Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica

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Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica, Vol. 36 (1997), No. 1, 163--166

Persistent URL: http://dml.cz/dmlcz/120363

Acta Univ. Palacki. Olomuc., Fac. rer. nat.,
Mathematica 36 (1997) 163–166

A Note on Relative Complements in Lattices

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(Received May 21, 1996)

Abstract

It is well-known that every modular complemented lattice is also relatively complemented. We set a weaker condition than modularity which yields the same construction of relative complements.

Key words: Complement, relative complement, modularity, modular triplets.

1991 Mathematics Subject Classification: 06C15, 06C20

If L is a complemented modular lattice and $[a,b]$ is an interval of L (with $a \leq b$) then for each $x \in [a,b]$ the element $z = a \vee (y \wedge b) = (a \vee y) \wedge b$ is a relative complement of x in $[a,b]$ whenever y is a complement of x in L, see e.g. [2] (or [1] for the original source). In what follows we show that the assumption of modularity can be omitted if y is substituted by an element of a special sort:

Theorem 1 *Let L be a lattice, let $x, a, b \in L$ with $a < b$ and $x \in [a,b]$. If $y \in L$ satisfies*

$$(a \vee y) \wedge x = a \qquad \text{and} \qquad x \vee (y \wedge b) = b$$

then the elements $e = (a \vee y) \wedge b$ and $f = a \vee (y \wedge b)$ are relative complements of x in $[a,b]$. Moreover, $f \leq e$.

Proof We infer directly

$$e \wedge x = ((a \vee y) \wedge b) \wedge x = (a \vee y) \wedge (b \wedge x) = (a \vee y) \wedge x = a$$
$$f \vee x = (a \vee (y \wedge b)) \vee x = (a \vee x) \vee (y \wedge b) = x \vee (y \wedge b) = b\,.$$

Since $a \leq b$, we have $a \leq (a \vee y) \wedge b$. Further, $y \wedge b \leq y \leq a \vee y$ imply $y \wedge b \leq (a \vee y) \wedge b$. Hence

$$f = a \vee (y \wedge b) \leq (a \vee y) \wedge b = e\,.$$

Thus $f \leq e$ and we obtain

$$b = ((a \vee y) \wedge b) \vee b = e \vee b \geq e \vee x \geq f \vee x = b$$
$$a = (a \vee (y \wedge b)) \wedge a = f \wedge a \leq f \wedge x \leq e \wedge x = a\,.$$

Hence $e \vee x = b$ and $f \wedge x = a$ thus e and f are relative complements of x in the interval $[a, b]$. □

Example 1 Consider the lattice L whose diagram is visualized in Fig. 1. Evidently, L is neither modular nor complemented. One can see that the element y satisfies the assumption of Theorem 1. It is worth to say that y is not a complement of x in L. However, it holds $(a \vee y) \wedge x = a$, $x \vee (y \wedge b) = b$, and $e = (a \vee y) \wedge b$ and $f = a \vee (y \wedge b)$ are relative complements of x in $[a, b]$.

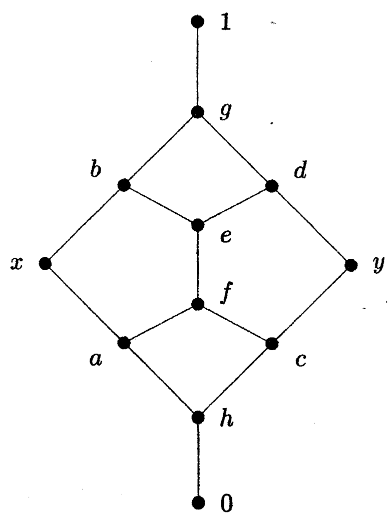

Fig. 1

Example 2 Let L be a lattice whose diagram is depicted in Fig. 2. Evidently, L is not modular. It is an easy exercise to verify that for every element x and for every interval $[a, b]$ there exists an element satisfying the assumption of Theorem 1.

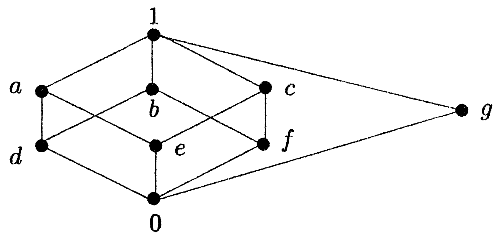

Fig. 2

Hence, L is complemented and relative complements of each x of every $[a, b]$ can be found by the prescribed construction.

We are going to show that if y is a complement of x in L then an easy generalization of modularity yields necessary and sufficient conditions for e and f to be relative complements of x in $[a, b]$ (notation of elements is the same as in Theorem 1).

Definition 1 Let L be a lattice and $a, b, c \in L$ with $a \leq c$. The triplet (a, b, c) is called *modular triplet* whenever $a \vee (b \wedge c) = (a \vee b) \wedge c$.

Of course, if L is modular then every triplet of its elements (a, b, c) with $a \leq c$ is a modular triplet.

Theorem 2 *Let L be a lattice with the least element* 0 *and the greatest element* 1. *Let $x, a, b \in L$ and $a < b$, $x \in [a, b]$. Let y is a complement of x in L. The following conditions are equivalent:*

(1) The elements $e = (a \vee y) \wedge b$ and $f = a \vee (y \wedge b)$ are relative complements of x in $[a, b]$;

(2) The triplet (a, y, x) and (x, y, b) are modular.

Proof (1) $\Rightarrow$ (2) If e and f are relative complements of x in the interval $[a, b]$ then

$$(a \vee y) \wedge x = (a \vee y) \wedge (x \wedge b) = ((a \vee y) \wedge b) \wedge x = e \wedge x = a = a \vee (y \wedge x).$$

Thus $(a \vee y) \wedge x = a \vee (y \wedge x)$. Since $a \leq x$, the triplet (a, y, x) is modular. For the element f we prove analogously

$$x \vee (y \wedge b) = (a \vee x) \wedge (y \wedge b) = x \vee (a \vee (y \wedge b)) = x \vee f = b = b \wedge (y \vee x).$$

Since $x \leq b$, also (x, y, b) is a modular triplet.

(2) $\Rightarrow$ (1) It is an easy computation

$$(a \vee y) \wedge x = a \vee (y \wedge x) = a \vee 0 = a$$
$$x \vee (y \wedge b) = (x \vee y) \wedge b = 1 \wedge b = b\,.$$

Thus y satisfies the assumption of Theorem 1 which proves (1). □

Example 3 Let L be a lattice with the diagram as shown in Fig. 3. Clearly y is a complement of x in L. It is an easy exercise to verify that (a, y, x) and (x, y, b) are modular triplets. Of course, L is not a modular lattice. Elements $e = (a \vee y) \wedge b$ and $f = a \vee (y \wedge b)$ are relative complements of x in the interval $[a, b]$.

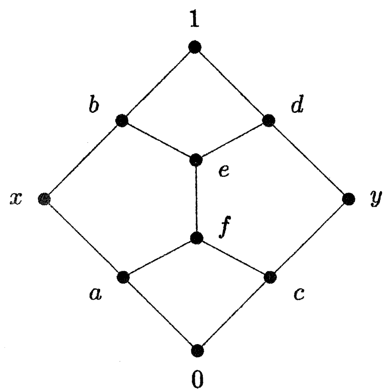

Fig. 3

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[1] Neumann, J.: *Lectures on continuous geometries.* Princeton Math. Ser. **25**, Princeton Univ. Press, 1960.

[2] Szász, G.: *Introduction to lattice theory.* Akad. Kiaidó, Budapest, 1963.