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One of the calibration problems
One of the Calibration Problems *

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Abstract

Two measurement devices are characterized by dispersions $\sigma_1^2$ and $\sigma_2^2$ of their registration. The ratio $\sigma_1^2/\sigma_2^2$ is a priori unknown and approximations $\sigma_{1,0}^2$ and $\sigma_{2,0}^2$ are at our disposal only.

The relation between errorless registrations of these devices is given by the calibration curve.

One of the procedures of constructing this curve is described in the paper. Further, accuracy characteristics of the calibration curve parameters estimators and the MINQUE $\hat{\sigma}_1^2$ and $\hat{\sigma}_2^2$ of the dispersions $\sigma_1^2$ and $\sigma_2^2$ are given.

Key words: BLUE, MINQUE, model with constraints.

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Introduction

The aim of the paper is to find explicite formulae for estimators of parameters $\beta_1$ and $\beta_2$, which link registrations of two measurement devices and estimators of the dispersions $\sigma_1^2$ and $\sigma_2^2$ of their registrations.
If an errorless registration of some value $v$ of the first device is $\mu$, then the errorless registration of the other device is $\nu$ and

$$\nu = \beta_1 + \beta_2 \mu$$

(calibration curve).

The registration $\mu$ of the first device is realized with an error characterized by the dispersion $\sigma_1^2$, the registration $\nu$ of the other device is realized with an error characterized by the dispersion $\sigma_2^2$.

1 Preliminaries

Let $n$ ($n \geq 2$) different values $v_1, \ldots, v_n$, of the measured quantity be registered by both devices. The results from the first device creates a realization of the $n$-dimensional random vector (observation vector) $Y_1$; an analogous meaning has $Y_2$ for the other device.

In our case we have the model

$$
\begin{pmatrix}
Y_1 \\
Y_2
\end{pmatrix}
\sim
\begin{pmatrix}
\mu \\
\nu
\end{pmatrix} +
\begin{pmatrix}
\sigma_1^2 I & 0 \\
0 & \sigma_2^2 I
\end{pmatrix}
$$

(1.1)

$\mu = (\mu_1, \ldots, \mu_n)'$, $\nu = (v_1, \ldots, v_n)'$ and $I$ denotes the $n \times n$ identity matrix (the notation $\eta \sim (a, W)$ means that the random vector $\eta$ has the mean value equal to $a$ and its covariance matrix is $W$). Further

$$\nu = \mathbf{1}\beta_1 + \mu \beta_2$$

(1.2)

must hold (here $\mathbf{1} = (1, \ldots, 1)' \in \mathbb{R}^n$ and $\mathbb{R}^n$ denotes the $n$-dimensional Euclidean space).

Lemma 1.1 Consider the model of incomplete direct measurements with a condition $Y \sim (\gamma_1, \Sigma)$,

$$
\begin{pmatrix}
\gamma_1 \\
\gamma_2
\end{pmatrix}
\in
\left\{ \begin{pmatrix}
u \\
\nu
\end{pmatrix} : b + B_1 u + B_2 v = 0 \right\};
$$

here $\gamma_1$ is an unknown $n$-dimensional parameter, $\gamma_2$ an unknown $k$-dimensional parameter, $\Sigma$ a given $n \times n$ p.d. matrix, $B_1$ a given $q \times n$ matrix and $B_2$ is a given $q \times k$ matrix; the ranks of the matrices $B_1$ and $B_2$ are characterized by the relations

$$r(B_1, B_2) = q, \quad r(B_2) = k, \quad k < q < n + k.$$

Then the BLUE (best linear unbiased estimator) of $\gamma_1$ and $\gamma_2$ is

$$
\begin{pmatrix}
\hat{\gamma}_1 \\
\hat{\gamma}_2
\end{pmatrix}
= 
\begin{pmatrix}
I & \Sigma B_1 Q_{1,1} B_1 \\
-Q_{2,1} B_1 & -Q_{2,1} B_1
\end{pmatrix}
Y +
\begin{pmatrix}
-\Sigma B_1' Q_{1,1} \\
-Q_{2,1} B_1
\end{pmatrix}
b,
$$

(1.3)

and the covariance matrix of this estimator is

$$
\text{Var}
\left(
\begin{pmatrix}
\hat{\gamma}_1 \\
\hat{\gamma}_2
\end{pmatrix}
\right)
= 
\begin{pmatrix}
\Sigma - \Sigma B_1' Q_{1,1} B_1 \Sigma, & -\Sigma B_1' Q_{1,2} \\
-Q_{2,1} B_1 \Sigma, & -Q_{2,2}
\end{pmatrix},
$$
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where

\[
\begin{pmatrix}
  Q_{1,1}, & Q_{1,2} \\
  Q_{2,1}, & Q_{2,2}
\end{pmatrix} = \begin{pmatrix}
  B_1 \Sigma B_1', & B_2 \\
  B_2', & 0
\end{pmatrix}^{-1}.
\]

(1.4)

Proof cf. in [2], pp. 138–143.

Remark 1.2 If in Lemma 1.1 \( \Sigma = \sigma^2 V \), where \( V \) is a given \( n \times n \) p.d. matrix and \( \sigma^2 \in (0, \infty) \) is an unknown parameter, then

\[
\begin{pmatrix}
  \tilde{\gamma}_1 \\
  \tilde{\gamma}_2
\end{pmatrix} = \begin{pmatrix}
  I - VB_1'\tilde{Q}_{1,1}B_1 \\
  -\tilde{Q}_{2,1}B_1
\end{pmatrix} Y + \begin{pmatrix}
  -VB_1'\tilde{Q}_{1,1} \\
  -\tilde{Q}_{2,1}
\end{pmatrix} b,
\]

\[
\text{Var} \begin{pmatrix}
  \tilde{\gamma}_1 \\
  \tilde{\gamma}_2
\end{pmatrix} = \sigma^2 \begin{pmatrix}
  V - VB_1'\tilde{Q}_{1,1}B_1V & -VB_1'\tilde{Q}_{1,2} \\
  -\tilde{Q}_{2,1}B_1V & -\tilde{Q}_{2,2}
\end{pmatrix},
\]

where

\[
\begin{pmatrix}
  \tilde{Q}_{1,1}, & \tilde{Q}_{1,2} \\
  \tilde{Q}_{2,1}, & \tilde{Q}_{2,2}
\end{pmatrix} = \begin{pmatrix}
  B_1 VB_1', & B_2 \\
  B_2', & 0
\end{pmatrix}^{-1}.
\]

In this case the parameter \( \sigma^2 \) can be estimated by the statistic

\[
\hat{\sigma}^2 = (Y - \tilde{\gamma}_1)'V^{-1}(Y - \tilde{\gamma}_1)/(q - k).
\]

In the case of normality of the vector \( Y \), the distribution of the statistic \( \hat{\sigma}^2 \) is

\[
\hat{\sigma}^2 \sim \sigma^2 \chi^2_{q-k}(0)/(q - k),
\]

where \( \chi^2_{q-k}(0) \) is the random variable with the central chi-square distribution and \( q - k \) degrees of freedom (in more detail cf. [2]).

The notation \( P^W_A \) (used in the following) means an orthogonal projection matrix \( A(A'WA)^{-1}A'W \) on the subspace \( \mathcal{M}(A_{n,k}) = \{Au, u \in R^k\} \) in the norm given by the p.d. \( n \times n \) matrix \( W \), i.e. \( ||y||_W = \sqrt{y'Wy}, y \in R^n \). Furthemore \( M^W_A = I - P^W_A \). If \( W = I \), then instead of \( P^I_A \) and \( M^I_A \) the notation \( P_A \) and \( M_A \), respectively, is used.

2 Estimators of \( \mu, \nu, \beta_1 \) and \( \beta_2 \)

Let \( \mu_0 \) and \( \beta_{2,0} \) be approximate values of \( \mu \) and \( \beta_2 \), respectively, such that the vector \( (\mu - \mu_0)(\beta_2 - \beta_{2,0}) \) can be neglected. Then the relation (1.2) can be substituted by

\[
\beta_{2,0}\mu_0 + (\beta_{2,0}I, -I) \begin{pmatrix}
  \delta \mu \\
  \delta \nu
\end{pmatrix} + (1, \mu_0) \begin{pmatrix}
  \beta_1 \\
  \delta \beta_2
\end{pmatrix} = 0,
\]

where \( \delta \mu = \mu - \mu_0 \) and \( \delta \beta_2 = \beta_2 - \beta_{2,0} \).

In order to obtain a little more general solution of our problem, let \( \sigma^2_2I \) be substituted by an \( n \times n \) p.d. matrix \( \Sigma_{1,1} \) and \( \sigma^2_2I \) by the \( n \times n \) p.d. matrix \( \Sigma_{2,2} \).
Under this assumptions, with respect to Lemma 1.1. our model can be rewritten under the following scheme

\[
Y \rightarrow \begin{pmatrix} Y_1 - \mu_0 \\ Y_2 \end{pmatrix}, \quad \gamma_1 \rightarrow \begin{pmatrix} \delta \mu \\ \nu \end{pmatrix}, \quad \gamma_2 \rightarrow \begin{pmatrix} \beta_1 \\ \delta \beta_2 \end{pmatrix},
\]
\[
b \rightarrow \beta_{2,0} \mu_0, \quad B_1 \rightarrow (\beta_{2,0} I, -I), \quad B_2 \rightarrow (1, \mu_0),
\]
\[
\Sigma \rightarrow \begin{pmatrix} \Sigma_{1,1} & 0 \\ 0 & \Sigma_{2,2} \end{pmatrix}.
\]

**Lemma 2.1** Let \( W \) be an \( n \times n \) p.d. matrix and \( A \) an \( n \times k \) matrix with the rank \( r(A) = k \). Then the matrix

\[
\begin{pmatrix} W & A \\ A' & 0 \end{pmatrix}
\]

is regular and its inverse is

\[
\left( \begin{array}{cc} W & A \\ A' & 0 \end{array} \right)^{-1} = \left( \begin{array}{cc} W^{-1} - W^{-1}A(A'W^{-1}A)^{-1}A'W^{-1}, & W^{-1}A(A'W^{-1}A)^{-1} \\ (A'W^{-1}A)^{-1}A'W^{-1}, & -(A'W^{-1}A)^{-1} \end{array} \right)
\]

**Proof** is obvious.

The matrix \( W^{-1} - W^{-1}A(A'W^{-1}A)^{-1}A'W^{-1} \) can be written in the form \((M_A WM_A)^+\) (+ denotes the Moore–Penrose g-inverse; in more detail cf. [3]).

**Theorem 2.2** In the model (cf. (2.1))

\[
\begin{pmatrix} Y_1 - \mu_0 \\ Y_2 \end{pmatrix} \sim \begin{pmatrix} \delta \mu \\ \nu \end{pmatrix}, \quad \begin{pmatrix} \Sigma_{1,1} & 0 \\ 0 & \Sigma_{2,2} \end{pmatrix},
\]

\[
\beta_{2,0} \mu_0 + (\beta_{2,0} I, -I) \begin{pmatrix} \delta \mu \\ \nu \end{pmatrix} + (1, \mu_0) \begin{pmatrix} \beta_1 \\ \delta \beta_2 \end{pmatrix} = 0,
\]

if we denote \( M_{1,\mu_0} = I - P_{1,\mu_0} \), where

\[
P_{1,\mu_0} = (1, \mu_0) \begin{pmatrix} n, & 1' \mu_0 \\ 1' \mu_0, & \mu_0 \mu \end{pmatrix}^{-1} \begin{pmatrix} 1' \\ \mu_0' \end{pmatrix}
\]

the following is valid

\[
Q_{1,1} = [M_{1,\mu_0}(\beta_{2,0}^2 \Sigma_{1,1} + \Sigma_{2,2})M_{1,\mu_0}]^+,
\]
\[
Q_{2,1} = -Q_{2,2} \begin{pmatrix} 1'(\beta_{2,0}^2 \Sigma_{1,1} + \Sigma_{2,2})^{-1} \\ \mu_0(\beta_{2,0}^2 \Sigma_{1,1} + \Sigma_{2,2})^{-1} \end{pmatrix},
\]
\[
Q_{2,2} = -\left( 1'(\beta_{2,0}^2 \Sigma_{1,1} + \Sigma_{2,2})^{-1}1', \quad 1'(\beta_{2,0}^2 \Sigma_{1,1} + \Sigma_{2,2})^{-1} \mu_0 \\ \mu_0(\beta_{2,0}^2 \Sigma_{1,1} + \Sigma_{2,2})^{-1}1', \quad \mu_0(\beta_{2,0}^2 \Sigma_{1,1} + \Sigma_{2,2})^{-1} \mu_0 \right)^{-1},
\]
\[
\hat{\mu} = \mu_0 + \delta \mu
\]
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\[ Y_1 + \beta_{2,0} \Sigma_{1,1}[M_{1,\mu_0}(\beta_{2,0} \Sigma_{1,1} + \Sigma_{2,2})M_{1,\mu_0}]^+(Y_2 - \beta_{2,0} Y_1), \]

\[ \hat{\nu} = Y_2 - \Sigma_{2,2}[M_{1,\mu_0}(\beta_{2,0} \Sigma_{1,1} + \Sigma_{2,2})M_{1,\mu_0}]^+(Y_2 - \beta_{2,0} Y_1), \]

\[ \begin{pmatrix} \hat{\beta}_1 \\ \delta \hat{\beta}_2 \end{pmatrix} = -Q_{2,2} \begin{pmatrix} 1'(\beta_{2,0}^2 \Sigma_{1,1} + \Sigma_{2,2})^{-1}(Y_2 - \beta_{2,0} Y_1) \\ \mu_0'(\beta_{2,0}^2 \Sigma_{1,1} + \Sigma_{2,2})^{-1}(Y_2 - \beta_{2,0} Y_1) \end{pmatrix}, \]

\[ \text{Var}(\hat{\mu}) = \Sigma_{1,1} - \beta_{2,0} \Sigma_{1,1} Q_{1,1} \Sigma_{1,1}, \]

\[ \text{cov}(\hat{\mu}, \hat{\nu}) = \beta_{2,0} \Sigma_{1,1} Q_{1,1} \Sigma_{2,2}, \]

\[ \text{Var}(\hat{\nu}) = \Sigma_{2,2} - \Sigma_{2,2} Q_{1,1} \Sigma_{2,2}, \]

\[ \text{Var} \left( \frac{\hat{\beta}_1}{\hat{\beta}_2} \right) = -Q_{2,2}. \]

\[ \text{cov} \left[ \left( \frac{\hat{\beta}_1}{\hat{\beta}_2} \right), \left( \hat{\mu} \hat{\nu} \right) \right] = Q_{2,2} \begin{pmatrix} 1'(\beta_{2,0}^2 \Sigma_{1,1} + \Sigma_{2,2})^{-1}\beta_{2,0} \Sigma_{1,1}, & -1'(\beta_{2,0}^2 \Sigma_{1,1} + \Sigma_{2,2})^{-1}\beta_{2,0} \Sigma_{1,1}, & -\mu_0'(\beta_{2,0}^2 \Sigma_{1,1} + \Sigma_{2,2})^{-1}\Sigma_{2,2} \end{pmatrix} \]

**Proof** The expression for \( Q_{1,1} \) is obvious with respect to Lemma 2.1 and the remark after it. Further it is necessary to use Lemma 1.1 under the scheme (2.1). Since this is simple however tedious, it is omitted. \( \square \)

**Corollary 2.3** If in Theorem 2.2, \( \Sigma_{1,1} = \sigma_1^2 I \) and \( \Sigma_{2,2} = \sigma_2^2 I \), then

\[ Q_{1,1} = \frac{1}{\beta_{2,0}^2 \sigma_1^2 + \sigma_2^2} M_{1,\mu_0}, \quad Q_{2,1} = \begin{pmatrix} n, & 1' \mu_0 \\ \mu_0'1, & \mu_0'\mu_0 \end{pmatrix} \begin{pmatrix} 1' \\ \mu_0' \end{pmatrix} \]

\[ Q_{2,2} = -(\beta_{2,0}^2 \sigma_1^2 + \sigma_2^2) \begin{pmatrix} n, & 1' \mu_0 \\ \mu_0'1, & \mu_0'\mu_0 \end{pmatrix}^{-1}. \]

Thus

\[ \hat{\mu} = Y_1 + \frac{\beta_{2,0}^2 \sigma_1^2}{\beta_{2,0}^2 \sigma_1^2 + \sigma_2^2} M_{1,\mu_0}(Y_2 - \beta_{2,0} Y_1), \]

\[ \hat{\nu} = Y_2 - \frac{\sigma_2^2}{\beta_{2,0}^2 \sigma_1^2 + \sigma_2^2} M_{1,\mu_0}(Y_2 - \beta_{2,0} Y_1), \]

\[ \begin{pmatrix} \hat{\beta}_1 \\ \delta \hat{\beta}_2 \end{pmatrix} = \begin{pmatrix} n, & 1' \mu_0 \\ \mu_0'1, & \mu_0'\mu_0 \end{pmatrix}^{-1} \begin{pmatrix} 1'(Y_2 - \beta_{2,0} Y_1) \\ \mu_0'(Y_2 - \beta_{2,0} Y_1) \end{pmatrix}, \]

\[ \text{Var}(\hat{\mu}) = \sigma_1^2 I - \frac{\beta_{2,0}^2 \sigma_1^4}{\beta_{2,0}^2 \sigma_1^2 + \sigma_2^2} M_{1,\mu_0}, \]

\[ \text{cov}(\hat{\mu}, \hat{\nu}) = \frac{\beta_{2,0} \sigma_1^2 \sigma_2^2}{\beta_{2,0}^2 \sigma_1^2 + \sigma_2^2} M_{1,\mu_0}, \]

\[ \text{Var}(\hat{\nu}) = \sigma_2^2 I - \frac{\sigma_2^4}{\beta_{2,0}^2 \sigma_1^2 + \sigma_2^2} M_{1,\mu_0}. \]
\[
\text{Var}\left( \begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{pmatrix} \right) = (\beta_{2,0}^2 \sigma_1^2 + \sigma_2^2) \left( \begin{array}{c} n, \\ \mu_0'1, \\ \mu_0^0\mu_0 \end{array} \right)^{-1} \\
= \left( \begin{array}{c} v_{1,1}, v_{1,2} \\ v_{2,1}, v_{2,2} \end{array} \right)
\]

and
\[
\text{cov} \left[ \left( \begin{array}{c} \hat{\beta}_1 \\ \hat{\beta}_2 \end{array} \right), \left( \begin{array}{c} \hat{\mu} \\ \hat{\nu} \end{array} \right) \right] = -\left( \begin{array}{c} n, \\ \mu_0'1, \\ \mu_0^0\mu_0 \end{array} \right)^{-1} \left( \begin{array}{c} \beta_{2,0}^2 \sigma_1^2 1', -\sigma_2^2 1' \\ \beta_{2,0}^2 \sigma_1^2 \mu_0', -\sigma_2^2 \mu_0' \end{array} \right).
\]

**Remark 2.4** The formulae given in Corollary 2.3 can be considered as a solution of the first problem, i.e. how to determine parameters of the calibration curve. It is to be remarked that for \( \sigma_1 = \sigma_2 \) and \( \delta_{2,0} = 0 \) (i.e. \( \beta_{2,0} = \hat{\beta}_2 \)) the equality
\[
\forall\{i = 1, \ldots, n\} (y_{2,i} - \hat{\nu}_i)/(y_{1,i} - \hat{\mu}_i) = -1/\beta_{2,0}
\]
is valid. This is a reason that the treated problem is sometimes called “orthogonal regression problem”.

It is not necessary to emphasize that in the case \( \hat{\delta}_{2} \neq 0 \) (or a negligible value) or \( \hat{\delta}_{\mu} \neq 0 \), the calculation must be repeated with the new starting vectors \( \hat{\mu} = \mu_0 + \hat{\delta}_{\mu} \) and \( \hat{\nu} \) instead of \( Y_1 \) and \( Y_2 \), respectively; also the value \( \beta_{2,0} \) must be substituted by \( \beta_{2,0} + \hat{\delta}_{2} \).

**Remark 2.5** If it is a priori known that \( \sigma = \sigma_1 = \sigma_2 \), where \( \sigma \) is unknown, then there is no problem to estimate it. With respect to Remark 1.2
\[
\hat{\sigma}^2 = [(Y_1 - \hat{\mu})'(Y_1 - \hat{\mu}) + (Y_2 - \hat{\nu})'(Y_2 - \hat{\nu})]/(n - 2).
\]

In the next section we shall see that the estimator of the vector \((\sigma_1^2, \sigma_2^2)'\) in the case \( \sigma_1 \neq \sigma_2 \) when simultaneously \( \sigma_1/\sigma_2 \) is unknown, cannot be found so simply.

### 3 MINQUE of \( \sigma_1^2 \) and \( \sigma_2^2 \)

In this section the model of incomplete direct measurement with a condition
\[
\begin{pmatrix} Y_1 - \mu_0 \\ Y_2 \end{pmatrix} \sim \begin{pmatrix} (\delta_{\mu})' \\ \nu \end{pmatrix}, \begin{pmatrix} (\sigma_1^2 I, 0) \\ 0, (\sigma_2^2 I) \end{pmatrix}, \quad (3.1)
\]
\[
\beta_{2,0}\mu_0 + (\beta_{2,0} I, -I) (\begin{pmatrix} \delta_{\mu} \\ \nu \end{pmatrix}) + (1, \nu_0) (\begin{pmatrix} \beta_1 \\ \delta_{2,0} \end{pmatrix}) = 0
\]
is under consideration.

**Lemma 3.1** Let in the model of incomplete direct measurement with a condition from Lemma 1.1 the matrix \( \Sigma \) be of the form \( \Sigma = \sum_{i=1}^p \theta_i V_i \), where \( V_1, \ldots, V_p \), are given symmetric \( n \times n \) matrices and \( \theta = (\theta_1, \ldots, \theta_p)' \in \theta \subset R^p \).
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is an unknown vector parameter; $\mathfrak{q}$ is an open set in $R^{p}$ such that $\vartheta \in \mathfrak{q}$ implies $\sum_{i=1}^{p} \vartheta_{i} V_{i}$ is p.d. Let $\text{Ker}(B_{1}, B_{2}) = M \begin{pmatrix} K_{1} \\ K_{2} \end{pmatrix}$, where $K_{1}$ is an $n \times (n+k-q)$ matrix and $K_{2}$ is a $k \times (n+k-q)$ matrix. Let $\gamma_{1,0}$ be any solution of the equation $b + B_{1} \gamma_{1} + B_{2} \gamma_{2} = 0$.

Then $\theta_{0}-\text{MINQUE}$ (for more detail cf. [4]) of a function $g' \vartheta$, $\vartheta \in \mathfrak{q}$, exists iff

$$g \in M \left( S(M_{K_{1}} \Sigma_{0} M_{K_{1}})^{+} \right),$$

where

$$\left\{ S(M_{K_{1}} \Sigma_{0} M_{K_{1}})^{+} \right\}_{i,j} = \text{Tr} \left[ (M_{K_{1}} \Sigma_{0} M_{K_{1}})^{+} V_{i} (M_{K_{1}} \Sigma_{0} M_{K_{1}})^{+} V_{j} \right],$$

$i, j = 1, \ldots, p$, and $\Sigma_{0} = \Sigma(\theta_{0}) = \sum_{i=1}^{p} \vartheta_{0,i} V_{i}$, $\vartheta_{0} \in \mathfrak{q}$.

If this condition is fulfilled, then $\theta_{0}-\text{MINQUE}$ of the function $g' \vartheta$, $\vartheta \in \mathfrak{q}$, is

$$\bar{g'} \vartheta = \sum_{i=1}^{p} \lambda_{i} (Y - \gamma_{1,0})' (M_{K_{1}} \Sigma_{0} M_{K_{1}})^{+} V_{i} (M_{K_{1}} \Sigma_{0} M_{K_{1}})^{+} (Y - \gamma_{1,0}),$$

where the vector $\lambda = (\lambda_{1}, \ldots, \lambda_{p})'$ is a solution of the equation

$$S(M_{K_{1}} \Sigma_{0} M_{K_{1}})^{+} \lambda = g.$$

**Proof** It is a consequence of Theorem 5.2.1 in [4].

**Corollary 3.2** Let the matrix $S(M_{K_{1}} \Sigma_{0} M_{K_{1}})^{+}$ be regular; thus there exists the $\theta_{0}-\text{MINQUE}$ of the whole vector $\vartheta$ and it is

$$\hat{\vartheta} = S^{-1}(M_{K_{1}} \Sigma_{0} M_{K_{1}})^{+} \begin{pmatrix} (Y - \gamma_{1,0})' (M_{K_{1}} \Sigma_{0} M_{K_{1}})^{+} V_{1} (M_{K_{1}} \Sigma_{0} M_{K_{1}})^{+} (Y - \gamma_{1,0}) \\ \vdots \\ (Y - \gamma_{1,0})' (M_{K_{1}} \Sigma_{0} M_{K_{1}})^{+} V_{p} (M_{K_{1}} \Sigma_{0} M_{K_{1}})^{+} (Y - \gamma_{1,0}) \end{pmatrix}.$$

**Theorem 3.3** (i) The matrix $(M_{K_{1}} \Sigma_{0} M_{K_{1}})^{+}$ can be expressed as

$$B_{1}' Q_{1,1} B_{1}$$

(in (1.4) the matrix $\Sigma_{0}$ is used instead of $\Sigma$).

(ii) The expression $(M_{K_{1}} \Sigma_{0} M_{K_{1}})^{+} (Y - \gamma_{1,0})$ is equal to $\Sigma_{0}^{-1} (Y - \hat{\gamma}_{1})$, with $\hat{\gamma}_{1}$ given by (1.3), where the matrix $\Sigma_{0}$ is used instead of $\Sigma$.

**Proof** (i) Obviously $B_{1} K_{1} + B_{2} K_{2} = 0 \Rightarrow K_{2} = -(B_{2}' B_{2})^{-1} B_{2}' B_{1} K_{1}$. Thus $[I - B_{2}(B_{2}' B_{2})^{-1} B_{2}'] B_{1} K_{1} = M_{B_{1}} B_{1} K_{1} = 0$. If $\gamma_{1} = \gamma_{1,0} + K_{1} \tau$, then the BLUE of $\beta_{1}$ is

$$\hat{\gamma}_{1} = \gamma_{0,1} + \overrightarrow{K_{1}} \tau = \gamma_{1,0} + \hat{\vartheta}_{0}^{-1} (Y - \gamma_{1,0})$$
and with respect to (1.3) also

\[ \hat{\gamma}_1 = (I - \Sigma_0 B_1 B_1') Y - \Sigma_0 B_1' Q_{1,1} b. \]

Thus \( I - \Sigma_0 B_1' Q_{1,1} B_1 = P_{K_1}^{\Sigma_0^{-1}} \).

To prove directly this equality it is necessary and sufficient to show that

(a) \( (I - \Sigma_0 B_1' Q_{1,1} B_1)^2 = I - \Sigma_0 B_1' Q_{1,1} B_1, \)

(b) \( \mathcal{M}(I - \Sigma_0 B_1' Q_{1,1} B_1) = \mathcal{M}(K_1), \)

(c) \( \Sigma_0^{-1} (I - \Sigma_0 B_1' Q_{1,1} B_1) = (I - B_1' Q_{1,1} B_1 \Sigma_0) \Sigma_0^{-1}. \)

The equality (a) is a direct consequence of (1.4).

(b) Obviously

\[ M_{B_1} B_1 (I - \Sigma_0 B_1' Q_{1,1} B_1) = M_{B_1} (B_1 - B_1 \Sigma_0 B_1' Q_{1,1} B_1) = 0, \]

since \( B_1 - B_1 \Sigma_0 B_1' Q_{1,1} B_1 = B_2 Q_{2,1} \) (cf. (1.4)). Thus

\[ \mathcal{M}(I - \Sigma_0 B_1' Q_{1,1} B_1) \subset \mathcal{M}(K_1). \]

Since \( I - \Sigma_0 B_1' Q_{1,1} B_1 \) is idempotent,

\[ r(I - \Sigma_0 B_1' Q_{1,1} B_1) = \text{Tr}(I - \Sigma_0 B_1' Q_{1,1} B_1) = \text{Tr}(I) - \text{Tr}(\Sigma_0 B_1' Q_{1,1} B_1). \]

Further \( B_1 \Sigma_0 B_1' Q_{1,1,1} + B_2 Q_{2,1} = I \) and thus

\[ \text{Tr}(\Sigma_0 B_1' Q_{1,1} B_1) = \text{Tr}(B_1 \Sigma_0 B_1' Q_{1,1}) = \text{Tr}(I - B_2 Q_{2,1}) = q - \text{Tr}(Q_{2,1} B_2) = q - \text{Tr}(I_{k,k}) = q - k. \]

Thus \( r(I - \Sigma_0 B_1' Q_{1,1} B_1) = n + k - q. \)

Further

\[ n + k - q = \dim \mathcal{M} \left( \begin{array}{c} K_1 \\ K_2 \end{array} \right) = \dim \mathcal{M} \left[ \begin{array}{c} I \\ - (B_2' B_2)^{-1} B_2' B_1 \end{array} \right] K_1 = \dim \mathcal{M}(K_1), \]

since the matrix \( \begin{pmatrix} I \\ - (B_2' B_2)^{-1} B_2' B_1 \end{pmatrix} \) is of the full rank in columns.

The implication

\[ \mathcal{M}(I - \Sigma_0 B_1' Q_{1,1} B_1) \subset \mathcal{M}(K_1) \quad \& \quad \dim \mathcal{M}(I - \Sigma_0 B_1' Q_{1,1} B_1) = \dim \mathcal{M}(K_1) \]

proves (b).

(c) is obvious.

Since

\[ (M_{K_1} \Sigma_0 M_{K_1})^+ = \Sigma_0^{-1} M_{K_1}^{\Sigma_0^{-1}} = \Sigma_0^{-1} (I - P_{K_1}^{\Sigma_0^{-1}}) = B_1' Q_{1,1} B_1, \]
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the statement (i) is proved.

(ii) The expression \((M_k \Sigma_0 M_k)^+(Y - \gamma_{1,0})\) can be rewritten as

\[
\Sigma_0^{-1}(I - P_{K_1}^{-1})(Y - \gamma_{1,0}) = \Sigma_0^{-1}(Y - \gamma_{1,0} - K_1) = \Sigma_0^{-1}(Y - \tilde{\gamma}_1).
\]

\[\square\]

**Theorem 3.4** In the framework of the model (3.1) we have

\[
S_{(M_k \Sigma_0 M_k)^+} = \frac{n - 2}{\left(\beta_{0,0}^2 \sigma_{1,0}^2 + \sigma_{2,0}^2\right)^2} \left(\begin{array}{c}
\beta_{2,0}^2 \\
\beta_{2,0}^2
\end{array}\right),
\]

and for \(g \in \mathcal{M}(S_{(M_k \Sigma_0 M_k)^+})\)

\[
g' \left(\begin{array}{c}
\sigma_1^2 \\
\sigma_2^2
\end{array}\right) = \lambda_1 \frac{1}{\sigma_{1,0}^2} (Y_1 - \hat{\mu})' (Y_1 - \hat{\mu}) + \lambda_2 \frac{1}{\sigma_{2,0}^2} (Y_2 - \hat{\nu})' (Y_2 - \hat{\nu}),
\]

where

\[
S_{(M_k \Sigma_0 M_k)^+} \left(\begin{array}{c}
\lambda_1 \\
\lambda_2
\end{array}\right) = \left(\begin{array}{c}
g_1 \\
g_2
\end{array}\right).
\]

**Proof** With respect to Theorem 2.2 and Corollary 2.3 we have

\[
Q_{1,1} = (M_{1,\mu_0} \left(\beta_{2,0}^2 \Sigma_{1,1} + \Sigma_{2,2}\right) M_{1,\mu_0})^+ = \frac{1}{\beta_{2,0}^2 \sigma_{1,0}^2 + \sigma_{2,0}^2} M_{1,\mu_0}
\]

(here the obvious equality \(M_{1,\mu_0}^+ = M_{1,\mu_0}\) is utilized) and thus

\[
(M_k \Sigma_0 M_k)^+ = B_1' Q_{1,1} B_1 = \frac{1}{\beta_{2,0}^2 \sigma_{1,0}^2 + \sigma_{2,0}^2} \left(\begin{array}{c}
\beta_{2,0}^2 M_{1,\mu_0}, \\
-\beta_{2,0}^2 M_{1,\mu_0},
\end{array}\right) M_{1,\mu_0}
\]

Since \(\text{Tr}(M_{1,\mu_0}) = n - 2\), we obtain

\[
\text{Tr} (B_1' Q_{1,1} V_1 B_1) = \{S_{(M_k \Sigma_0 M_k)^+}\}_{1,1} = \frac{n - 2}{\left(\beta_{2,0}^2 \sigma_{1,0}^2 + \sigma_{2,0}^2\right)^2} \beta_{2,0}^4
\]

\[
\text{Tr} (B_1' Q_{1,1} V_1 B_1 Q_{1,1} B_1) = \{S_{(M_k \Sigma_0 M_k)^+}\}_{1,2} = \{S_{(M_k \Sigma_0 M_k)^+}\}_{2,1} = \frac{n - 2}{\left(\beta_{2,0}^2 \sigma_{1,0}^2 + \sigma_{2,0}^2\right)^2} \beta_{2,0}^2
\]

\[
\text{Tr} (B_1' Q_{1,1} V_1 B_2 B_1) = \{S_{(M_k \Sigma_0 M_k)^+}\}_{2,2} = \frac{n - 2}{\left(\beta_{2,0}^2 \sigma_{1,0}^2 + \sigma_{2,0}^2\right)^2} \beta_{2,0}^2
\]

As far as the terms

\[
(Y - \gamma_{1,0})' (M_k \Sigma_0 M_k)^+ V_i (M_k \Sigma_0 M_k)^+ (Y - \gamma_{1,0}), \quad i = 1, \ldots, p,
\]

are concerned, it is sufficient to realize the relationships (implied by Theorem 3.3. (ii))

\[
(Y - \gamma_{1,0})' (M_k \Sigma_0 M_k)^+ V_1 (M_k \Sigma_0 M_k)^+ (Y - \gamma_{1,0}) \rightarrow \left(\begin{array}{c}
Y_1 - \hat{\mu} \\
Y_2 - \hat{\nu}
\end{array}\right)' \left(\begin{array}{cc}
\sigma_{1,0}^{-2} I, & 0 \\
0, & \sigma_{2,0}^{-2} I
\end{array}\right) \left(\begin{array}{cc}
I, & 0 \\
0, & 0
\end{array}\right) \left(\begin{array}{cc}
\sigma_{1,0}^{-2} I, & 0 \\
0, & \sigma_{2,0}^{-2} I
\end{array}\right) \left(\begin{array}{c}
Y_1 - \hat{\mu} \\
Y_2 - \hat{\nu}
\end{array}\right) = \frac{1}{\sigma_{1,0}^2} (Y_1 - \hat{\mu})' (Y_1 - \hat{\mu})
\]
and

\[(Y - \gamma_{1,0})' (M_{K_1} \Sigma_0 M_{K_1})' V_2 (M_{K_1} \Sigma_0 M_{K_1})' (Y - \gamma_{1,0}) \rightarrow \]

\[
\rightarrow \left( \begin{array}{c} Y_1 - \hat{\mu} \\ Y_2 - \hat{\nu} \end{array} \right)' \left( \begin{array}{cc} \sigma_{1,0}^{-2} I & 0 \\ 0, & \sigma_{2,0}^{-2} I \end{array} \right) \left( \begin{array}{cc} 0, & 0 \\ 0, & I \end{array} \right) \left( \begin{array}{cc} \sigma_{1,0}^{-2} I & 0 \\ 0, & \sigma_{2,0}^{-2} I \end{array} \right) \left( \begin{array}{c} Y_1 - \hat{\mu} \\ Y_2 - \hat{\nu} \end{array} \right) =
\]

\[
= \frac{1}{\sigma_{2,0}^4} (Y_2 - \hat{\nu})'(Y_2 - \hat{\nu}).
\]

\[\square\]

**Corollary 3.5** The \((\sigma_{1,0}^2, \sigma_{2,0}^2)\)-MINQUE of the function

\[g'(\sigma_1^2, \sigma_2^2), (\sigma_1^2, \sigma_2^2)' \in (0, \infty) \times (0, \infty), \]

exists iff \(g = c(\beta_{2,0}^2, 1)'\), where \(c \in R^1\); in this case the estimator is \((c = 1)\)

\[(\beta_{2,0}^2 \sigma_1^2 + \sigma_2^2)^2 \]

\[
= \frac{(\beta_{2,0}^2 \sigma_1^2 + \sigma_2^2)^2}{(n - 2)(1 + \beta_{2,0}^4)} \left\{ \beta_{2,0}^2 \sigma_1^2 (Y_1 - \hat{\mu})'(Y_1 - \hat{\mu}) + \frac{1}{\sigma_{2,0}^4} (Y_2 - \hat{\nu})'(Y_2 - \hat{\nu}) \right\}.
\]

**Proof** With respect to Lemma 3.1. and Theorem 3.3.

\[g \in \mathcal{M}(S_{(M_{K_1} \Sigma_0 M_{K_1})}+ ) \Rightarrow \overline{\mathcal{g}}^T = g'S_{(M_{K_1} \Sigma_0 M_{K_1})}^-, \hat{\kappa}, \]

where

\[\hat{\kappa}_i = (Y - \hat{\gamma}_1)' \Sigma_0^{-1} V_1 \Sigma_0^{-1} (Y - \hat{\gamma}_1), \quad i = 1, \ldots, p.\]

In our case

\[S_{(M_{K_1} \Sigma_0 M_{K_1})}^- = \frac{(\beta_{2,0}^2 \sigma_1^2 + \sigma_2^2)^2}{n - 2} \left( \begin{array}{cc} \beta_{2,0}^4, & \beta_{2,0}^2 \\ \beta_{2,0}^2, & 1 \end{array} \right)^-\]

\[
= \frac{(\beta_{2,0}^2 \sigma_1^2 + \sigma_2^2)^2}{(n - 2)(1 + \beta_{2,0}^4)^2} \left( \begin{array}{cc} 2\beta_{2,0}^4 + 1, & -\beta_{2,0}^6 \\ -\beta_{2,0}^6, & 1 + \beta_{2,0}^2 + \beta_{2,0}^6 \end{array} \right).
\]

(Here a special version of a \(g\)-inverse of the matrix \(S_{(M_{K_1} \Sigma_0 M_{K_1})}^+\) is given, i.e. the Moore - Penrose \(g\)-inverse.) Since \(\Sigma_0^{-1} V_1 \Sigma_0^{-1} = \frac{1}{\sigma_{1,0}^2} \left( \begin{array}{cc} I, & 0 \\ 0, & 0 \end{array} \right)\) and

\[\Sigma_0^{-1} V_2 \Sigma_0^{-1} = \frac{1}{\sigma_{2,0}^2} \left( \begin{array}{cc} 0, & 0 \\ 0, & I \end{array} \right),\]

we obtain the equality in the statement. \[\square\]

The estimator \((\beta_{2,0}^2 \sigma_1^2 + \sigma_2^2)^2\) is a good check of the value \(\beta_{2,0}^2 \sigma_1^2 + \sigma_2^2\) in the case that in Corollary 2.3 the value \(\beta_{2,0}^2 \sigma_1^2 + \sigma_2^2\) must be used instead of \(\beta_{2,0}^2 \sigma_1^2 + \sigma_2^2\).

Under the design of experiment till now considered, the vector \((\sigma_1^2, \sigma_2^2)'\) cannot be estimated. Thus it is necessary to change the design. The simplest way how to do it is to replicate the measurement.
Let $Y_{(1)}, \ldots, Y_{(m)}$ be $m$ replications of the observation vector $Y$ in the model

$$(Y, \gamma_1, \Sigma(\vartheta)), \quad \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \in \mathcal{V},$$

$$\mathcal{V} = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} : u \in \mathbb{R}^n, v \in \mathbb{R}^k, b + B_1 u + B_2 v = 0 \right\}.$$ 

If $Y = (Y_{(1)}', \ldots, Y_{(m)}')'$ is the observation vector after $m$ replications, we have the model

$$\left( Y, (I_m \otimes I)\gamma_1, I \otimes \sum_{i=1}^p \vartheta_i V_i \right), \quad \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \in \mathcal{V},$$

which is equivalent to

$$\left( Y - (I \otimes I)\gamma_{1,0}, (I \otimes K_1)\tau, I \otimes \sum_{i=1}^p \vartheta_i V_i \right), \quad \tau \in \mathbb{R}^{n+k-q}, \quad (3.2)$$

where $\gamma_{1,0}$ is any vector satisfying the equality $b + B_1 \gamma_{1,0} + B_2 \gamma_{2,0} = 0$ and the matrix $K_1$ is the same as in Lemma 3.1.

**Lemma 3.6** In the model (3.2) the $\vartheta_0$-MINQUE of a function $g^t \vartheta$, $\vartheta \in \mathcal{V}$, is

$$\hat{g}^t \vartheta = \sum_{i=1}^p \lambda_i \left\{ \text{Tr} \left[ \Sigma_0^{-1} V_i \Sigma_0^{-1} \sum_{j=1}^m (Y_{(j)} - \bar{Y})(Y_{(j)} - \bar{Y})' \right] + m(\bar{Y} - \gamma_{1,0})'(M_{K_1} \Sigma_0 M_{K_1})^+ V_i (M_{K_1} \Sigma_0 M_{K_1})^+(\bar{Y} - \gamma_{1,0}) \right\},$$

where $\bar{Y} = \frac{1}{m} \sum_{j=1}^m Y_{(j)}$ and the vector $\lambda = (\lambda_1, \ldots, \lambda_p)'$ is a solution of the equation

$$\left[ (m - 1)S_{\Sigma_0}^{-1} + S_{(M_{K_1} \Sigma_0 M_{K_1})}^+ \right] \lambda = g.$$

If $V_1, \ldots, V_p$ are linearly independent, then $S_{\Sigma_0}$ is p.d. (thus regular).

**Proof** It is a consequence of Theorem 6.1.1 in [4].

**Theorem 3.7** Let the model (3.1) be $m$-times replicated. Then the $(\sigma_{1,0}^2, \sigma_{2,0}^2)$-MINQUE of the vector $(\sigma_1^2, \sigma_2^2)'$ is

$$\left( \hat{\sigma}_1^2, \hat{\sigma}_2^2 \right) = \frac{1}{(m - 1)n} \left[ I - c(m, n, \beta_{2,0}, \sigma_{1,0}^2, \sigma_{2,0}^2) \left( \begin{array}{ll} \sigma_{1,0}^2 & \sigma_{1,0}^2 \\ \sigma_{2,0}^2 & \sigma_{2,0}^2 \end{array} \right) \right] \begin{pmatrix} \hat{\kappa}_1 \\ \hat{\kappa}_2 \end{pmatrix},$$

where

$$c(m, n, \beta_{2,0}, \sigma_{1,0}^2, \sigma_{2,0}^2) = \frac{n - 2}{(\beta_{2,0}^2 \sigma_{1,0}^2 + \sigma_{2,0}^2)((m - 1)n + n - 2) + 2\beta_{2,0}^2 \sigma_{1,0}^2 \sigma_{2,0}^2 (m - 1)n}.$$
and
\[
\begin{align*}
\kappa_1 &= \sum_{j=1}^{m} (Y_{1,(j)} - \bar{Y}_1)'(Y_{1,(j)} - \bar{Y}_1) + m(\bar{Y}_1 - \mu)'(\bar{Y}_1 - \mu), \\
\kappa_2 &= \sum_{j=1}^{m} (Y_{2,(j)} - \bar{Y}_2)'(Y_{2,(j)} - \bar{Y}_2) + m(\bar{Y}_2 - \mu)'(\bar{Y}_2 - \mu).
\end{align*}
\]

**Proof** Since \( V_1 = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \) and \( V_2 = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \), are linearly independent, the matrix \( S_{\Sigma_0^{-1}} \) is regular and since both matrices \( S_{\Sigma_0^{-1}} \) and \( S_{(M_K, \Sigma_0 M_K)} + \) are at least p.s.d., the criterion matrix

\[
(m - 1)S_{\Sigma_0^{-1}} + S_{(M_K, \Sigma_0 M_K)} + \]

is p.d.

The matrix \( S_{(M_K, \Sigma_0 M_K)} + \) is of the form \( hh' \), where

\[
h = \frac{\sqrt{n-2}}{\beta_2 \sigma_{1,0}^2 + \sigma_{2,0}^2} \begin{pmatrix} \beta_2 \\ 1 \end{pmatrix},
\]

thus the relationship

\[
[(m - 1)S_{\Sigma_0^{-1}} + hh']^{-1} = \frac{1}{m - 1} S_{\Sigma_0^{-1}}^{-1} - \frac{1}{m - 1} \frac{S_{\Sigma_0^{-1}, hh'S_{\Sigma_0^{-1}}}}{m - 1 + h'hS_{\Sigma_0^{-1}, h}}
\]
enables us to express the matrix \([ (m - 1)S_{\Sigma_0^{-1}} + S_{(M_K, \Sigma_0 M_K)} + ]^{-1}\) in our case in the form

\[
[(m - 1)S_{\Sigma_0^{-1}} + S_{(M_K, \Sigma_0 M_K)} + ]^{-1} = \frac{1}{(m - 1)n} \begin{pmatrix} \sigma_{1,0}^4 & 0 \\ 0 & \sigma_{2,0}^4 \end{pmatrix} - \frac{1}{(m - 1)n} c(m, n, \beta_2, \sigma_{1,0}^2, \sigma_{2,0}^2) \begin{pmatrix} \sigma_{1,0}^4 \beta_2 \sigma_{1,0}^2, \sigma_{1,0}^4 \beta_2 \sigma_{2,0}^2 \\ \sigma_{2,0}^2 \beta_2 \sigma_{1,0}^2, \sigma_{2,0}^2 \beta_2 \sigma_{2,0}^2 \end{pmatrix} \begin{pmatrix} \sigma_{1,0}^4 & 0 \\ 0 & \sigma_{2,0}^4 \end{pmatrix}.
\]

Further Theorem 3.3 (ii) must be used; with respect to it we obtain

\[
(\bar{Y} - \gamma_{1,0})'(M_K, \Sigma_0 M_K)^+ V_i (M_K, \Sigma_0 M_K)^+ (\bar{Y} - \gamma_{1,0}) \rightarrow \left( \begin{array}{c} Y_1 - \mu \\ Y_2 - \hat{\gamma} \end{array} \right)' \begin{pmatrix} \sigma_{1,0}^2 I & 0 \\ 0 & \sigma_{2,0}^2 I \end{pmatrix} V_i \begin{pmatrix} \sigma_{1,0}^2 I & 0 \\ 0 & \sigma_{2,0}^2 I \end{pmatrix} \left( \begin{array}{c} Y_1 - \hat{\mu} \\ Y_2 - \hat{\gamma} \end{array} \right)
\]

and obviously

\[
\text{Tr}[\Sigma_0^{-1} V_1 \Sigma_0^{-1} \sum_{j=1}^{m} (Y_{j,1} - \bar{Y}_1)'(Y_{j,1} - \bar{Y}_1)] \rightarrow \frac{1}{\sigma_{1,0}^4} \sum_{j=1}^{m} (Y_{1,(j)} - \bar{Y}_1)'(Y_{1,(j)} - \bar{Y}_1)
\]

\[
\text{Tr}[\Sigma_0^{-1} V_2 \Sigma_0^{-1} \sum_{j=1}^{m} (Y_{j,2} - \bar{Y}_2)'(Y_{j,2} - \bar{Y}_2)] \rightarrow \frac{1}{\sigma_{2,0}^4} \sum_{j=1}^{m} (Y_{2,(j)} - \bar{Y}_2)'(Y_{2,(j)} - \bar{Y}_2).
\]

Now it is obvious how to finish the proof. □
4 Conclusion remarks

If it is necessary to characterize the random variable $\xi$ given by the expression $(\eta - \bar{\beta}_1)/\bar{\beta}_2$, where $\eta$ is a registration of the value $\nu$ given by the second measurement device and $\bar{\beta}_1$ and $\bar{\beta}_2$ are given by Corollary 2.3, then it is necessary to determine the bias

$$b = E(\xi) - \frac{\nu - \bar{\beta}_1}{\bar{\beta}_2} = f(\nu, \bar{\beta}_1, \bar{\beta}_2)$$

and the variance $\text{Var}(\xi)$.

Here the formulae given in [1] can be used. Since $\eta$ is stochastically independent of $\left(\frac{\bar{\beta}_1}{\bar{\beta}_2}\right)$ we obtain an approximate value of $b$ in the form [1], Corollary 3.5 (the value $v_{i,j}$ cf. Corollary 2.3)

$$b = \frac{1}{2} \text{Tr} \left[ \frac{\partial^2 f(\nu, \bar{\beta}_1, \bar{\beta}_2)}{\partial(\nu, \bar{\beta}_1, \bar{\beta}_2)} \begin{pmatrix} \sigma_2^2, & 0, & 0 \\ 0, & v_{1,1}, & v_{1,2} \\ 0, & v_{2,1}, & v_{2,2} \end{pmatrix} \right] + \ldots$$

$$= \frac{1}{2} \text{Tr} \left[ \begin{pmatrix} 0, & 0, & -1/\beta_2^2 \\ 0, & 0, & 1/\beta_2^2 \\ -1/\beta_2^2, & 1/\beta_2^2, & 2(\nu - \bar{\beta}_1)/\bar{\beta}_2^2 \end{pmatrix} \begin{pmatrix} \sigma_2^2, & 0, & 0 \\ 0, & v_{1,1}, & v_{1,2} \\ 0, & v_{2,1}, & v_{2,2} \end{pmatrix} \right] + \ldots$$

$$= \frac{v_{1,2}}{\beta_2^2} + \frac{\nu - \bar{\beta}_1}{\beta_2^3} v_{2,2} + \ldots$$

Further (in the case of normally distributed errors, cf. Corollary 3.6 in [1]) an approximate value of the variance is

$$\text{Var} \left( \frac{\eta - \bar{\beta}_1}{\bar{\beta}_2} \right) =$$

$$= \frac{\partial f(\nu, \bar{\beta}_1, \bar{\beta}_2)}{\partial(\nu, \bar{\beta}_1, \bar{\beta}_2)} \begin{pmatrix} \sigma_2^2, & 0, & 0 \\ 0, & v_{1,1}, & v_{1,2} \\ 0, & v_{2,1}, & v_{2,2} \end{pmatrix} \left[ \frac{\partial^2 f(\nu, \bar{\beta}_1, \bar{\beta}_2)}{\partial(\nu, \bar{\beta}_1, \bar{\beta}_2)} \begin{pmatrix} \sigma_2^2, & 0, & 0 \\ 0, & v_{1,1}, & v_{1,2} \\ 0, & v_{2,1}, & v_{2,2} \end{pmatrix} \right]^2 + \ldots$$

$$= \frac{\sigma_2^2}{\beta_2^2} + v_{1,1} \frac{1}{\beta_2^2} + 2v_{1,2} \frac{\nu - \bar{\beta}_1}{\beta_2^3} + v_{2,2} \frac{(\nu - \bar{\beta}_1)^2}{\beta_2^4} + \ldots$$
The given formula reflects reality adequately if the distribution of the errors can be approximated by the normal distribution however the support of the probability measure of the actual errors is included into the domain of convergence of the Taylor series considered. By a simulation it was verified that in the case $\frac{\sqrt{v_{2,2}}}{\beta_2} < 0.1$ the given approximation is sufficient for practical purposes (in more detail cf. [1]).

In the practice the polynomial

$$v_i = \beta_1 + \mu_i \beta_2 + \mu_i^2 \beta_3 \ldots + \mu_i^{k-1} \beta_k$$

$i = 1, \ldots, n$, $k < n$, and $\mu_i \neq \mu_j$ for more than $k$ couples is frequently used.

If there exists a special reason to use a nonlinear (in parameters) calibration curve, then the procedures demonstrated in preceding sections can also be used.

Let, e.g. the calibration curve be of the form

$$v = f(\mu, \beta_1, \beta_2).$$

Then the linearized model can be written in the form

$$\begin{bmatrix}
Y_1 - \mu_0 \\
Y_2
\end{bmatrix} \sim \begin{bmatrix}
\delta \mu \\
\nu
\end{bmatrix} = \begin{bmatrix}
\Sigma_{1,1} & 0 \\
0 & \Sigma_{2,2}
\end{bmatrix},$$

$$\begin{pmatrix}
\vdots \\
f(\mu_{0,1}, \beta_{0,1}, \beta_{0,2}) \\
\vdots \\
f(\mu_{0,n}, \beta_{0,1}, \beta_{0,2})
\end{pmatrix} + (A, -I) \begin{pmatrix}
\delta \mu \\
\nu
\end{pmatrix} +$$

$$\begin{pmatrix}
\partial f(\mu_{0,1}, \beta_1, \beta_2)/\partial (\beta_1, \beta_2) \\
\vdots \\
\partial f(\mu_{0,n}, \beta_1, \beta_2)/\partial (\beta_1, \beta_2)
\end{pmatrix} \bigg|_{\beta_1 = \beta_{0,1}; \beta_2 = \beta_{0,2}} \begin{pmatrix}
\delta \beta_1 \\
\delta \beta_2
\end{pmatrix} = 0,$$

where

$$A = \text{Diag}(\partial f(\mu_{1}, \beta_{0,1}, \beta_2)/\partial \mu_1 |_{\mu_1 = \mu_{0,1}}, \ldots, \partial f(\mu_{n}, \beta_{0,1}, \beta_2)/\partial \mu_n |_{\mu_n = \mu_{0,n}})$$

and now we can proceed analogously as it was already demonstrated.

References