Kazimierz Głazek
A certain Galois connection and weak automorphisms


Persistent URL: [http://dml.cz/dmlcz/120367](http://dml.cz/dmlcz/120367)

Terms of use:
© Palacký University Olomouc, Faculty of Science, 1997

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these Terms of use.
A Certain Galois Connection and Weak Automorphisms*

KAZIMIERZ GLAZEK

Institute of Mathematics, Technical University of Zielona Góra,
ul. Podgórna 50, PL-65 246 Zielona Góra, Poland

(Received September 18, 1996)

Abstract

It is a survey of results on the so called weak automorphisms. Connections between bijections of a set \( A \) and families of operations on \( A \) are described. It could be interested from the point of view of universal algebra as well as of that of multiple-valued logic.

Key words: Weak automorphism, operation, iterative Post algebra, Galois connection.

1991 Mathematics Subject Classification: 08A35, 08A40

Introduction

In this paper we will try to describe a certain Galois connection between bijections of a set \( A \) and families of finitary operations on \( A \). These investigations are situated on the borderline between Universal Algebra and Multiple-valued Logics. Topics of the paper are related to the important notion of weak automorphism of general algebras. Weak automorphisms of an algebra (with the carrier \( A \)) induce so-called inner automorphisms of the iterative Post algebra (in the sense of A. I. Mal'cev [Ma66]) of operations on the set \( A \), of the Menger algebras (or \( n \)-clones) of \( n \)-ary operations on \( A \), and of the Menger system of all operations on \( A \) (see, e.g., [Whi64] and [ScT79]). We have payed attention to importance of the considered Galois connection in our lecture during the ICM-90 in Kyoto (see [Gi90]). Almost all of the results, presented here, was announced (in Polish) in the book [Gi94] (MR 96b:08006).

*This paper was finally completed during the author's stay in Palacky University of Olo­mouc in 1996 (supported by a grant of The Ministry of Education of the Czech Republic).
1 Preliminaries

Let $A$ be a non-empty set, $\Omega(A)$ be the set of all finitary operations over the set $A$, and let $\sigma \in S_A$ (the set of all bijections of the set $A$ onto itself). For every $f \in \Omega(A)$ (say: $n$-ary), consider a new ($n$-ary) operation $\bar{\sigma}(f)$ defined by the equality

$$\bar{\sigma}(f)(a_1, \ldots, a_n) = \sigma(f(\sigma^{-1}(a_1), \ldots, \sigma^{-1}(a_n))).$$

(1)

Similarly, we can define a new operation $(\sigma^{-1})^\sim(f)$. Thus, we have the mapping $\bar{\sigma} : \Omega(A) \to \Omega(A)$ (and also the mapping $(\sigma^{-1})^\sim : \Omega(A) \to \Omega(A)$) induced by $\sigma$.

Of course, $(\bar{\sigma})^{-1} = (\sigma^{-1})^\sim$.

Such mappings $\bar{\sigma}$ or $(\sigma^{-1})^\sim$ have been used by several authors in different investigations, first – according to the best of my knowledge – about 1905 by C.L. Bouton and E.V. Huntington (see [Hu05], p. 226) in the case of the algebra of complex numbers (for $\sigma$ being a homography). Mappings $\bar{\sigma}$ (or $(\sigma^{-1})^\sim$) also play an essential role in the theory of formal groups and so-called "annalysers" (see [Laz55], p. 338, [Laz75], p. 34). The theory of abstract mean values (e.g., the Kolmogoroff-Nagumo Theorem, [Ko30], [Na30], and the de Finetti-Kitagawa Theorem, [Fi31], [K34]) also uses suitable mappings $(\sigma^{-1})^\sim$ (see also, e.g., [Ac48], [Ry49], [AcW80], and references in Aczél’s book [Ac66]).

Mappings $\bar{\sigma}$ and $(\sigma^{-1})^\sim$ also appear in a natural manner in theories of several functional equations (see, e.g., [Ac49], [Ac61], [Ac66], [Ac69], [Ho53], [Ho54], [Kn49], [Vi59], [Vi61]). For some other applications see, e.g., [KaT79] and [Ri48].

An operation $f \in \Omega(A)$ is said to be self-dual with respect to a permutation $\sigma \in S_A$ if the equality

$$\bar{\sigma}(f) = f$$

is fulfilled. Several authors have investigated self-dual operations with respect to different permutations (see, for instance, [DHM81], [DR83], [EvH57], [Lei72], [Mar79], [Mar82], [MarDH80], [Mi71], [Mu59], [PöK79], p. 87, [Ro61], [St86], [StM86], [Ya58]).

If for $f, g \in \Omega(A)$ we have $g = (\sigma^{-1})^\sim(f)$, then – sometimes in the theory of multiple-valued logics – the operation $g$ is called similar to $f$ (this notion is a natural generalization of the duality for Boolean functions in two-valued logic; cf. [Pos41], [Ya58], [Ya66], [Ly51], [Mi71]).

The mapping $\bar{\sigma}$ is a so-called inner automorphism of the iterative Post algebra $P_A = (\Omega(A); *, \zeta, \tau, \Delta, \nabla)$ in the sense of A.I. Mal’cev, and of the pre-iterative Post algebra $P^*_A = (\Omega(A); *, \zeta, \tau, \Delta)$ (see [Ma66], [Ma76], and also [Mal72], [La79], [Ba80], [Ba81], [GoL83], [G192]). Moreover $\bar{\sigma}$ is an (inner) automorphism of the (full) Menger algebra (or the $n$-clone – in the terminology of T. Evans; see [Me46], [Me61], [Wh64], [LaN73], and [Ev81]).

Recall that, if a subset $\mathcal{A}$ of $\Omega(A)$ is closed under the compositions of functions, then $\mathcal{A}$ is called a closed class of functions in the sense of E.L. Post (see [Pos20], [Pos41], [Ya58], [Ya66]). If, besides, $\mathcal{A}$ contains all trivial operations $e_x^{(n)}(x_1, \ldots, x_n) = x_i$ ($i = 1, \ldots, n$; $n = 1, 2, \ldots$), then $\mathcal{A}$ is a clone in the sense of Ph. Hall (see [Co65], [McMT87], and [Sz86]). A closed class (or a clone) $\mathcal{A}$ is called self-dual if the inclusion $\bar{\sigma}(\mathcal{A}) \subseteq \mathcal{A}$ holds true for all bijections.
A certain Galois connection and weak automorphisms

σ: A → A. Such classes have been considered by several authors (see [DH79], [DHR83], [DR84], [Mi71]).

2 Weak automorphisms

Let now A = (A; ⨁) be a general algebra, A(⊂ O(A)) be the clone of all term operations of A (see [MMT87]), and let σ ∈ S_A. If

\[ \sigma(A) = A, \]

then σ is said to be a weak automorphism of the general algebra A = (A; ⨁) (see [Se70]; this notion is a special case of the notion of the weak isomorphism defined by A. Goetz [Go66]). Equivalently, in another terminology, σ is a cryptoautomorphism (as a special case of the notion of the cryptomorphism in the sense of G. Birkhoff, see [Bi71], [Bi82], [Pô85]). It is worth adding, that – in the definition of the weak automorphism – it is not enough to assume the inclusion \( \sigma(A) \subseteq A \).

As an example, we consider a weak automorphism σ of an infinite integral domain \((R; +, -, 0, \cdot, e)\) with the unity e treated as a constant fundamental operation. Then σ determines new ring operations ⊕ and ⊙ defined by the formulas:

\[ x \oplus y = x + y - \sigma(0) \]

and

\[ x \otimes y = (x \cdot y - \sigma(0) \cdot (x + y) + \sigma(0) \cdot \sigma(e)) \cdot (\sigma(e) - \sigma(0))^{-1}, \]

where \( \sigma(0) \) and \( \sigma(e) \) belong to the subring \((e)\) of \( R \) generated by e, and \( \sigma(e) - \sigma(0) \) belongs to \( R^* \) (the set of all units, i.e. invertible elements of \( R \)). Moreover the rings \((R; +, \cdot)\) and \((R; \oplus, \otimes)\) are isomorphic. This result, proved in [Gi70], is a generalization of some well-known results for infinite fields ([Lev45], [HNE64]; see also [ZaS58], p. 11). If we take a bijection σ of the ring \( R \) onto itself, such that \( \sigma(0) = e \) and \( \sigma(e) = 0 \), then we get a case considered by A.L. Foster and B.A. Bernstein (see [FoB44]). Considering the mappings \( x \mapsto x + e \) or \( x \mapsto -x + e \) (in rings with the unity e treated as fundamental constant operation) leads to some generalization of the Principle of Duality for Boolean rings and Boolean algebras (see [Fo45], [FoB44], [FoB45], [Yaq56]).

We will now give some examples of new field operations in finite fields (for more details see [Gi81]). Consider a new addition \( \oplus_1 \) in \( F = GF(7) \):

\[ x \oplus_1 y = x + y + 5x^2y^2(x^3 + y^3) + 3x^3y^3(x + y). \]

Then \((F; +, \cdot) \simeq (F; \oplus_1, \cdot)\). In the same field we can define the new operations:

\[ x \oplus_2 y = x + y + x^2y^2 + 3x^5y^5 + 6x^3y^3(x + y) + 5xy(x^2 + y^2) + 2x^2y^2(x^3 + y^3) \]

and

\[ x \otimes y = 3x^4y^4 + 3x^4y + 3xy^4 + xy. \]
Then we similarly have \((F;+,\cdot) \simeq (F;\oplus_2,\circ)\). These new field operations can be obtained by using suitable weak automorphisms of \(GF(7)\) (which can be represented as permutation polynomials; see, e.g., [Ca63], [LaN73], [LN83] and [Gl81]). Namely, for the bijections \(\sigma_1(x) = x^5\) and \(\sigma_2(x) = x^5 + 2x^2\) of \(f = GF(7)\) onto itself we have \(\sigma_1(+) = \oplus_1, \sigma_1(\cdot) = \cdot, \sigma_2(+) = \oplus_2\), and \(\sigma_2(\cdot) = \circ\). Observe that the induced mapping for the first of those weak automorphisms preserves multiplication “\(\cdot\)”.

Such weak automorphisms \(\sigma\) of field \(F\), for which the induced mappings \(\tilde{\sigma}\) preserve multiplication, form a normal subgroup of the group \(WAut(F)\) of all weak automorphisms of the field \(F\). Denote by the symbol \(AM(F)\) the set of all weak automorphisms \(\sigma\) for which the mappings \(\tilde{\sigma}\) preserve field multiplication. Then we have

\[
\text{The sequence of normal subgroups } \quad Aut(F) < AM(F) < WAut(F). 
\]

(6)

If \(F = GF(q)\) with \(q = p^n\), then \(\sigma \in AM(F)\) iff there exists a natural number \(k \leq p^n - 2\) such that \((k, q - 1) = 1\) and \(\sigma(x) = x^k\) for every \(x \in F\). Of course, for \(\sigma \in AM(F)\) we have \(\sigma(e) = e\) and \(\sigma(0) = 0\).

It is worth adding that for finite fields we have a generalization (announced in [Gl94]) of well-known Dedekind Independence Theorem:

**Proposition 1** Let \(\sigma_1, \ldots, \sigma_n\) be pair-wise distinct weak automorphisms of finite field \(F\), such that induced mappings \(\tilde{\sigma}_i\) \((i = 1, \ldots, n)\) preserve field multiplication, i.e. \(\sigma_i \in AM(F)\). Then \(\sigma_1, \ldots, \sigma_n\) are linearly independent (as elements of linear space \(F^F\) over the field \(F\)).

Indeed, we should prove that if \(\sigma_1, \ldots, \sigma_n \in AM(F)\), \(\sigma_i \neq \sigma_j\) for \(i \neq j\), and \(\lambda_1, \ldots, \lambda_n \in F\), then the following implication

\[
(\forall x \in F) \ (\lambda_1 \sigma_1(x) + \ldots + \lambda_n \sigma_n(x) = 0) \Rightarrow \lambda_1 = \ldots = \lambda_n = 0
\]

holds true. We will prove it induction with respect to \(n\). Let \(\lambda \sigma(x) = 0\) for every \(x \in F\). Then for \(x = e\) we obtain \(\lambda = \lambda \sigma(e) = 0\), which is the first step of the inductive proof. Consider \(n + 1\) distinct weak automorphisms \(\sigma_i\) and assume

\[
(\forall x \in F) \ (\lambda_1 \sigma_1(x) + \ldots + \lambda_n \sigma_n(x) = 0). 
\]

(7)

The mappings \(\sigma_1\) and \(\sigma_{n+1}\) are distinct, thus there exists \(b \in F \setminus \{0\}\), such that \(\sigma_1(b) \neq \sigma_{n+1}(b)\), and for arbitrary \(x \in F\) there is \(y \in F\) with \(x = y \cdot b\). Therefore we have

\[
\lambda_1 \sigma_1(y) \sigma_1(b) + \lambda_2 \sigma_2(y) \sigma_2(b) + \ldots + \lambda_{n+1} \sigma_{n+1}(y) \sigma_{n+1}(b) = 0
\]

and

\[
\lambda_1 \sigma_1(y) \sigma_1(b) + \lambda_2 \sigma_2(y) \sigma_1(b) + \ldots + \lambda_{n+1} \sigma_{n+1}(y) \sigma_1(b) = 0.
\]

Further we infer that

\[
\lambda_2 (\sigma_2(b) - \sigma_1(b)) \sigma_2(y) + \ldots + \lambda_{n+1} (\sigma_{n+1}(b) - \sigma_1(b)) \sigma_{n+1}(y) = 0.
\]
A certain Galois connection and weak automorphisms

By the assumption of validity of our proposition for \( n \) we have \( \lambda_{n+1} = 0 \), and from (7) we get \( \lambda_1 \sigma_1(x) + \ldots + \lambda_n \sigma_n(x) = 0 \) for any \( x \in F \). Thus, using once more our inductive assumption, we infer \( \lambda_1 = \ldots = \lambda_n = 0 \), which completes the proof of Proposition 1.

We recall that a more general notion of the \( \gamma \)-weak automorphism (with respect to some composition closure \( \gamma \) over the set \( \mathcal{O}(A) \)) was introduced in [G193] (see also [G194]). Namely, a permutation \( \sigma \in S_A \) is said to be a \( \gamma \)-weak automorphism of a general algebra \( A = (A; F) \) if

\[
\tilde{\sigma}(\gamma(F)) = \gamma(\tilde{F}) \quad (= \gamma(\tilde{\sigma}(F))).
\]

Denoting by \( WAut(A) \) and \( \gamma WAut(A) \) the groups of, respectively, all weak automorphisms and all \( \gamma \)-weak automorphisms of \( A \), one can verify that \( WAut(A) \) is a normal subgroup of the group \( \gamma WAut(A) \). So, we have

\[
Aut(A) < \gamma WAut(A) < WAut(A).
\]

It is easy to observe, that if \( \sigma \in S_A \), then for every composition closure \( \gamma \), the mapping \( \tilde{\sigma} \) is a monomorphisms of the \( \gamma \)-closure space \( (\mathcal{O}(A); \gamma) \), i.e. \( \tilde{\sigma} \) is \( \gamma \)-closure automorphism.

3 A certain Galois connection

Consider a set \( A \), with \( \text{card}(A) > 1 \), and the set \( \mathcal{O}(A) \) of all (finitary) operations on the set \( A \). Let now \( B \subseteq \mathcal{O}(A) \), \( \sigma \in S_A \), and let \( \tilde{\sigma} \in S_{\mathcal{O}(A)} \) be defined by (1). Define the relation

\[
\rho_\sigma \subseteq S_A \times 2^{\mathcal{O}(A)}
\]

by the equality

\[
B = \tilde{\sigma}(B).
\]

The relation \( \rho_\sigma \) determines a Galois connection or a polarity in the sense of G. Birkhoff ([Bi40]; see also [Or44]). Investigations of such a connection for the relation \( \rho_\sigma \) were initiated by us in 1989 and reported during ICM-90 in Kyoto, Japan (see [Gh90] and [Gh94]), but we are still in the initial stages of investigations. The suitable Galois correspondence in the sense of 0. Ore (see [Or44]) between subsets \( G \subseteq S_A \) and families \( \mathcal{F} \) of subsets of \( \mathcal{O}(A) \) are given by two mappings:

\[
G \mapsto \tilde{\mathcal{F}}(G) = \{ F \subseteq \mathcal{O}(A) \mid (\forall \sigma \in G)(\tilde{F} = \tilde{\sigma}(F)) \}\]

and

\[
F \mapsto \tilde{G}(F) = \{ \sigma \in S_A \mid (\forall B \in \mathcal{F})(\tilde{B} = \tilde{\sigma}(B)) \}.
\]

Note some simple properties of mappings (11) and (12), and a relation the notion to the notions of weak automorphism (see [Se70]) and of \( \gamma \)-weak automorphism (see [Gh93] and [Gh94]). The following statements are easy to verify:

\[
(i) \tilde{G}(\{E\}) = \tilde{G}(\{\mathcal{O}(1)(A)\}) = S_A.
\]
(ii) \( E, \mathcal{O}(A) \in \hat{\mathcal{F}}(G) \) for every \( G \subseteq S_A \).

(iii) Let \( B = \{f\} \) and \( A = (A; f) \). Then \( \hat{G}({B}) = \text{Aut}(A) \).

(iv) Let \( A = (A; B) \) for some \( B \subseteq \mathcal{O}(A) \). Then \( \hat{G}({B}) \subseteq \text{WAut}(A) \). Moreover, if \( B = (B) = T(A) \) is a clone of operations over \( A \), then \( \hat{G}({B}) = \text{WAut}(A) \). More generally, if \( B = \gamma(B) \) for some composition closure \( \gamma \) on \( \mathcal{O}(A) \) (see [Gla93]), then \( \hat{G}({B}) = \gamma \text{WAut}(A) \).

(v) Let \( \sigma \in S_A \). If \( B \in \hat{\mathcal{F}}(\{\sigma\}) \) and \( A = (A; B) \), then \( \sigma \in \text{WAut}(A) \). Moreover, if \( B = (B) \), then \( B \in \hat{\mathcal{F}}(\{\sigma\}) \) iff \( \sigma \in \text{WAut}(A) \). More generally, if \( B = \gamma(B) \) (for some composition closure \( \gamma \)), then \( B \in \hat{\mathcal{F}}(\{\sigma\}) \) iff \( \sigma \in \gamma \text{WAut}(A) \).

(vi) Let \( G = \langle G \rangle \) be a subgroup of \( S_A \) and \( A = (A; B) \). If \( B \in \hat{\mathcal{F}}(G) \), then \( G < \text{WAut}(A) \). Moreover, if \( B = (B) \), and \( G < \text{WAut}(A) \), then \( B \in \hat{\mathcal{F}}(G) \).

(vii) If \( \gamma: 2^{\mathcal{O}(A)} \rightarrow 2^{\mathcal{O}(A)} \) is a composition closure on \( \mathcal{O}(A) \) (i.e. for every \( B \subseteq \mathcal{O}(A) \) we have \( B \subseteq \gamma(B) \subseteq (B) \) and \( B \in \hat{\mathcal{F}}(G) \), then \( \gamma(B) \in \hat{\mathcal{F}}(G) \).

Property (vii) shows that the family \( \hat{\mathcal{F}}(G) \), where \( G \subseteq S_A \), is very extensive. The next two properties also emphasize this fact:

(viii) If \( B \subseteq \hat{\mathcal{F}}(G) \), then also \( B^{(n)} \subseteq \hat{\mathcal{F}}(G) \) for every \( n = 0, 1, \ldots \).

(ix) If \( B_1, B_2, B_i \in \hat{\mathcal{F}}(G) \) (\( i \in I \)), then \( B_1 \cup B_2 \in \hat{\mathcal{F}}(G) \) and \( \bigcup_{i \in I} B_i \in \hat{\mathcal{F}}(G) \).

It is worth noting that:

(x) \( \hat{\mathcal{F}}(G) = \hat{\mathcal{F}}(\langle G \rangle) = \bigcup_{\sigma \in G} \hat{\mathcal{F}}(\{\sigma\}) \), where \( \langle G \rangle \) is the subgroup of \( S_A \) generated by the set \( G \) of permutations.

(xi) \( \hat{G}(F) = \bigcup_{B \in \mathcal{F}} \hat{G}({B}) \subseteq S_A \).

(xii) \( G \subseteq \bigcup_{B \in \mathcal{F}(G)} \text{WAut}((A; B)) \).

(xiii) \( \langle \hat{\mathcal{F}}(\text{Sub}(S_A)) \rangle \) is a complete lattice with the lower bound \( \hat{\mathcal{F}}(S_A) \) and the upper bound \( \hat{\mathcal{F}}(\{\text{id}_A\}) = 2^{2^{\mathcal{O}(A)}} (= \mathcal{F}(\emptyset)) \).

Taking into account the results of G. Birkhoff and O. Ore we immediately have
A certain Galois connection and weak automorphisms

**Proposition 2** The mappings (11) and (12) establish a Galois connection between subsets $G \subset S_A$ and subsets of $2^{O(A)}$, i.e. we have:

$$G_1 \subset G_2 \subset S_A \Rightarrow \hat{F}(G_2) \subset \hat{F}(G_1) \subset 2^{O(A)},$$

$$\mathcal{F}_1 \subset \mathcal{F}_2 \subset 2^{O(A)} \Rightarrow \hat{G}(\mathcal{F}_2) \subset \hat{G}(\mathcal{F}_1),$$

$$G \subset \hat{G}({\hat{F}(G)}),$$

$$F \subset \hat{F}({\hat{G}(F)}),$$

$$\hat{F}(\hat{F}(G)) = \hat{F}(G),$$

$$\hat{G}(\hat{G}(\mathcal{F})) = \hat{G}(\mathcal{F}).$$

It is easy to observe that equalities (17) and (18) follow from (13)–(16). Define the operators $\nabla$ on $2^S_A$ and $\Delta$ on $2^{2^{O(A)}}$ in the following way:

$$(\nabla(G)) = \hat{G}(\hat{F}(G)) = \{ \sigma \in S_A | (\forall B \subset O(A)) ((\forall \tau \in G) (\hat{\tau}(B) = B) \Rightarrow (\hat{\sigma}(B) = B)) \},$$

$$(\Delta(\mathcal{F})) = \hat{F}(\hat{G}(\mathcal{F})) = \{ B \subset O(A) | (\forall \tau \in S_A)((\forall F \in \mathcal{F}) (\hat{\tau}(F) = F) \Rightarrow (\hat{\sigma}(B) = B)) \}. $$

Like in the classical Galois theory, we can easily verify that the operators $\Delta$ and $\nabla$ are closure operators over $2^S_A$ and $2^{2^{O(A)}}$, respectively. Moreover, the closed elements with respect to these operators are of the form $\hat{G}(\mathcal{F})$ and $\hat{F}(G)$. Taking into account the general theory described by O. Ore (see [Or44]) we get the following results (announced in [Gi90] and appeared in [Gi94]):

**Proposition 3** The mappings (11) and (12) determine one-to-one correspondence between families of sets $\nabla(G)$ and $\Delta(\mathcal{F})$, defined by (18) and (19), respectively. Moreover the families

$$\{ \nabla(G) | G \subset S_A \} \quad \text{and} \quad \{ \Delta(\mathcal{F}) | \mathcal{F} \subset 2^{O(A)} \}$$

form complete lattices with respect to suitable inclusions, and these lattices are dually isomorphic, i.e. the following rules:

$$\hat{F}(\nabla(G_1) \cap \nabla(G_2)) = \Delta(\hat{F}(\nabla(G_1))) \cup \hat{F}(\nabla(G_2))) = \Delta(\hat{F}(G_1) \cup \hat{F}(G_2)), \quad (21)$$

and

$$\hat{F}(\nabla(G_1) \cup \nabla(G_2)) = \Delta(\hat{F}(\nabla(G_1)) \cap \hat{F}(\nabla(G_2))) = \Delta(\hat{F}(G_1) \cap \hat{F}(G_2)) \quad (22)$$

for the operator $\hat{F}$ hold, and the analogous rules for the operator $\hat{G}$ hold.
4 Some stabilizers

Finally, for any family \( \mathcal{F} \subseteq 2^\mathcal{O}(A) \), define the “stabilizer” of it:

\[
G_0(\mathcal{F}) = \{ \sigma \in S_A \mid (\forall f \in \{B \mid B \in \mathcal{F}\}) \quad (\tilde{\sigma}(f) = f) \}, \tag{23}
\]

i.e. the largest subset of \( S_A \) such that every operation \( f \) from any family \( B \) of \( \mathcal{F} \in 2^{\mathcal{O}(A)} \) is self-dual with respect to each permutation \( \sigma \in G_0(\mathcal{F}) \). Then we obtain a generalization of the well-known fact, proved independently by J. R. Senft ([Se70]) and E. Plonka (see [DuP71]), that for an arbitrary general algebra \( A \) the group of all automorphisms of \( A \) is a normal subgroup of the group of all weak automorphisms of \( A \), namely:

**Proposition 4** Let \( A \) be a set with \( \text{card}(A) > 1 \) and let \( G_0(\mathcal{F}) \) and \( \hat{G}(\mathcal{F}) \) be defined by (23) and (12), respectively. Then the sets \( G_0(\mathcal{F}) \) and \( \hat{G}(\mathcal{F}) \) are subgroups of the group \( S_A \) of all permutations of the set \( A \), and \( G_0(\mathcal{F}) \) is a normal subgroup of \( \hat{G}(\mathcal{F}) \).

Indeed, it is clear that the sets \( G_0(\mathcal{F}) \) and \( \hat{G}(\mathcal{F}) \) are subgroups of \( S_A \). Let now \( \sigma \in G_0(\mathcal{F}), \tau \in \hat{G}(\mathcal{F}) \) and let \( f \in B^{(n)} \), where \( B \in \mathcal{F} \). Then we have

\[
\hat{\tau}(f) = g \in B = \hat{\tau}(B), \quad \tilde{\sigma}(g) = g
\]

and

\[
\begin{align*}
((\tau^{-1} \circ \sigma \circ \tau)^{-1}(f))(x_1, \ldots, x_n) &= \tau^{-1}(((\sigma \circ \tau)^{-1}(f))(\tau(x_1), \ldots, \tau(x_n))) = \\
&= \tau^{-1}((\sigma \circ \tau)(f((\tau^{-1} \circ \sigma^{-1} \circ \tau)(x_1), \ldots, (\tau^{-1} \circ \sigma^{-1} \circ \tau)(x_n)))) = \\
&= (\tau^{-1} \circ \sigma)((\tilde{\tau}(f)){((\sigma^{-1} \circ \tau)(x_1), \ldots, (\sigma^{-1} \circ \tau)(x_n))) = \\
&= \tau^{-1}((\tilde{\sigma}(g))(\tau(x_1), \ldots, \tau(x_n))) = ((\tau^{-1} \circ g)(x_1, \ldots, x_n)) = f(x_1, \ldots, x_n).
\end{align*}
\]

Therefore \( \tau^{-1} \circ \sigma \circ \tau \in G_0(\mathcal{F}) \), which completes the proof of our proposition.

Let \( A = (A; F) \) be an algebra. Take \( \mathcal{F} = \{B\} \), where \( B \) is the set of all term operation of the algebra \( A \). Then we can get—as an easy corollary from Proposition 4—that \( \text{Aut}(A) \) is a normal subgroup of \( W\text{Aut}(A) \).

**References**


A certain Galois connection and weak automorphisms


A certain Galois connection and weak automorphisms


