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The Degrees of Regularity in Varieties

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(Received May 15, 1996)

Abstract

Congruence regular varieties are characterized by a Mal’cev condition containing \( m \)-ary terms. We prove that this number \( m \) is the degree of regularity, i.e. the number of elements which generate the congruence class of every principal congruence.

Key words: Mal’cev condition, regularity, 0-regularity, regularity with respect to \( g_1, \ldots, g_n \).

1991 Mathematics Subject Classification: 08B05, 08A30

Recall from [1] that if \( g_1(x), \ldots, g_n(x) \) are unary terms, we say that an algebra \( A \) is regular with respect to \( g_1, \ldots, g_n \) if for any \( a \in A \), \( \Theta = \Phi \) for \( \Theta, \Phi \in \text{Con} A \) whenever \( [g_i(a)]_\Theta = [g_i(a)]_\Phi \) for \( i = 1, \ldots, n \). A variety \( V \) is regular with respect to \( g_1, \ldots, g_n \) if all its members have this property.

Let us remark that if \( g_i(x) = x \) for \( i = 1, \ldots, n \) then it gives the common concept of regularity. If \( g_i(x) = 0 \) for \( i = 1, \ldots, n \) (where 0 is a nullary term) then we obtain the concept of 0-regularity alias weak regularity. Moreover, the concept of regularity with respect to \( g_1, \ldots, g_n \) coincides with that of subregularity introduced by J. Duda, [4], see [1] for some details. The following statement was proven in [1]:

**Proposition 1** The following conditions on a variety \( V \) with unary terms

\[ g_1(x), \ldots, g_n(x) \]

are equivalent:
(1) \( \mathcal{V} \) is regular with respect to \( g_1, \ldots, g_n \);

(2) for some positive integer \( m \), there exist ternary terms \( p_1, \ldots, p_m \) and a function \( r \mapsto i_r \) from \( \{1, \ldots, m\} \) to \( \{1, \ldots, n\} \) such that \( \mathcal{V} \) satisfies
\[
[p_1(x,y,z) = g_{i_1}(z) \& \cdots \& p_m(x,y,z) = g_{i_m}(z)] \Rightarrow x = y;
\]

(3) for some positive integers \( m, k \) there exist ternary terms \( p_1, \ldots, p_m \), \((m + 3)\)-ary terms \( t_1, \ldots, t_k \) and a function \( r \mapsto i_r \) from \( \{1, \ldots, m\} \) to \( \{1, \ldots, n\} \) such that for \( j = 1, \ldots, k - 1 \) and \( r = 1, \ldots, m \), \( \mathcal{V} \) satisfies
\[
p_r(x,x,z) = g_{i_r}(z) \text{ and } t_j(x,y,z,p_1(x,y,z),\ldots,p_m(x,y,z)) = t_{j+1}(x,y,z,g_{i_1}(z),\ldots,g_{i_m}(z)),
\]
\[
y = t_k(x,y,z,p_1(x,y,z),\ldots,p_m(x,y,z)).
\]

Moreover, if one of the foregoing equivalent conditions holds for (*) then \( k \) is the smallest integer for which \( \mathcal{V} \) is \((k+1)\)-permutable.

Hence, the Proposition characterizes the degree of permutability by the number of terms \( t_i \) in (*). On the other hand, it was not clear what is the dependence of the integer \( m \) in (ii) or (iii). We are going to introduce a degree of regularity which relates this \( m \).

At first, we will solve the simplest case for \( k = 1 \) and \( m = 1 \), i.e. for permutable varieties.

**Definition 1** Let \( g_1(x), \ldots, g_n(x) \) be unary terms. An algebra \( A \) has transferable congruences with respect to \( g_1, \ldots, g_n \) if for any \( a, b, x \in A \) there exist \( c_1, \ldots, c_n \in A \) such that \( \Theta(a,b) = \Theta(g_i(x),c_i) \) holds for each \( i \in \{1, \ldots, n\} \). A variety \( \mathcal{V} \) has transferable congruences with respect to \( g_1, \ldots, g_n \) if each \( A \in \mathcal{V} \) has this property.

**Theorem 1** The following conditions are equivalent for a variety \( \mathcal{V} \) with unary terms \( g_1(x), \ldots, g_n(x) \):

(1) \( \mathcal{V} \) has transferable congruences with respect to \( g_1, \ldots, g_n \);

(2) for each \( i \in \{1, \ldots, n\} \) there exists an integer \( k \) and a ternary term \( p_i \) and \( 5\)-ary terms \( t_1, \ldots, t_k \) such that \( p_i(x,x,z) = g_i(z) \) and
\[
x = t_1(x,y,z,g_i(z),p_i(x,y,z)),
\]
\[
t_j(x,y,z,p_i(x,y,z),g_i(z)) = t_{j+1}(x,y,z,g_i(z),p_i(x,y,z)) \text{ for } j = 1, \ldots, k - 1,
\]
\[
y = t_k(x,y,z,p_i(x,y,z),g_i(z));
\]

(3) for each \( i \in \{1, \ldots, n\} \) there exists a ternary term \( p_i \) such that
\[
p_i(x,x,z) = g_i(z) \iff x = y.
\]
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Proof (1) ⇒ (2): Put \( A = F_V(x, y, z) \). By (1), for each \( i \in \{1, \ldots, n\} \) there exists \( c_i \in A \) with \( \Theta(x, y) = \Theta(g_i(z), c_i) \). Hence, \( c_i = p_i(x, y, z) \) for some 3-ary term \( p_i \) and, immediately, \( p_i(x, x, z) = g_i(z) \). Since \( (x, y) \in \Theta(g_i(z), p_i(x, y, z)) \), there exist 5-ary terms \( t_1, \ldots, t_k \) satisfying (2).

(2) ⇒ (1): Let \( A \in \mathcal{V} \) and \( a, b, x \in A \). By (2) we have

\[
(a, b) \in \Theta(g_i(x), p_i(a, b, x)).
\]

Further, \( (g_i(x), p_i(a, b, x)) = (p_i(a, a, x), p_i(a, b, x)) \in \Theta(a, b) \), i.e. \( \Theta(a, b) = \Theta(g_i(x), p_i(a, b, x)) \) proving (1).

(1) ⇒ (3) is implicitly contained in (1) ⇒ (2) since for those \( p_i \) we have \( p_i(x, y, z) = g_i(z) \) iff \( x = y \).

(3) ⇒ (1): Let \( A \in \mathcal{V} \) and \( x, y, z \in A \). Put \( c_i = p_i(x, y, z) \). Then \( (g_i(z), c_i) = (p_i(x, x, z), p_i(x, y, z)) \in \Theta(x, y) \). Denote by \( \Theta = \Theta(g_i(z), p_i(x, y, z)) \). Then in \( A/\Theta \) we have \( [g_i(z)]_\Theta = [p_i(x, y, z)]_\Theta = [x]_\Theta [y]_\Theta [z]_\Theta \). However, \( A/\Theta \in \mathcal{V} \), thus also \( A/\Theta \) satisfies (3), i.e. we obtain \( [x]_\Theta = [y]_\Theta \) giving

\[
(x, y) \in \Theta = \Theta(g_i(z), c_i).
\]

Altogether, \( \Theta(x, y) = \Theta(g_i(z), c_i) \) proving (1).

By (iii) of the Proposition, we conclude

Corollary 1 If a variety \( \mathcal{V} \) has transferable congruences with respect to \( g_1, \ldots, g_n \) then \( \mathcal{V} \) is regular with respect to \( g_1, \ldots, g_n \).

Now, we can characterize the simplest case:

Theorem 2 For a variety \( \mathcal{V} \), the following are equivalent:

(1) \( \mathcal{V} \) is permutable and has transferable congruences with respect to \( g_1, \ldots, g_n \);

(2) for each \( i \in \{1, \ldots, n\} \) there exists a 3-ary term \( p_i \) and a 4-ary term \( t_i \) such that \( p_i(x, x, z) = g_i(z) \) and \( x = t_i(x, y, z, g_i(z)) \), \( y = t_i(x, x, g_i(z)) \).

Proof (1) ⇒ (2): Consider again \( F_V(x, y, z) \) and \( \Theta = \Theta(x, y) \). For each \( i \in \{1, \ldots, n\} \) there exists \( c_i \in F_V(x, y, z) \) with \( \Theta(x, y) = \Theta(g_i(z), c_i) \). Hence \( c_i = p_i(x, y, z) \) for some 3-ary term \( p_i(x, y, z) \) and \( p_i(x, x, z) = g_i(z) \). Moreover, the permutability implies

\[
\Theta(g_i(z), p_i(x, y, z)) = R(g_i(z), p_i(x, y, z))
\]

whence \( (x, y) \in R(g_i(z), p_i(x, y, z)) \). It is a routine way to prove (2).

(2) ⇒ (1): for permutability, put \( m(x, y, z) = t_i(x, z, y, p_i(y, z, y)) \). Then \( m(x, y, z) \) is a Mal’cev term, i.e. \( \mathcal{V} \) is permutable.

Prove transferability: let \( A \in \mathcal{V} \) and \( a, b, x \in A \). Then \( \langle g_i(x), p_i(a, b, x) \rangle = \langle p_i(a, a, x), p(a, b, x) \rangle \in \Theta(a, b), \langle a, b \rangle = \langle t_i(a, b, x, g_i(x)), t_i(a, b, x, p_i(a, b, x)) \rangle \in \Theta(g_i(x), p_i(a, b, x)) \) thus \( \Theta(a, b) = \Theta(g_i(x), p_i(a, b, x)) \).
Remark 1 By Theorem 2, if regularity is replaced by transferability in a permutable variety, then $m = 1$ in the Proposition. Hence, this condition has an influence on this number. We can generalize the concept of transferability to obtain a full characterization of this $m$. Theorem 2 is a generalization of the result of [2], [3] for regular and permutable varieties.

Definition 2 An algebra $A$ is said to have $m$-transferable congruences with respect to $g_1, \ldots, g_n$ if for any $a, b, x$ of $A$ there exist $c_1, \ldots, c_m \in A$ such that

$$\Theta(a, b) = \Theta(g_{i_1}(x), c_1) \vee \cdots \vee \Theta(g_{i_m}(x), c_m)$$

for any subset $\{i_1, \ldots, i_m\} \subseteq \{1, \ldots, n\}$. A variety $V$ has $m$-transferable congruences w.r.t. $g_1, \ldots, g_n$ if each $A \in V$ has this property.

Theorem 3 A variety $V$ has $m$-transferable congruence with respect to $g_1, \ldots, g_n$ if and only if $V$ satisfies (ii) of the Proposition.

Proof Consider $F_V(x, y, z)$ of $V$. By the definition, there exist $c_1, \ldots, c_m \in F_V(x, y, z)$ with

$$\Theta(x, y) = \Theta(g_{i_1}(z), c_1) \vee \cdots \vee \Theta(g_{i_m}(z), c_m)$$

for any $\{i_1, \ldots, i_m\} \subseteq \{1, \ldots, n\}$. Hence, $c_j = p_j(x, y, z)$ ($i = 1, \ldots, m$) and

$$[p_1(x, y, z) = g_{i_1}(z) \& \cdots \& p_m(x, y, z) = g_{i_m}(z)] \iff x = y.$$

The converse implication can be shown similarly as in the proof of Theorem 1.

Corollary 2 A variety $V$ is regular with respect to $g_1, \ldots, g_n$ if and only if $V$ has $m$-transferable congruences w.r.t. $g_1, \ldots, g_n$ for some integer $m \geq 1$.

Combining the approach developed in [1] with the foregoing results, we can easily prove:

Theorem 4 If a variety $V$ satisfies (*) of the Proposition for some integers $m, k$, then $k$ is the smallest integer for which $V$ is $(k+1)$-permutable and $V$ has $m$-transferable congruences with respect to $g_1, \ldots, g_n$.

Let us remark that if $g_i(z) = \cdots = g_n(z) = z$ then $V$ has $m$-transferable congruences, i.e. $\forall A \in V$ and for each $a, b, d \in A$ there exist $c_1, \ldots, c_m \in A$ with

$$\Theta(a, b) = \Theta(d, c_1, \ldots, c_m).$$

If $g_1(z) = \cdots = g_n(z) = 0$ then $V$ has $m$-transferable congruences at 0, i.e. for each $A \in V$, any $a, b \in A$ there are $c_1, \ldots, c_m \in A$ with $\Theta(a, b) = \Theta(0, c_1, \ldots, c_m)$. Hence, a variety $V$ is regular (or 0-regular) if and only if $V$ has $m$-transferable congruences (at 0, respectively) for some integer $m \geq 1$. 

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References


