Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica

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Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica, Vol. 36 (1997), No. 1, 187--196

Persistent URL: `http://dml.cz/dmlcz/120366`

Acta Univ. Palacki. Olomuc., Fac. rer. nat.,
Mathematica 36 (1997) 187–196

Polynomial Stuctures with Double Roots *

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(Received June 23, 1994)

Abstract

Our aim is to investigate integrability of a polynomial structures the characteristic polynomial of which has at most double real roots. The general case can be regarded as a "refinement" of the special case
$h(h-I)^2(h^2+I)=0$.

Key words: Projector, manifold, polynomial structure, integrability.

1991 Mathematics Subject Classification: 53C05

We will formulate integrability conditions for a polynomial structure the characteristic polynomial of which has at most double roots. The well-known examples of such structures are almost tangent structures, or f-structures (almost contact structures) which satisfy $f^3+f=0$. The case of single roots was completely solved in [9], [11].

Suppose that all objects under consideration (manifolds, tensor fields etc.) are of the class C^∞. The Nijehuis bracket (tensor) is denoted by $[,]$.

*Supported by grant No. 201/96/0027 of The Grant Agency of Czech Republic

1 Almost tangent structures

Recall some well-known facts. *An almost tangent structure* is given by a $(1,1)$-tensor field h of constant rank which is nilpotent, $h^2 = 0$. The integrability conditions were found by J. Lehmann-Lejeune, [7]. Note that the case $h^n = 0$ with $n \geq 3$ was not solved in general, it is more complicated from the technical point of view

At any point x of an almost tangent manifold the inclusion im $h_x \subseteq$ ker h_x is satisfied. If the "dimension regularity" conditions dim im $h = p$, dim ker $h = q$ with $p, q \geq 0$ real constants are satisfied then the image im h (respectively the kernel ker h) is a p-dimensional (respectively $(p+q)$-dimensional) distribution, and $\dim M = m = 2p + q$. A frame $(x; X_1, \dots, X_m)$ is called *h-adapted* if $X_{i+p+q} = h(X_i)$, $i = 1, \dots, p$, X_{i+p+q}, $i = 1, \dots, p$ is a basis of im h_x and $X_{1+p}, \dots, X_m$ is a basis of ker h_x. The matrix representation of $h_x \in \mathrm{End}(T_x M)$ with respect to the h-adapted frame is of the form

$$\begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \boldsymbol{I} & \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{matrix} \}p \\ \}q\,. \\ \}p \end{matrix} \tag{1}$$

The family of all h-adapted frames form a G-structure for which G is a Lie subgroup of $GL(m, R)$ formed by all square (m, m)-matrices of the form

$$\begin{pmatrix} \boldsymbol{A}_{11} & \mathbf{0} & \mathbf{0} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} & \mathbf{0} \\ \boldsymbol{A}_{31} & \boldsymbol{A}_{32} & \boldsymbol{A}_{11} \end{pmatrix} \begin{matrix} \}p \\ \}q\,. \\ \}p \end{matrix}$$

The almost tangent structure h is called *integrable* if the corresponding G-structure is integrable, i.e. if there are local "h-adapted" coordinates on a nbd of each point with respect to which the matrix of h_x is (1). Another speaking the holonomic frame $(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m})$ is h-adapted. For a nilpotent polynomial structure, $h^2 = 0$, the following conditions are equivalent, [6]:

ker h is integrable, and $[h, h] = 0$;

h is integrable;

there exists a symmetric connection ∇ on M such that $\nabla h = 0$.

2 Complex almost product structures

If $(D_1, \dots, D_t)$ is an almost product structure and J a complex structure[1] satisfying $JD_i = D_i$, $i = 1, \dots, t$, $(J; D_1, \dots, D_t)$ is *a complex almost product structure.* The structure $(J; D_1, \dots, D_t)$ is *integrable* if J can be written

[1] A complex structure is an almost complex structure, $J^2 + I = 0$, satisfying the integrability condition $[J, J]{=}0$.

locally in the form

$$J = \begin{pmatrix} \mathbf{0}_{n_1} & \boldsymbol{I}_{n_1} & & & \\ -\boldsymbol{I}_{n_1} & \mathbf{0}_{n_1} & & \mathbf{0} & \\ & & \ddots & & \\ & \mathbf{0} & & \mathbf{0}_{n_t} & \boldsymbol{I}_{n_t} \\ & & & -\boldsymbol{I}_{n_t} & \mathbf{0}_{n_t} \end{pmatrix}$$

where $\dim D_i = 2n_i$. Through the corresponding projectors P_i, the integrability condition can be refomulated as $[P_i, P_j] = [P_i, J] = 0$, [9].

3 Almost tangent almost product structures

In [10], the problem of simultaneous integrability of an almost tangent structure and a distribution was solved.

We will need here a generalization: a simultaneous integrability of an almost tangent and an almost product structure. Suppose that $(D_1, \dots, D_t)$ is an almost product structure on M with projectors $P_1, \dots, P_t$, and at the same time, let M be endowed with an almost tangent structure g such that $(g - I)^2 = 0$. Let us assume that $g \circ P_i = P_i \circ g$, $i = 1, \dots, t$. Then

$$(g - I)D_i \subseteq D_i,$$

and $(g; D_1, \dots, D_t)$ will be called *an almost tangent almost product structure.* Let us use the notation

$$g_i = g \mid D_i, \quad \dim\ \ker D_i = p_i + q_i, \quad \dim\ \operatorname{im} g_i = p_i,$$
$$\dim D_i = n_i, \quad n_i = 2p_i + q_i.$$

Now it is natural to define:

Definition 1 We say that $(g; D_1, \dots, D_t)$ is *integrable* if there are local coordinates such that g is represented by

$$g = \begin{pmatrix} \boldsymbol{I}_{p_1} & \mathbf{0} & \mathbf{0}\ \}p & & & & \\ \mathbf{0} & \boldsymbol{I}_{q_1} & \mathbf{0}\ \}q & & & \mathbf{0} & \\ \boldsymbol{I}_{p_1} & \mathbf{0} & \boldsymbol{I}_{p_1}\ \}p & & & & \\ & & & \ddots & & & \\ & & & & \boldsymbol{I}_{p_t} & \mathbf{0} & \mathbf{0}\ \}p \\ & \mathbf{0} & & & \mathbf{0} & \boldsymbol{I}_{q_t} & \mathbf{0}\ \}q \\ & & & & \boldsymbol{I}_{p_t} & \mathbf{0} & \boldsymbol{I}_{p_t}\ \}p \end{pmatrix} \tag{2}$$

where $\boldsymbol{I}_s$ denotes a unit matrix of the type (s, s).

By standard methods, we can prove the following [11].

Proposition 1 *An almost tangent almost product structure* $(g; D_1, \ldots, D_t)$ *is integrable if and only if the following conditions are satisfied:*

(i) $[P_i, P_j] = 0$ *for* $i, j = 1, \ldots, t,$

(ii) $[g, g] = 0,$

(iii) $\ker\,(g - I)$ *is integrable,*

(iv) $[P_i, g] = 0$ *for* $i = 1, \ldots, t.$

4 The case $h(h - I)^2(h^2 + I) = 0$

Now let us consider a polynomial structure h satisfying

$$h(h - I)^2(h^2 + I) = 0. \tag{3}$$

Suppose that $D_1 = \ker h$, $D_2 = \ker(h - I)^2$, $D_3 = \ker(h^2 + I)$ are of constant ranks on M, $\dim D_1 = p$, $\dim D_2 = q$, $\dim D_3 = 2s$, where $q = 2k+l$, $p+q+2s = m$. Then the tangent space is a Whitney sum $TM = D_1 \oplus D_2 \oplus D_3$. The correspending projectors are $P_1 = (h-I)^2(h^2+I)$, $P_2 = I-(h-I)^2(h^2-\frac{1}{2}h+I)$, $P_3 = \frac{1}{2}h(h - I)^2$. It is natural to define

Definition 2 A polynomial structure h satisfying (3) on M is *integrable* if there are local coordinates with respect to which the matrix representation of h is

$$h = \begin{pmatrix} \mathbf{0}_p & & & & & \mathbf{0} \\ & \boldsymbol{I}_k & \mathbf{0} & \mathbf{0} & & \\ & \mathbf{0} & \boldsymbol{I}_l & \mathbf{0} & & \\ & \boldsymbol{I}_k & \mathbf{0} & \boldsymbol{I}_k & & \\ & & & & \mathbf{0} & \boldsymbol{I}_s \\ \mathbf{0} & & & & -\boldsymbol{I}_s & \mathbf{0} \end{pmatrix}. \tag{4}$$

The following technical lemma is useful in the next proof.

Lemma 1 *Let* f *be a* $(1,1)$*-tensor field satisfying* $[f, f] = 0$*. Then for any natural* $a, b \geq 0$

$$[f^a, f^b] = 0.$$

Theorem 1 *A polynomial structure (3) is integrable if and only if the following conditions are satisfied:*

(i) $[h, h] = 0,$

(ii) $\ker(h - I)$ *is integrable.*

Proof The conditions are necessary as it can be verified. So let them be satisfied. To prove that (D_1, D_2, D_3) is integrable we will verify $[P_i, P_j] = 0$, $i, j = 1, 2, 3$. Since the projectors are polynomials in h, the brackets $[P_i, P_j]$ can be expressed as linear combinations of terms of the form $[h^a, h^b]$ with natural exponents a, b. So all couples of projectors vanish. Now we can find local coordinates in a nbd of any point

$$(x_1, \dots, x_p, v_1, \dots, v_q, y_1, \dots, y_{2s})$$

such that

$$h = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{F} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{G} \end{pmatrix},$$

where $\boldsymbol{F}$ and $\boldsymbol{G}$ are matrices of the type (p,p) and $(2s, 2s)$ respectively, depending on $(x_1, \dots, x_{2s})$. Let us denote their entries by (F^t_j) or (G^r_k), respectively. We will prove that $\boldsymbol{F}$ depends in fact only on $v_1, \dots, v_q$ and, $\boldsymbol{G}$ depends on $y_1, \dots, y_{2s}$. Let $1 \le i \le p$, $1 \le j \le q$. By (i)

$$0 = \frac{1}{2}[h,h]\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial v_j}\right) = -h\left[\frac{\partial}{\partial x_i}, h\frac{\partial}{\partial v_j}\right] = -h\left[\frac{\partial}{\partial x_i}, F^t_j \frac{\partial}{\partial v_t}\right] = -h\left(\frac{\partial F^t_j}{\partial x_i} \cdot \frac{\partial}{\partial v_t}\right).$$

On D_2, $(h - I)^2 = 0$ is satisfied. We obtain that $h \,|\, D_2$ is an automorphism since $h(2I - h) = I$ on D_2. It follows $\frac{\partial F^t_j}{\partial x_i} = 0$. The equality $\frac{\partial G^k_j}{\partial x_i} = 0$ can be proved for $1 \le j \le q$, $1 \le i \le p$ in a similar way: h is an automorphism on D_3 since $h(-h) = I$ is satisfied on D_3, and $\frac{1}{2}[h,h]\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_j}\right) = -h\left(\frac{\partial G^k_j}{\partial x_i} \cdot \frac{\partial}{\partial y_k}\right)$. Now let $1 \le i \le q$, $1 \le j \le 2s$. It can be easily verified that $[h^2 + I, h] = 0$ follows as a consequence of our assumption $[h,h] = 0$. We evaluate

$$[h,h]\left(\frac{\partial}{\partial y_j}, \frac{\partial}{\partial v_i}\right) = 2\left[h\frac{\partial}{\partial y_j}, h\frac{\partial}{\partial v_i}\right] - 2h\left[h\frac{\partial}{\partial y_j}, \frac{\partial}{\partial v_i}\right] - 2h\left[\frac{\partial}{\partial y_j}, h\frac{\partial}{\partial v_i}\right] = 0, \tag{5}$$

$$\begin{aligned}[h^2 + I, h]\left(\frac{\partial}{\partial y_j}, \frac{\partial}{\partial v_i}\right) = {} & \left[h\frac{\partial}{\partial y_j}, (h^2 + I)\frac{\partial}{\partial v_i}\right] - (h^2 + I)\left[\frac{\partial}{\partial y_j}, h\frac{\partial}{\partial v_i}\right] \\ & - h\left[\frac{\partial}{\partial y_j}, (h^2 + I)\frac{\partial}{\partial v_i}\right].\end{aligned} \tag{6}$$

On $D_2 = \ker\,(h - I)^2$, the equality $h^2 + I = 2h$ is satisfied. So (6) can be written as $2[h\frac{\partial}{\partial y_j}, h\frac{\partial}{\partial v_i}] - (h^2 + I)[\frac{\partial}{\partial y_j}, h\frac{\partial}{\partial v_i}] - 2h[\frac{\partial}{\partial y_j}, h\frac{\partial}{\partial v_i}] = 0$. Combining (5) and (6) gives $-2h\left[h\frac{\partial}{\partial y_j}, \frac{\partial}{\partial v_i}\right] + (h^2 + I)\left[\frac{\partial}{\partial y_j}, h\frac{\partial}{\partial v_i}\right] = 0$,

$$2h\left(\frac{\partial}{G}{}^{k}_{j}\partial v_i \cdot \frac{\partial}{\partial}y_k\right) + (h^2 + I)\left(\frac{\partial}{F}{}^{r}_{i}\partial y_j \cdot \frac{\partial}{\partial}v_r\right) = 0. \tag{7}$$

We apply the automorphism $h^2 + I$ on both sides of the equality (7) to obtain $(h^2+I)^2(\frac{\partial F_i^r}{\partial y_j}\cdot\frac{\partial}{\partial v_r}) = 0$ which gives $\frac{\partial F_i^r}{\partial y_j} = 0$ since $(h^2+I)^2$ is again an automorphism. Similarly, an application of $(h-I)^2$ on (7) gives $h(h-I)^2(\frac{\partial G_j^k}{\partial v_i}\cdot\frac{\partial}{\partial y_k}) = 0$. But $h(h-I)^2 \mid D_3$ is an automorphism since on $D_3 = \ker(h^2 + I)$, $h(h-I)^2 = h^3 - 2h^2 + h = -2h^2 = 2I$. It follows $\frac{\partial G_j^k}{\partial v_i} = 0$. By our assumptions and the above results, F is an integrable almost tangent structure on integral submanifolds of the distribution D_2, and G is a complex structure on integral submanifolds of D_3. So there exists a coordinate transformation $x_j = \varphi_j(v_1,\ldots,v_q)$, $x_l = \varphi_l(y_1,\ldots,y_{2s})$, where $p+1 \le j \le p+q$, $p+q+1 \le l \le p+q+2s = m$ such that with respect to the corresponding holonomic frame, the matrix of h admits the desired form. □

5 The general case

More generally, let us consider a polynomial structure (M, f) satisfying the polynomial equation with at most double real roots of the characteristic polynomial R

$$R(f) = \prod_{i=1}^{r}(f - b_iI)\prod_{j=1}^{R}(f - B_j)^2\prod_{k=1}^{s}(f^2 + 2c_kf + d_kI) = 0, \qquad b_i, B_j, c_k, d_k \in R, \qquad c_i^2 - d_j < 0 \tag{8}$$

with pairwise distinct factors. The decomposition of the tangent bundle is $TM = \bigoplus_{i=1}^{r} D_i' \oplus \bigoplus_{j=1}^{R}\tilde{D}_j \oplus \bigoplus_{k=1}^{s} D_k''$ where $D_i' = \ker(f - b_iI)$, $i = 1,\ldots,r$, $\tilde{D}_j = \ker(f - B_j)^2$, $j = 1,\ldots,R$, $D_k'' = \ker(f^2 + 2c_kf + d_kI)$, $k = 1,\ldots,s$ are distributions on M invariant under f, of constant dimensions, [9], $n_i' = \dim D_i'$, $m_j = \dim\tilde{D}_j$, $2n_k'' = \dim D_k''$, $\sum n_i' = \tilde{n}$, $\sum m_j = \tilde{m}$, $\sum n_j'' = \tilde{\tilde{n}}$, $\dim M = m = \tilde{m} + \tilde{n} + 2\tilde{\tilde{n}}$. We obtain an almost product structure

$$(D_1', \ldots, D_r', \tilde{D}_1, \ldots, \tilde{D}_R, D_1'', \ldots, D_s'') \tag{9}$$

associated with f. Denote by P_i', $\tilde{P}_j$, P_k'' the corresponding projectors.

Let us define integrability of the structure (8). We can introduce an almost tangent structure on each $\tilde{D}_j$, $j = 1,\ldots,R$, and an almost complex structure on each D_k'', $k = 1,\ldots,s$ as follows. Denote $\tilde{f}_j = f \mid \tilde{D}_j$, $I_j = I \mid \tilde{D}_j$. The equality $(\tilde{f}_j - B_jI_j)^2 = 0$ can be written as $((\tilde{f}_j - B_jI_j + I_j) - I_j)^2 = 0$. So the formula $S_j = \tilde{f}_j - (B_j - 1)I_j$ defines an almost tangent structure S_j on $\tilde{D}_j$, and $\tilde{f}_j$ can be evaluated by $\tilde{f}_j = S_j + (B_j - 1)I$. Similarly, $f_k'' = f \mid D_k''$ satisfies $f_k''^2 + 2c_kf_k'' + d_kI_k = 0$, and an almost complex structure J_k'' is introduced on D_k'' by $J_k'' = \dfrac{1}{\sqrt{d_k - c_k^2}}(f_k'' + c_kI_k)$. Obviously, $f_k'' = \sqrt{d_k - c_k^2}\,J_k'' - c_kI_k$.

Definition 3 A polynomial structure (8) is *integrable* if on some nbd of each point $x \in M$, there are local coordinates $(x_1, \dots, x_m)$ with respect to which f has a representation

$$f = \begin{pmatrix} b_1 \boldsymbol{I}_{n_1'} & & & & & & & & \\ & \ddots & & & & & & \mathbf{0} & \\ & & b_r \boldsymbol{I}_{n_r'} & & & & & & \\ & & & \boldsymbol{L}_1 & & \mathbf{0} & & & \\ & & & & \ddots & & & & \\ & & & \mathbf{0} & & \boldsymbol{L}_R & & & \\ & & & & & & \ddots & & \\ & & & & & & & \boldsymbol{K}_1 & \mathbf{0} \\ & \mathbf{0} & & & & & & & \ddots \\ & & & & & & & \mathbf{0} & \boldsymbol{K}_s \end{pmatrix}$$

where $\boldsymbol{L}_j$ are block matrices of the form

$$\boldsymbol{L}_j = \begin{pmatrix} B_j \boldsymbol{I}_{p_j} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & B_j \boldsymbol{I}_{q_j} & \mathbf{0} \\ \boldsymbol{I}_{p_j} & \mathbf{0} & B_j \boldsymbol{I}_{p_j} \end{pmatrix},$$

$p_j = \dim \operatorname{im}\,(f - B_j)$, $p_j + q_j = \dim\ \ker\,(f - B_j)$, $2p_j + q_j = m_j$, and $\boldsymbol{K}_k$ are of the form

$$\boldsymbol{K}_k = \begin{pmatrix} -c_k \boldsymbol{I}''_{n_k} & \sqrt{d_k - {c_k}^2}\boldsymbol{I}''_{n_k} \\ -\sqrt{d_k - {c_k}^2}\boldsymbol{I}''_{n_k} & -c_k \boldsymbol{I}''_{n_k} \end{pmatrix}.$$

To give necessary and sufficient integrability conditions let us associate with f a $(1,1)$-tensors field Φ introduced by

$$\Phi = \sum_{j=1}^{R} (f - B_j I + I) \circ \tilde{P}_j + \sum_{k=1}^{s} \frac{1}{\sqrt{d_k - {c_k}^2}} (f + c_k I) \circ P''_k$$

which satisfies on M the equation

$$\Phi(\Phi - I)^2(\Phi^2 + I) = 0.$$

The original tensor field f can be evaluated by the formula

$$f = \sum_{i=1}^{r} b_i P_i + \sum_{j=1}^{R} (\Phi + B_j I - I)\tilde{P}_j + \sum_{k=1}^{S} (\sqrt{d_k - {c_k}^2}\, \Phi - c_k I) P''_k. \tag{10}$$

Let us denote $\tilde{D} = \bigoplus_{j=1}^{R} \tilde{D}_j$, $D'' = \bigoplus_{k+1}^{s} D''_k$.

Theorem 2 *A polynomial structure (M, f) satisfying (8) is integrable if and only if the following conditions hold:*

(i) The Nijenhuis brackets of all couples of projectors vanish.

(ii) ker $(\Phi - I)$ *is integrable, and* $[\Phi, \Phi] = 0$.

(iii) $\{\tilde{P}_j, \Phi\} = 0, \quad j = 1, \dots, R, \quad \{P''_k, \Phi\} = 0, \quad k = 1, \dots, s.$

Remark 1 The condition (i) is equivalent with integrability of the associated almost product structure (9); (iii) means integrability of $\tilde{D}_j \oplus D''$ and $\tilde{D} \oplus D''_k$.

Proof It can be verified that the above conditions are necessary. Let us prove that they are sufficient. By (i), there are local coordinates

$$(x_1, \dots, x_{\tilde{n}}, v_1, \dots, v_{\tilde{m}}, y_1, \dots, y_{2\tilde{\tilde{n}}})$$

in a nbd of any point such that $(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{n'}})$ form a basis of D'_1 etc., the middle part of the coordinate frame $(\frac{\partial}{\partial v})$ form a basis of $\tilde{D}$, and the last part $(\frac{\partial}{\partial y})$ is a basis of D''. The endomorphisms P'_i, $\tilde{P}_j$, P''_k, Φ have representations

$$P'_i = \begin{pmatrix} \mathbf{0}_{n'_1} & \mathbf{0} & & & \\ \mathbf{0} & \boldsymbol{I}_{n'_i} & & \mathbf{0} & \\ & & \ddots & & \\ & \mathbf{0} & & \boldsymbol{I}_{n'_r} & \mathbf{0} \\ & & & \mathbf{0} & \mathbf{0}_{2\tilde{\tilde{n}}} \end{pmatrix},$$

$$\tilde{P}_j = \begin{pmatrix} \mathbf{0}_{\tilde{n}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{Q}_j & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0}_{2\tilde{\tilde{n}}} \end{pmatrix} \quad P''_k = \begin{pmatrix} \mathbf{0}_{\tilde{n}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{Q}_{\tilde{m}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{P}_k \end{pmatrix} \quad \Phi = \begin{pmatrix} \mathbf{0}_{\tilde{n}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{F} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{G} \end{pmatrix}$$

where $\boldsymbol{Q}_j$, $\boldsymbol{F}$ are $(\tilde{m}, \tilde{m})$-matrices, and $\boldsymbol{P}_k$, $\boldsymbol{G}$ are $(2\tilde{\tilde{n}}, 2\tilde{\tilde{n}})$ - matrices the entries of which are functions depending on $x_1, \dots, y_{2\tilde{\tilde{n}}}$. In the proof of Theorem 1, we found that $\boldsymbol{F}$ depends only on the coordinates $v_1, \dots, v_{\tilde{m}}$, while $\boldsymbol{G}$ depends on $y_1, \dots, y_{2\tilde{\tilde{n}}}$. In a similar way, we can verify that $\boldsymbol{Q}_j$ are matrix functions of variables $v_1, \dots, v_{\tilde{m}}$, and $\boldsymbol{P}_k$ are functions in $y_1, \dots, y_{2\tilde{\tilde{n}}}$. It suffices to use the equations

$$[P'_i, \tilde{P}_k]\left(\frac{\partial}{\partial} x^t, \frac{\partial}{\partial} v^l\right) = 0, \qquad [P''_j, \tilde{P}_k]\left(\frac{\partial}{\partial y^t}, \frac{\partial}{\partial v^l}\right) = 0,$$

$$\text{where } \frac{\partial}{\partial x^t} \in D_i, \quad \frac{\partial}{\partial v^l} \in \tilde{D}, \quad \frac{\partial}{\partial y^t} \in D'',$$

$$[P'_i, P''_j]\left(\frac{\partial}{\partial x^h}, \frac{\partial}{\partial y^l}\right) = 0, \qquad [\tilde{P}_i, P''_j]\left(\frac{\partial}{\partial v^k}, \frac{\partial}{\partial y^l}\right) = 0,$$

$$\text{where } \frac{\partial}{\partial x^h} \in D_i, \quad \frac{\partial}{\partial v^k} \in \tilde{D}, \quad \frac{\partial}{\partial y^l} \in D''.$$

The matrices of projectors P'_i indicate that they depend only on $x_1, \dots, x_{\tilde{n}}$. In a natural way, a coordinate neighborhood N is foliated into three systems of leaves. The leaves of the first foliation are given by $x_1 = \text{const}, \dots, y_{2\tilde{\tilde{n}}} = \text{const}$;

the second system of leaves is given by $x_1 = \text{const}, \dots, x_{\tilde{n}} = \text{const}, y_1 = \text{const}, \dots, y_{2\tilde{\tilde{n}}} = \text{const}$. The third foliaton is defined by $x_1, \dots, v_{\tilde{m}}$ constant. We restrict Φ onto each leaf of the second family to obtain an integrable almost tangent structure independent of parameters x_i, y_j determining the leaf. At the same time, the restrictions of projectors $\tilde{P}_k$ is independent of these parameters. By Proposition 1 we can find a coordinate transformation $\tilde{x}_{\tilde{n}+1} = \varphi_1(v_1, \dots, v_{\tilde{m}})$, $\tilde{v}_{\tilde{n}+\tilde{m}} = \varphi_{\tilde{m}}(v_1, \dots, v_{\tilde{m}})$ such that with respect to the new coordinate frame $\left(\frac{\partial}{\partial \tilde{x}_{\tilde{n}+1}}, \dots, \frac{\partial}{\partial x_{\tilde{n}+\tilde{m}}}\right)$,

$$F = \begin{pmatrix} \begin{matrix} I & 0 & 0 \\ 0 & I & 0 \\ I & 0 & 0 \end{matrix} & 0 \\ 0 & \begin{matrix} I & 0 & 0 \\ 0 & I & 0 \\ I & 0 & 0 \end{matrix} \end{pmatrix}.$$

Similarly, the restriction of Φ onto each leaf of the third foliation defines an integrable almost complex structure which is independent of the parameters x_i, v_k determining a leaf. The restrictions of P''_j onto the leaves of this last foliation are also independ on the variables x_i, v_k. So there exists a coordinate transformation $\tilde{\tilde{x}}_{\tilde{n}+\tilde{m}+1} = \psi_1(y_1, \dots, y_{2\tilde{\tilde{n}}})$, $\tilde{\tilde{x}}_{\tilde{n}+\tilde{m}+2\tilde{\tilde{n}}} = \psi_{2\tilde{\tilde{n}}}(y_1, \dots, y_{2\tilde{\tilde{n}}})$ such that

$$G = \begin{pmatrix} \begin{matrix} 0 & I \\ -I & 0 \end{matrix} & 0 \\ 0 & \begin{matrix} 0 & I \\ -I & 0 \end{matrix} \end{pmatrix}.$$

Now it is obvious that the coordinate transormation

$$\begin{aligned} x'_i &= x_i, & 1 \le i \le \tilde{n}, \\ x'_{\tilde{n}+k} &= \varphi_k(v_1, \dots, v_{\tilde{m}}), & 1 \le k \le \tilde{m}, \\ x_{\tilde{n}+\tilde{m}+j} &= \psi_j(y_1, \dots, y_{2\tilde{\tilde{n}}}), & 1 \le j \le 2\tilde{\tilde{n}} \end{aligned}$$

yields a coordinate frame with respect to which in the representation of Φ, exactly blocks of the form

$$\mathbf{0}_{\tilde{n}}, \qquad \begin{matrix} I & 0 & 0 \\ 0 & 0 & 0 \\ I & 0 & I \end{matrix}, \qquad \text{and} \qquad \begin{matrix} 0 & I \\ -I & 0 \end{matrix}$$

occur on the diagonal. It follows that f admits the desired matrix representation. □

6 Polynomial structures and webs

A non-holonomic 3-web can be defined by a couple of (1,1)-tensor fields P, B such that P is idempotent, $P^2 - P = 0$, B is involutive, $B^2 - I = 0$, and

$PB + BP = B$. Web distributions are given by $\ker P$, $\ker(I - P)$, $\ker(B - I)$. They are integrable (and the web is holonomic) iff $\ker(B - I)$ is involutive and $[P, P] = 0$. The condition $[P, B] = 0$ is satisfied exactly for webs which are paratactical (the torsion tensor of which vanishes identically). A 3-web is parallelizable (equivalent with three systems of parallel r-planes in R^{2r}) iff all three couples of almost product structures formed by web-distributions are simultaneously integrable.

More generally, a non-holonomic $(n+1)$-web of dimension r on a nr-dimensional manifold can be described by a family of $(1,1)$-tensor fields $\{\overset{\alpha}{\underset{\beta}{H}},\ \alpha, \beta = 1, \dots, n\}$ which satisfy $\sum_\alpha \overset{\alpha}{\underset{\alpha}{H}} = I$, $\overset{\gamma}{\underset{\beta}{H}}\overset{\alpha}{\underset{\kappa}{H}} = \delta^\gamma_\kappa \overset{\alpha}{\underset{\beta}{H}}$. The mappings with different indexis are nilpotents, and $\{\overset{\alpha}{\underset{\alpha}{H}}\}$ is a family of mutually orthogonal projectors onto web distributions $D_\alpha = \operatorname{im} \overset{\alpha}{\underset{\alpha}{H}}$. The remaining distribution is given by $D_0 = \operatorname{im} \overset{0}{\underset{0}{H}}$ where $\overset{0}{\underset{0}{H}} = \frac{1}{n} \sum_{\alpha,\beta} \overset{\alpha}{\underset{\beta}{H}}$ is the remaining projector. The kernels of the above projectors form a web of codimenison r. We will discuss these examples on some other place.

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