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Probabilistic Model of School-achievement Test with the Correction Account of the Random Choice and its Statistical Analysis

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Abstract

In this paper a school-achievement test with the possible choice of one correct response among q offered alternatives is considered, where the missing answer is evaluated in an other way as the incorrect one. The constructed probabilistic model takes into account a certain difference between the real knowledges of the examined person and the results of the test evaluated by the examiner. The parameters of this model were estimated by the maximum likelihood method.

Key words: School-achievement test with the correction of the random choice, probabilistic model of a school-achievement test and its statistical analysis.

1991 Mathematics Subject Classification: 62P10, 62P15

1 Introduction

In 1978 the papers published by S. Komenda have reported on the probabilistic model of school-achievement tests with a compulsory choice of one correct response among q offered alternatives, where the missing answer is evaluated as the incorrect one.

The aim of this paper is a construction of probabilistic model of the school-achievement test with the possible choice of one correct response among q offered alternatives, where the missing ansver is evaluated in the other way as the incorrect one. The construction of this model take again into account a certain difference between the real knowledges of the examined person and the results of the test evaluated by the examiner and the attention has been paid to the statistical estimation of this difference.On the basis of this estimation the rules for unbiased evaluation as well as the rules for classification strategy can be stated.

This paper follows the series of articles describing the probability structure of school-achievement tests with multiple choice response, published by author of this paper in 1992–95.

2 Model of probabilistic structure of the test

Let us consider the school-achievement test with possible choice of one correct response among $q > 2$ response alternatives with the correction of conjectures so that the missing answer is evaluated in the other way as an incorrect one. The examined persons are informed about this fact before testing. It is also presumed that the test consists of n independent questions of the same difficulty a that the number of offered alternatives is the same in all questions of the test. Answers to one question should be chosen in order to avoid similarity and discrepancy.

The probabilistic model of this kind of the school-achievement test contains two basic parameters ν and δ. Parameter ν expresses the relative part of real unfamiliarity which the tested person realizes, the parameter δ expresses the part of the real unfamiliarity which the tested person does not realize. The whole part of the real unfamiliarity with the examined topic then can be expressed by the value of the parameter $\tau = \nu+\delta$, while the relative part of the real familiarity is expressed by $1-\tau$. When the difficulty of the individual questions is identical, the parameter ν also expresses probable occurrence of the question within the person's realized unfamiliarity with the examined topic and the parameter δ expresses probable occurrence of the question within the person's unrealized unfamiliarity with the examined topic. Parameter $1 - \nu - \delta$ then expresses probable occurrence of the question within the person's real familiarity.

The description of the probabilistic structure of the test comprises the following notation for random events relative to i-th question of the test:

- Z_i i-th question belongs to familiarity with the topic
- N_i i-th question belongs to the unfamiliarity with the topic, which the tested person realizes
- D_i i-th question belongs to the unfamiliarity with the topic, which the tested person does not realize.

Random events Z_i, N_i, D_i form a complete system of events with probabilities

$$P(Z_i) = 1 - \nu - \delta, \qquad P(N_i) = \nu, \qquad P(D_i) = \delta.$$

In relation to the registered results of the test the following three random events are considered:

S_{i0} no answer was given to i-th question
S_{i1} the correct answer was given to i-th question
S_{i2} an uncorrect answer was given to i-th question

In case of familiarity with the topic tested by the i-th question the examined person will certainly give the correct response. The following relations are in accordance with this presumption.

$$P(S_{i0}|Z_i) = 0, \qquad P(S_{i1}|Z_i) = 1, \qquad P(S_{i2}|Z_i) = 0.$$

In case of the realized unfamiliarity with this topic the tested person will not give any response to i-th question. So the following relations hold

$$P(S_{i0}|N_i) = 1, \qquad P(S_{i1}|N_i) = 0, \qquad P(S_{i2}|N_i) = 0.$$

In case of the unrealized unfamiliarity the tested person will use the possibility of random choice. With regard to the presence of one correct response among the offered alternatives the following relations

$$P(S_{i0}|D_i) = 0, \qquad P(S_{i1}|D_i) = \frac{1}{q}, \qquad P(S_{i2}|D_i) = \frac{q-1}{q}$$

hold for selection possibility in each question.

Unconditional probabilities of events S_{i0}, S_{i1}, S_{i2} can be calculated according to the theorem of total probability:

$$\begin{aligned}
P(S_{i0}) &= P(S_{i0} \cap Z_i) + P(S_{i0} \cap N_i) + P(S_{i0} \cap D_i) = \\
&= P(Z_i)P(S_{i0}|Z_i) + P(N_i)P(S_{i0}|N_i) + P(D_i)P(S_{i0}|D_i) = \\
&= 0\,.\,(1-\nu-\delta) + 1\,.\,\nu + 0\,.\,\delta = \nu \\
P(S_{i1}) &= P(S_{i1} \cap Z_i) + P(S_{i1} \cap N_i) + P(S_{i1} \cap D_i) = \\
&= P(Z_i)P(S_{i1}|Z_i) + P(N_i)P(S_{i1}|N_i) + P(D_i)P(S_{i1}|D_i) = \\
&= 1\,.\,(1-\nu-\delta) + 0\,.\,\nu + \tfrac{1}{q}\cdot\delta = 1-\nu-\delta\tfrac{q-1}{q} \\
P(S_{i2}) &= P(S_{i2} \cap Z_i) + P(S_{i2} \cap N_i) + P(S_{i2} \cap D_i) = \\
&= P(Z_i)P(S_{i2}|Z_i) + P(N_i)P(S_{i2}|N_i) + P(D_i)P(S_{i2}|D_i) = \\
&= 0\,.\,(1-\nu-\delta) + 0\,.\,\nu + \tfrac{q-1}{q}\cdot\delta = \delta\tfrac{q-1}{q}\,.
\end{aligned} \tag{1}$$

The assumption of the same difficulty of questions and the same schema of offered alternatives allows us to omit the index of the question and to introduce the following notation $p_{i0} = p_0$, $p_{i1} = p_1$, $p_{i2} = p_2$, $S_{i0} = S_0$, $S_{i1} = S_1$, $S_{i2} = S_2$, similarly as $Z_i = Z$, $N_i = N$ for all considered i.

Now we can associate a random vector $M = (M_0, M_1, M_2)$ to the whole test of n independent questions, where the random variable M_0 represents the number of questions, to which no response was given by the tested person, the variable M_1 represents the number of questions to which the correct response was given and the variable M_2 represents the number of questions to which an uncorrect response was given. The random vector M has the multinomial probability distribution defined by the relation

$$P(M_0 = m_0, M_1 = m_1, M_2 = m_2) = \tag{2}$$
$$= \frac{n!}{m_0!m_1!m_2!}\nu^{m_0}\left(1-\nu-\delta\frac{q-1}{q}\right)^{m_1}\left(\delta\frac{q-1}{q}\right)^{m_2}$$

The marginal probability distributions of the individual components of the random vector M are binomial with the expectations

$$E(M_0) = n\nu, \qquad E(M_1) = n\left(1-\nu-\delta\frac{q-1}{q}\right), \qquad E(M_2) = n\left(\delta\frac{q-1}{q}\right),$$

and with the variances

$$D(M_0) = n\nu(1-\nu), \qquad D(M_1) = n\left(1-\nu-\delta\frac{q-1}{q}\right)\left(\nu+\delta\frac{q-1}{q}\right),$$
$$D(M_2) = n\left(\delta\frac{q-1}{q}\right)\left(1-\delta\frac{q-1}{q}\right).$$

The statistical analysis following this probabilistic model makes possible to estimate the parameters ν and δ, and in this way also the parameter τ, resp. $1-\tau$, and to evaluate their properties.

3 Point estimation of parameters of the model

The point estimators of the parameters ν and δ representing a part of the tested person's realized or unrealized unfamiliarity with the examined topic, will be performed at first by the moment method using the theoretical and empirical moments of the variable M_0 or M_2 on the basis of a single application of the test, respectively. Comparing the theoretical and empirical mean values the following equations

$$E(M_0) = n\nu = m_0\,, \qquad E(M_2) = n\delta\frac{q-1}{q} = m_2\,,$$

are derived. These equations provide estimators of the parameters ν and δ in the form

$$\hat{\nu} = \frac{m_0}{n}\,, \qquad \hat{\delta} = \frac{m_2}{n}\frac{q}{q-1}\,. \tag{3}$$

The same solution is reached by the maximum likelihood method on the basis of a single application of the test, i.e. on the basis of values (m_0, m_1, m_2) of the random vector M.

The probability distribution (2) of the random vector M will be selected for the likelihood function

$$L(\tau, m) = \frac{n!}{m_0! m_1! m_2!} \nu^{m_0} \left(1 - \nu - \delta \frac{q-1}{q}\right)^{m_1} \left(\delta \frac{q-1}{q}\right)^{m_2}.$$

The maximum likelihood estimators of the parameters ν and δ, i.e. the values of the parameters ν and δ at which the likelihood function reaches its maximum, is solved by the likelihood equations

$$\frac{\partial \ln L(\nu, m)}{\partial \nu} = 0, \qquad \frac{\partial \ln L(\delta, m)}{\partial \delta} = 0.$$

Considering the logarithm of the likelihood function

$$\ln L(m, \nu, \delta) =$$
$$= \ln \frac{n!}{m_0! m_1! m_2!} + m_0 \ln \nu + m_1 \ln \left(1 - \nu - \delta \frac{q-1}{q}\right) + m_2 \ln \left(\delta \frac{q-1}{q}\right)$$

and its partial derivatives we get the likelihood equations in the form

$$\frac{m_0}{\nu} - \frac{m_1}{1 - \nu - \delta \frac{q-1}{q}} = 0, \qquad - \frac{m_1 \frac{q-1}{q}}{1 - \nu - \delta \frac{q-1}{q}} + \frac{m_2}{\delta} = 0,$$

modified as follows

$$m_0 \left(1 - \nu - \delta \frac{q-1}{q}\right) = m_1 \nu, \qquad m_2 \left(1 - \nu - \delta \frac{q-1}{q}\right) = m_1 \delta \frac{q-1}{q}.$$

From these equations we obtain the expression of the maximum likelihood estimators $\hat{\nu}$ and $\hat{\delta}$ of parameters ν and δ

$$\hat{\nu} = \frac{m_0}{n}, \qquad \hat{\delta} = \frac{m_2}{n} \frac{q}{q-1}. \tag{4}$$

These results coincide the results obtained by the moment method.

However, the expressions (2) or (4) are usefull only under the condition that the values of the estimators do not exceed the rank of realy possible values of parameters ν and δ, i.e. under the conditions that

$$0 \le \hat{\nu} \le 1, \qquad 0 \le \hat{\delta} \le 1, \qquad 0 \le \hat{\nu} + \hat{\delta} \le 1.$$

These conditions equal the situation that

$$m_0 + m_2 \frac{q}{q-1} \le n.$$

From the expressions (4) of the estimators of the parameters ν and δ also the maximum likelihood estimator $\hat{\tau}$ of the parameter τ can be found in the form

$$\hat{\tau} = \hat{\nu} + \hat{\delta} = \begin{cases} \frac{m_0}{n} + \frac{m_2}{n} \frac{q}{q-1} & \text{for } m_0 + m_2 \frac{q}{q-1} \le n \\ 1 & \text{for } m_0 + m_2 \frac{q}{q-1} > n \end{cases}.$$

The values of the estimator $1-\hat{\tau}$ of the real familiarity with the tested topic

$$1-\hat{\tau}=1-\hat{\nu}-\hat{\delta}=\begin{cases}1-\frac{m_0}{n}-\frac{m_2}{n}\frac{q}{q-1} & \text{for } m_0+m_2\frac{q}{q-1}\le n\\ 0 & \text{for } m_0+m_2\frac{q}{q-1}>n\end{cases}.$$

equal to the relative frequencies

$$\frac{m_1-\frac{m_2}{q-1}}{n}$$

of the achievement m_1 in the test after the correction $-\frac{m_2}{q-1}$ account of the random choice.

The values of the estimator $1-\hat{\tau}$ in comparation to the relative frequencies $\frac{m_1}{n}$ of the achievement without correction are summarized for case of $n=10$, $q=4$ or $q=6$ and for the different results m_0, m_1, m_2 of the test in the following table.

4 Point estimator of parameter $1-\tau$

				estimator $1-\hat{\tau}$	
m_0	m_1	m_2	$\frac{m_1}{n}$	$q=6$	$q=4$
0	0	10	0	0	0
	1	9	0.1	0	0
	2	8	0.2	0.04	0
	3	7	0.3	0.16	0.07
	4	6	0.4	0.28	0.20
	5	5	0.5	0.40	0.33
	6	4	0.6	0.52	0.47
	7	3	0.7	0.64	0.60
	8	2	0.8	0.76	0.73
	9	1	0.9	0.88	0.86
	10	0	1	1	1
1	0	9	0	0	0
	1	8	0.1	0	0
	2	7	0.2	0.06	0
	3	6	0.3	0.18	0.10
	4	5	0.4	0.30	0.24
	5	4	0.5	0.42	0.37
	6	3	0.6	0.54	0.50
	7	2	0.7	0.66	0.64
	8	1	0.8	0.78	0.76
	9	0	0.9	0.90	0.90

2	0	8	0	0	0
	1	7	0.1	0	0
	2	6	0.2	0.08	0
	3	5	0.3	0.20	0.14
	4	4	0.4	0.32	0.27
	5	3	0.5	0.44	0.40
	6	2	0.6	0.56	0.54
	7	1	0.7	0.68	0.66
	8	0	0.8	0.80	0.80
3	0	7	0	0	0
	1	6	0.1	0	0
	2	5	0.2	0.1	0.04
	3	4	0.3	0.22	0.17
	4	3	0.4	0.34	0.30
	5	2	0.5	0.46	0.44
	6	1	0.6	0.58	0.57
	7	0	0.7	0.70	0.70
4	0	6	0	0	0
	1	5	0.1	0	0
	2	4	0.2	0.12	0.07
	3	3	0.3	0.24	0.20
	4	2	0.4	0.36	0.33
	5	1	0.5	0.48	0.47
	6	0	0.6	0.60	0.60
5	0	5	0	0	0
	1	4	0.1	0.02	0
	2	3	0.2	0.14	0.10
	3	2	0.3	0.26	0.23
	4	1	0.4	0.38	0.37
	5	0	0.5	0.50	0.50
6	0	4	0	0	0
	1	3	0.1	0.04	0
	2	2	0.2	0.16	0.13
	3	1	0.3	0.28	0.27
	4	0	0.4	0.40	0.40

7	0	3	0	0	0
	1	2	0.1	0.06	0.03
	2	1	0.2	0.18	0.17
	3	0	0.3	0.30	0.30
8	0	2	0	0	0
	1	1	0.1	0.08	0.07
	2	0	0.2	0.20	0.20
9	0	1	0	0	0
	1	0	0.1	0.10	0.10
10	0	0	0	0	0

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