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Simple Balanced Groupoids

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Abstract

A class of simple groupoids derived from transitive permutation groups is described.

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In the description of (finite) simple medial groupoids (as given in [1]), it turned out that an important class of such groupoids is formed by those that are (in a certain natural way) constructed from transitive permutation groups. Now, the medial case leads to abelian groups, but in this short note we are considering a similar construction for general groups. The results may be useful for the description of (finite) simple zeropotent groupoids satisfying some linear identities.

1 Introduction

By a *groupoid* we mean a non-empty set with one binary operation. If G is a groupoid then an element $o \in G$ is said to be *absorbing* if $xo = o = ox$ for every $x \in G$. A groupoid G with an absorbing element o is said to be *zeropotent* if $xx = o$ for every $x \in G$ and G is said to be a *Z-semigroup* if $xy = o$ for all $x, y \in G$.

Let G be a groupoid. A non-empty subset I of G is said to be an *ideal* of G if $GI \subseteq I$ and $IG \subseteq G$. The groupoid G is said to be *ideal-simple* if $I = G$ whenever I is an ideal of G such that I contains at least two elements (obviously, if $I = \{o\}$ is a one-element ideal of G then o is an absorbing element).

A groupoid G is said to be *(congruence-)simple* if it is non-trivial and id_G and $G \times G$ are the only congruences of G.

2 Balanced groupoids

Throughout this section, let G be a non-trivial groupoid with an absorbing element o and let $G^* = G - \{o\}$. We shall say that G is

- *left (right) semibalanced* if for every $a \in G^*$ there is at most one $b \in G$ such that $ab \neq o$ ($ba \neq o$);
- *semibalanced* if it is both left and right semibalanced;
- *left (right) balanced* if for every $a \in G^*$ there is just one $b \in G$ such that $ab \neq o$ ($ba \neq o$);
- *balanced* if it is both left and right balanced.

Now assume that G is left (right) semibalanced. Then there is a partial transformation φ (ψ) of G^* such that $a\varphi(a) \neq o$ ($\psi(a)a \neq o$); thus $\varphi(a)$ ($\psi(a)$) is defined iff $aG \neq o$ ($Ga \neq o$). Put $f(a) = a\varphi(a)$ ($g(a) = \psi(a)a$); again, f (g) is a partial transformation of G^*.

Let G be semibalanced. If $a \in G^*$ is such that $\varphi(a)$ is defined then $\psi\varphi(a)$ is defined, $\psi\varphi(a) = a$ and $f(a) = a\varphi(a) = g\varphi(a)$. Similarly, if $b \in G^*$ and $\psi(b)$ is defined then $\varphi\psi(b) = b$ and $g(b) = \psi(b)b = f\psi(b)$. If $a, b \in G^*$ are such that $ab \neq o$ then $a = \psi(b)$, $b = \varphi(a)$ and $ab = f(a) = g(b) = g\varphi(a) = f\psi(b)$.

The following lemma is now clear:

Lemma 2.1 *Suppose that G is balanced. Then:*

(i) *φ and ψ are permutations of G^*, $\varphi = \psi^{-1}$ and $\psi = \varphi^{-1}$.*

(ii) *f and g are transformations of G^*, $f = g\varphi$ and $g = f\psi$.*

Lemma 2.2 *Suppose that $G = GG$ and G is left (right) semibalanced. Then:*

(i) *$f(G^*) = G^*$ ($g(G^*) = G^*$).*

(ii) *If G is finite then G is left (right) balanced and f (g) is a permutation of G^*.*

Proof (i) If $a \in G^*$ then $a = bc$ for some $b, c \in G$. Clearly, $b, c \in G^*$, $c = \varphi(b)$ and $a = f(b)$.

(ii) G^* is finite and hence from (i) follows that f is a permutation of G^*. Now, let $n = \mathrm{card}(G)$, $M = \{(a, b)\,;\, a, b \in G^*,\ ab \neq o\}$ and $N = \{a\,;\,(a, b) \in M\}$. Since G is left semibalanced, we have $\mathrm{card}(M) = \mathrm{card}(N)$. On the other hand, $\mathrm{card}(N) \leq n - 1$ and $\mathrm{card}(M) \geq \mathrm{card}(GG) - 1 = n - 1$. Thus $\mathrm{card}(N) = n - 1$, $N = G^*$ and G is left balanced. □

Lemma 2.3 *Suppose that G is simple, left (right) semibalanced and finite with at least three elements. Then G is left (right) balanced and f (g) is a permutation of G^*.*

Proof The relation $r = (GG \times GG) \cup \mathrm{id}_G$ is a congruence of G. If $r = \mathrm{id}_G$ then $GG = \{o\}$, G is a Z-semigroup and, since it is simple, it contains just two elements, a contradiction. Thus $r = G \times G$, and so $GG = G$ and the result follows from Lemma 2.2. □

Now, if G is balanced then both f and g are transformations of G^* and we denote by $\mathcal{T}$ the transformation semigroup generated by $\{f, g\}$. Further, we shall consider the corresponding biunar $G^*(f, g)$ (an algebra with two unary operations).

Proposition 2.4 *If G is balanced then the following conditions are equivalent:*

(i) *G is (congruence-)simple.*

(ii) *G is ideal-simple.*

(iii) *$\mathcal{T}$ acts transitively on G^*.*

(iv) *The biunar $G^*(f, g)$ is generated by any of its elements.*

Proof (i) implies (ii). If I is an ideal of G then $r = (I \times I) \cup \mathrm{id}_G$ is a congruence of G. Then either $r = \mathrm{id}_G$ and $I = \{o\}$ or $r = G \times G$ and $I = G$.

(ii) implies (iii). Let $a \in G^*$ and $I = \mathcal{T}(a) \cup \{o\}$. If $h \in \mathcal{T}$ and $x \in G$ are such that $h(a)x \neq o$ $(xh(a) \neq o)$ then $h(a)x = fh(a) \in I$ $(xh(a) = gh(a) \in I)$. We have checked that I is an ideal of G and $I \neq \{o\}$, since $o \neq f(a) \in I$. Thus $I = G$ and $\mathcal{T}(a) = G^*$.

(iii) implies (iv). Obvious.

(iv) implies (i). Let $r \neq \mathrm{id}_G$ be a congruence of G. There are $u, v \in G$ such that $u \neq v$ and $(u, v) \in r$. We can assume that $u \neq o$. Then $o \neq u\varphi(u) = f(u)$, $o = v\varphi(u)$ and $(f(u), o) \in r$. Consequently, $I \neq \emptyset$, where $I = \{a \in G^* ; (a, o) \in r\}$. Clearly, $f(I) \subseteq I$ and $g(I) \subseteq I$. This implies that $I = G$ and $r = G \times G$. □

Lemma 2.5 *If G is balanced then the following conditions are equivalent:*

(i) *f is a permutation of G^*.*

(ii) *g is a permutation of G^*.*

(iii) *Both f and g are permutations of G^*.*

(iv) *If $a, b, c, d \in G^*$ are such that $ab = cd \neq o$ then $a = c$ and $b = d$.*

Proof Use Lemma 2.1. □

We shall say that G is *strongly balanced* if it is balanced and satisfies the equivalent conditions of Lemma 2.5.

Remark 2.6 Suppose that G is strongly balanced. Then both f and g are permutations of G^* and we shall define an addition on G as follows:

(1) $o+o=o$;
(2) $o+a=o=a+o$ for every $a \in G^*$;
(3) $a+b=f^{-1}(a)g^{-1}(b)$ for all $a,b \in G^*$.

Clearly, $a+b \neq o$ iff $\varphi f^{-1}(a)=g^{-1}(b)$. But $g^{-1}=\varphi f^{-1}$, and so $a+b \neq o$ iff $a=b$; then we have $a+b=a+a=f^{-1}(a)\varphi f^{-1}(a)=ff^{-1}(a)=a$. This means that $x+x=x$ and $x+y=o$ for all $x,y \in G$, $x \neq y$. Now, it is clear that $G(+)$ is a semilattice. Further, setting $\overline{f}(o)=o=\overline{g}(o)$, $\overline{f}\,|\,G^*=f$ and $\overline{g}\,|\,G^*=g$, we get automorphisms $\overline{f}$ and $\overline{g}$ of $G(+)$ and $xy=\overline{f}(x)+\overline{g}(y)$ for all $x,y \in G$.

3 Transitive permutation groups

Let $\mathcal{G}$ be a transitive permutation group on a non-empty finite set G^* such that $\mathcal{G}$ is generated by elements f and g. Let $o \notin G^*$ and $G=G^* \cup \{o\}$. Now, define a multiplication on G as follows:

(1) $oo=o$;
(2) $ox=o=xo$ for every $x \in G^*$;
(3) $xy=o$ for all $x,y \in G^*$, $f(x) \neq g(y)$;
(4) $xy=f(x)$ $(=g(y))$ for all $x,y \in G^*$, $f(x)=g(y)$.

In this way, we get a groupoid $G=[\mathcal{G},G^*,f,g,o]$.

Proposition 3.1 (i) *G is a simple balanced groupoid and o is an absorbing element of G.*
(ii) *G is zeropotent iff $f(a) \neq g(a)$ for every $a \in G^*$.*

Proof (i) It is easy to see that G is a balanced groupoid. Now, G is simple by Proposition 2.4.
(ii) Easy. □

Lemma 3.2 *Let $G_1=[\mathcal{G}_1,G_1^*,f_1,g_1,o_1]$, $G_2=[\mathcal{G}_2,G_2^*,f_2,g_2,o_2]$ and let $\varrho^*: G_1^* \longrightarrow G_2^*$ be a bijection. Then $\varrho: G_1 \longrightarrow G_2$, where $\varrho\,|\,G_1^*=\varrho^*$ and $\varrho(o_1)=o_2$, is a groupoid isomorphism if and only if $\varrho^* f_1=f_2\varrho^*$ and $\varrho^* g_1=g_2\varrho^*$.*

Proof Easy. □

Now, let $\varrho: G_1 \longrightarrow G_2$ be an isomorphism. Then $\varrho(o_1)=o_2$ and $\varrho^* f_1=f_2\varrho^*$, $\varrho^* g_1=g_2\varrho^*$, where $\varrho^*=\varrho\,|\,G_1^*$. Define a mapping σ of $\mathcal{G}_1$ into the symmetric group on G_2^* by $\sigma(h)(\varrho^*(a))=\varrho^* h(a)$ for all $h \in \mathcal{G}_1$ and $a \in G_1^*$. Clearly, σ is an injective group homomorphism, $\sigma(f_1)=f_2$ and $\sigma(g_1)=g_2$. Consequently, σ is an isomorphism of $\mathcal{G}_1$ onto $\mathcal{G}_2$ (in particular, the permutation groups $\mathcal{G}_1$ and $\mathcal{G}_2$ are similar).

Let $\mathcal{A}$ denote the class of ordered quadruples (A,B,a,b), where A is a finite group, B is a core-free subgroup of A and $a,b \in A$ are such that $A=\langle a,b\rangle$. If $\alpha_i=(A_i,B_i,a_i,b_i) \in \mathcal{A}$, $i=1,2$, then we shall write $\alpha_1 \sim \alpha_2$ (and we shall say that α_1, α_2 are *equivalent*) iff there is an isomorphism $\lambda : A_1 \longrightarrow A_2$ such that $\lambda(a_1) = a_2$, $\lambda(b_1) = b_2$ and $\lambda(B_1)$ is a conjugate of B_2.

Let $\alpha = (A, B, a, b) \in \mathcal{A}$. Put $A/B = \{xB\,;\, x \in A\}$. For every $x \in A$, we have a permutation $\pi(x)$ of A/B defined by $\pi(x)(yB) = xyB$, $y \in A$. Then π is an injective group homomorphism of A into the symmetric group on A/B and we put $\Phi(\alpha) = \Phi(\alpha, o) = [\pi(A), A/B, \pi(a), \pi(b), o]$, where $o \notin A/B$.

Let $\alpha_1, \alpha_2 \in \mathcal{A}$. First, assume that $\alpha_1 \sim \alpha_2$. Then there is an isomorphism $\lambda : A_1 \longrightarrow A_2$ with the properties as above. Define $\varrho : \Phi(\alpha_1) \longrightarrow \Phi(\alpha_2)$ by $\varrho(o_1) = o_2$ and $\varrho(xB_1) = \lambda(x)w^{-1}B_2$ for every $x \in A_1$, where $w \in A_2$ is such that $\lambda(B_1) = B_2^w$. It is easy to check that ϱ is a bijection,

$$\varrho(\pi_1(a_1)(xB_1)) = \varrho(a_1xB_1) = a_2\pi(x)w^{-1}B_2 = \pi_2(a_2)(\varrho(xB_1))$$

and $\varrho(\pi_1(b_1)(xB_1)) = \pi_2(b_2)(\varrho(xB_2))$ for every $x \in A_1$. By 3.2, ϱ is a groupoid isomorphism.

Conversely, let $\varrho : \Phi(\alpha_1) \longrightarrow \Phi(\alpha_2)$ be a groupoid isomorphism. Then $\varrho(o_1) = o_2$ and we can find a mapping $\tau : A_1 \longrightarrow A_2$ such that $\varrho(xB_1) = \tau(x)B_2$ for every $x \in A_1$ and $\tau(x) = \tau(y)$, whenever $x^{-1}y \in B_1$. Put $w = \tau(1)^{-1} \in A_2$. Now, there is an isomorphism $\sigma : \pi_1(A_1) \longrightarrow \pi_2(A_2)$ such that $\sigma(\pi_1(a_1)) = \pi_2(a_2))$, $\sigma(\pi_1(b_1)) = \pi_2(b_2))$ and $\sigma(\pi_1(x))(\varrho(yB_1)) = \varrho(\pi_1(x)(yB_1))$ for all $x, y \in A_1$. Put $\lambda = \pi_2^{-1}\sigma\pi_1$. Then $\lambda : A_1 \longrightarrow A_2$ is an isomorphism and we have $\lambda(a_1) = a_2$, $\lambda(b_1) = b_2$ and $\lambda(x)\tau(y)B_2 = \tau(xy)B_2$ for all $x, y \in A_1$. That is, $\tau(xy)^{-1}\lambda(x)\tau(y) \in B_2$ and, for $x \in B_1$, $y = 1$, we get $\tau(1)^{-1}\lambda(x)\tau(1) = \tau(x)^{-1}\lambda(x)\tau(1) \in B_2$. Thus $\lambda(B_1) \subseteq B_2^w$. On the other hand, if $x \in A_1$ is such that $\lambda(x) \in B_2^w$ then $\tau(1)^{-1}\tau(x) = \tau(1)^{-1}\lambda(x)\tau(1) \cdot \tau(1)^{-1}\lambda(x)^{-1}\tau(x) \in B_2$, and hence $\tau(1) = \tau(x)$ and $x \in B_1$. We have proved that $\lambda(B_1) = B_2^w$. (Notice that the latter equality follows also from the inclusion $\lambda(B_1) \subseteq B_2^w$ and the facts that λ is injective and $\mathrm{card}(A_1) = \mathrm{card}(A_2)$, $\mathrm{card}(A_1/B_1) = \mathrm{card}(A_2/B_2)$ and $\mathrm{card}(B_1) = \mathrm{card}(B_2)$ are finite numbers.)

We have proved the following result:

Proposition 3.3 *Let $\alpha_1, \alpha_2 \in \mathcal{A}$. Then α_1 is equivalent to α_2 if and only if the groupoids $\Phi(\alpha_1)$ and $\Phi(\alpha_2)$ are isomorphic.*

4 Main result—the finite case

Let G be a finite simple balanced groupoid (see the second section). Then f, g are permutations of G^* and the transformation semigroup $\mathcal{T}$ (see Proposition 2.4) is a finite group acting transitively on G^*. For $u \in G^*$, denote by $\mathcal{H}_u$ the stabilizer of u in $\mathcal{T}$ and put $\alpha_u = (\mathcal{T}, \mathcal{H}_u, f, g) \in \mathcal{A}$ (see the preceding section). If $v \in G^*$ then $\alpha_u \sim \alpha_v$, since the stabilizers $\mathcal{H}_u$ and $\mathcal{H}_v$ are conjugate in $\mathcal{T}$. Finally, define a mapping $\varrho : G \longrightarrow \Phi(\alpha_u)$ by $\varrho(o) = o$ and $\varrho(a) = h\mathcal{H}_u$ for $a \in G^*$, $h \in \mathcal{T}$, $a = h(u)$. Clearly, ϱ is a groupoid isomorphism and we put $\Psi(G) = \Psi(G, u) = \alpha_u$.

Theorem 4.1 *The mappings Φ and Ψ yield a one-to-one correspondence between isomorphism classes of finite simple balanced groupoids and equivalence classes of quadruples from $\mathcal{A}$.*

Proof If $\alpha \in \mathcal{A}$ then $\Phi(\alpha)$ is a finite simple balanced groupoid and $\Phi(\alpha) \simeq \Phi(\beta)$ whenever $\alpha \sim \beta$. Conversely, if G is a finite simple balanced groupoid and $u, v \in G^*$ then $\Psi(G,u) \sim \Psi(G,v) \in \mathcal{A}$. Moreover, the groupoids G and $\Phi\Psi(G)$ are isomorphic. Now, if $G \simeq H$ then $\Phi\Psi(G) \simeq G \simeq H \simeq \Phi\Psi(H)$, and hence $\Psi(G) \sim \Psi(H)$. Finally, it is easy to see that $\alpha \sim \Phi\Psi(\alpha)$ for every $\alpha \in \mathcal{A}$. □

Remark 4.2 Let $\mathcal{B}$ denote the class of all quadruples $(A, B, a, b) \in \mathcal{A}$ such that $b^{-1}a \notin B^x$ for every $x \in A$. Then, by 4.1, we get a one-to-one correspondence between isomorphism classes of finite simple zeropotent balanced groupoids and equivalence classes of quadruples from $\mathcal{B}$.

5 Transitive transformation semigroups

Let $\mathcal{T}$ be a transitive transformation semigroup on an infinite set G^* such that $\mathcal{T} = \langle f, g\rangle$, where f and g are mappings of G^* onto G^*. Let $o \notin G^*$ and $G = G^* \cup \{o\}$. Now, define a multiplication on G as follows:

(1) $oo = o$;
(2) $ox = o = xo$ for every $x \in G^*$;
(3) $xy = o$ for all $x, y \in G^*$, $f(x) \neq g(y)$;
(4) $xy = f(x)$ $(= g(y))$ for all $x, y \in G^*$, $f(x) = g(y)$.

In this way, we get a groupoid $G = [\mathcal{T}, G^*, f, g, o]$.

Proposition 5.1 (i) *o is an absorbing element of G.*
(ii) *G is balanced iff both f and g are permutations of G^*.*
(iii) *G is zeropotent iff $f(a) \neq g(a)$ for every $a \in G^*$.*
(iv) *G is simple iff* $\ker(f) \cap \ker(g) = \mathrm{id}_{G^*}$.

Proof The first three assertions are easy. Now, assume that G is simple and put $r = \mathrm{id}_G \cup (\ker(f) \cap \ker(g))$. Then r is a congruence of G. If $r = \mathrm{id}_G$ then $\ker(f) \cap \ker(g) = \mathrm{id}_{G^*}$. If $r \neq \mathrm{id}_G$ then $r = G \times G$ and $\ker(f) = \ker(g) = G^* \times G^*$, which is impossible.

Conversely, let $\ker(f) \cap (ker(g)) = \mathrm{id}_{G^*}$ and let r be a congruence of G, $r \neq \mathrm{id}_G$. If $(x, o) \in r$ for some $x \in G^*$ then $r = G \times G$, since $\mathcal{T}$ acts transitively on G^*. Hence, assume that $(a, b) \in r$, $a, b \in G^*$, $a \neq b$. Then either $f(a) \neq f(b)$ or $g(a) \neq g(b)$ and we can use the preceding observation to show that $r = G \times G$. □

Lemma 5.2 *Let $G_1 = [\mathcal{T}_1, G_1^*, f_1, g_1, o_1]$, $G_2 = [\mathcal{T}_2, G_2^*, f_2, g_2, o_2]$ and let $\varrho^* : G_1^* \longrightarrow G_2^*$ be a bijection. Then $\varrho : G_1 \longrightarrow G_2$, where $\varrho | G_1^* = \varrho^*$ and $\varrho(o_1) = o_2$, is a groupoid isomorphism iff $\varrho^* f_1 = f_2 \varrho^*$ and $\varrho^* g_1 = g_2 \varrho^*$.*

Proof Easy. □

Let $\varrho : G_1 \longrightarrow G_2$ be an isomorphism. Then $\varrho(o_1) = o_2$, $\varrho^* f_1 = f_2 \varrho^*$, $\varrho^* g_1 = g_2 \varrho^*$, $\varrho^* = \varrho \,|\, G_1^*$ and we get an isomorphism $\sigma : \mathcal{T}_1 \longrightarrow \mathcal{T}_2$ such that $\sigma(f_1) = f_2$ and $\sigma(g_1) = g_2$.

Now, suppose that both f and g are permutations of G^* and denote by $\mathcal{G}$ the permutation group generated by f, g. Then $\mathcal{T} \subseteq \mathcal{G}$ and $\mathcal{G}$ is a transitive permutation group on G^*.

Lemma 5.3 *For all $a \in G^*$ and $h \in \mathcal{G}$ there is $k \in \mathcal{T}$ such that $k^{-1}h(a) = a$.*

Proof Use the fact that $\mathcal{T}$ is transitive on G^*. □

If f_1, g_1, f_2, g_2 are permutations and if $\varrho : G_1 \longrightarrow G_2$ is an isomorphism then ϱ induces an isomorphism $\sigma : \mathcal{G}_1 \longrightarrow \mathcal{G}_2$ such that $\sigma(\mathcal{T}_1) = \mathcal{T}_2$, $\sigma(f_1) = f_2$ and $\sigma(g_1) = g_2$.

Let $\mathcal{C}$ denote the class of ordered quadruples (A, B, a, b), where $A = \langle a, b \rangle$ is a group, B is a core-free subgroup of A such that the index $[A : B]$ is infinite and $a, b \in A$ are such that for every $x \in A$ there are elements r, s in the subsemigroup generated by $\{a, b\}$ such that $xr, sx \in B$. Further, we shall define an equivalence relation $\sim$ on $\mathcal{C}$ in the same way as in the third section.

Let $A = \langle a, b \rangle$ be a group and let B be a core-free subgroup of A such that $[A : B]$ is infinite. Denote by S the subsemigroup of A generated by $\{a, b\}$; we have $S = \{a^i b^j \,;\, i, j \in \mathbb{Z},\ i, j \geq 0,\ i + j \geq 1\}$. Then A acts as a transitive permutation group on A/B (left cosets) and we have an injective homomorphism π of A into the symmetric group on A/B. Now, it is easy to see that S is transitive on A/B iff for all $x, y \in A$ there is $s \in S$ with $xsy \in B$. But this condition is clearly equivalent to the fact that $(A, B, a, b) \in \mathcal{C}$.

Let $\alpha = (A, B, a, b) \in \mathcal{C}$. We put $\Phi(\alpha) = [\pi(S), A/B, \pi(a), \pi(b), o]$, $o \notin A/B$.

Proposition 5.4 *Let $\alpha_1, \alpha_2 \in \mathcal{C}$. Then $\alpha_1 \sim \alpha_2$ iff $\Phi(\alpha_1) \simeq \Phi(\alpha_2)$.*

Proof We may proceed similarly as in the proof of Proposition 3.3. □

6 Main result—the infinite case

Let G be an infinite simple strongly balanced groupoid. Then f, g are permutations of G^* and the permutation group $\mathcal{G} = \langle f, g \rangle$ is transitive on G^*. Now, similarly as in the fourth section, we define a quadruple $\Psi(G) \in \mathcal{C}$.

Theorem 6.1 *The mappings Φ and Ψ yield a one-to-one correspondence between isomorphism classes of infinite simple strongly balanced groupoids and equivalence classes of quadruples from $\mathcal{C}$.*

Proof Similar to that of Theorem 4.1. □

Remark 6.2 Let $\mathcal{D}$ denote the class of all quadruples $(A, B, a, b) \in \mathcal{C}$ such that $b^{-1}a \notin B^x$ for every $x \in A$. Then, by Theorem 6.1, we get a one-to-one correspondence between isomorphism classes of infinite simple zeropotent strongly balanced groupoids and equivalence classes of quadruples from $\mathcal{D}$.

Example 6.3 Let

$$G^* = \{(i,j)\,;\, i = 0,1,\ j \in \mathbb{Z},\ j \geq 1\} \cup \{(0,-j)\,;\, j \in \mathbb{Z},\ j \geq 0\}\,.$$

Define transformations f, ψ of the set G^* as follows:

(1) $f(i,j) = (i, j-1)$ for $(i,j) \neq (1,1)$;
(2) $f(1,1) = (0,0)$;
(3) $\psi(0,j) = (0,-2j)$, $\psi(0,-2j) = (0,j)$, $\psi(1,j) = (0,-2j-1)$, $\psi(0,-2j-1) = (1,j)$ for $j \geq 1$;
(4) $\psi(0,0) = (0,-1)$, $\psi(0,-1) = (0,0)$.

Further, put $g = f\psi$, denote by $\mathcal{T}$ the subsemigroup of the transformation semigroup of G^* generated by f, g and choose $o \notin G^*$. Then $G = [\mathcal{T}, G^*, f, g, o]$ is a zeropotent simple balanced groupoid which is not strongly balanced.

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