq \Theta \vee \Phi\,.$$

Suppose $\langle a, b\rangle \in \Theta \vee \Phi$. Then there exist $c_0, c_1, \dots, c_n \in G$ with $c_0 = a, c_n = b$ and $\langle c_{i-1}, c_i\rangle \in \Theta$ or $\langle c_{i-1}, c_i\rangle \in \Phi$ for $i = 1, \dots, n$. Since $\omega_G \prec \Theta$, Θ must be a principal congruence on $\mathcal{G}$. By Proposition 3, Θ has exactly one non-singleton congruence class, i.e. only one pair of $\langle c_{i-1}, c_i\rangle$ can be contained in Θ, whence $\langle a, b\rangle \in \Phi \bullet \Theta \bullet \Phi$. Moreover, Proposition 3 yields $\Theta \bullet \Phi \bullet \Theta \subseteq \Phi \bullet \Theta \bullet \Phi$. The converse inclusion is trivial thus

$$\Theta \vee \Phi = \Phi \bullet \Theta \bullet \Phi\,.$$

Suppose now $\langle x, y\rangle \in \Psi - \Phi$. Then $\langle x, y\rangle \in \Theta \vee \Phi$ and, by the foregoing equality, $\langle x, y\rangle \in \Phi \bullet \Theta \bullet \Phi$, i.e. there exist $u, v \in G$ with

$$x \Phi u \Theta v \Phi y \tag{3}$$

This gives $u \Phi x \Psi y \Phi v$. However, $\Phi \subset \Psi$, thus also $u \Psi x \Psi y \Psi v$, i.e. $\langle u, v\rangle \in \Psi$. Together with (3) we infer $\langle u, v\rangle \in \Theta \cap \Psi$.

Since Θ is an atom of Con $\mathcal{G}$ and Θ, Φ fail (1), we conclude $\Theta \cap \Psi = \omega_G$, i.e. $u = v$. By (3), it implies $\langle x, y\rangle \in \Phi$, a contraction. □

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