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Coordinatization of Projective Planes by Special Planar Ternary Rings

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Abstract

Planar ternary rings under consideration lie between general ones ([1]) and natural ones ([3]). The aim of the present paper is to find algebraic counterparts to various transitivities of convenient collineation subgroups.

Key words: Projective plane, flag, planar ternary ring, coordinatization, algebraic description of transitivities.

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1 Admissible planar ternary rings

Our starting point is the notion of a planar ternary ring: An ordered couple $(M, t)$ consisting of a set $M, \#M \geq 2$ and a ternary operation $t$ on $M$ is said to be a planar ternary ring (PTR) if it satisfies following conditions:

(A1) $\forall x, m, y \in M \exists! b \in M: t(x, m, b) = y$;
(A2) $\forall m, b, \bar{m}, \bar{b} \in M, m \neq \bar{m} \exists! x \in M: t(x, m, b) = t(x, \bar{m}, \bar{b})$;
(A3) $\forall x, y, \bar{x}, \bar{y} \in M, x \neq \bar{x} \exists! (m, b) \in M \times M: t(x, m, b) = y \wedge t(\bar{x}, m, b) = \bar{y}$.

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A PTR \((M, t)\) is said to be admissible (APTR) if

(A4) there is an element \(0_L \in M\) and a permutation \(*: b \mapsto b^*\) of \(M\) such that for all \(m, b \in M\) the equality

\[ t(0_L, m, b^*) = b \]

holds and moreover the following condition is fulfilled:

(A5) \(\forall a \in M, a \neq 0_L \; \exists! n_a \in M \forall b \in M: t(a, n_a, b^*) = b\).

Replacing (A4) and (A5) by

(A) there are elements \(0_L, 0_R\) and a permutation \(*: b \mapsto b^*\) of \(M\) such that for all \(m, b, x \in M\) the equalities

\[ t(0_L, m, b^*) = b \quad \text{and} \quad t(x, 0_R, b^*) = b \]

hold, we get a natural PTR (NPTR). Any NPTR is a special case of an APTR. In fact, it satisfies the condition \(n_a = 0_R\) for all \(a \in M \setminus \{0_L\}\).

The element \(0_L\) from (A4) is uniquely determined ([4], proposition 2.3) and is called the left quasizero of the given APTR \((M, t)\). In the sequel we will write briefly \(0\) instead of \(0_L\). For any \(a, b, c \in M, a \neq 0\), where \((M, t)\) is an APTR there exists just one \(x \in M\) such that \(t(a, x, b) = c\) (see [4], proposition 2.3). Hence for any \(a \in M \setminus \{0\}\) there is exactly one \(e_a \in M\) such that \(t(a, e_a, 0^*) = a\) is valid. When \(a = 0\) then we put \(e_a = 0\). Now we are able to define two binary operations \((a, b) \mapsto a + b\) (addition) and \((a, b) \mapsto a \cdot b\) (multiplication) on \(M\) such that

\[ a + b = t(a, e_a, b^*), \quad a \cdot b = t(a, b, 0^*). \]

Further we recall some fundamental properties of both operations + and · ([4], proposition 2.4):

(a) \(\forall a \in M:\) \quad a + 0 = 0 + a = a;
(b) \(\forall a, b \in M \; \exists! x \in M:\) \quad a + x = b,
   hence \(\forall a, x, y \in M:\) \quad a + x = a + y \implies x = y;
(c) \(\forall a \in M:\) \quad 0 \cdot a = 0, a \cdot n_a = 0;
(d) \(\forall a, b \in M, a \neq 0 \; \exists! x \in M:\) \quad a \cdot x = b,
   thus \(\forall a, x, y \in M, a \neq 0:\) \quad a \cdot x = a \cdot y \implies x = y;
(e) \(\forall a \in M:\) \quad a \cdot e_a = a.

If \(a \cdot x = b\) and \(a \neq 0\) we will write \(x = a \backslash b\). Thus we have \(a \cdot (a \backslash b) = b\) for all \(a, b \in M, a \neq 0\).

2 Coordinatization of projective planes by planar ternary rings

Consider a projective plane \(P = (U, L, e)\) and call a flag every couple consisting of a point and a line through this point. A projective plane together with a
distinguished flag \((V, n)\) will be denoted by \(P(V, n)\). Points of \(U \setminus n\) are said to be affine and those of \(n\) ideal. For any ideal point \(N\) the set \((N)\) of all lines containing \(N\) is said to be a direction. Especially the direction \((V)\) is called vertical and lines of \((V)\) are called vertical too. All the remaining directions are said to be skew and lines not going through \(V\) are said also to be skew.

Let \(A\) denote the set of all affine points and \(B\) the set of all skew lines. As it is well known the equality

\[
\text{card } A = \text{card } B = (\text{ord } P)^2
\]

is valid.

Now investigate a \(\text{PTR } (M, t)\) with \(\text{card } M = \text{ord } P\). By a frame of \(P\) we understand a couple \(S\) of bijections

\[
M \times M \rightarrow A, \ (x, y) \mapsto (x, y)_S \quad \text{and} \quad M \times M \rightarrow B, \ (m, b) \mapsto [m, b]_S
\]

such that

\[
y = t(x, m, b) \iff (x, y)_S \in [m, b]_S
\]

for all \(x, y, m, b \in M\).

We see that for all \(a \in M\) the set

\[
[a]_S = \{(x, y)_S \in A \mid x = a\} \cup \{V\}
\]

is a vertical line different from \(n\). Dually, for all \(u \in M\) the set

\[
(u)_S = \{[m, b]_S \in B \mid m = u\} \cup \{n\}
\]

is a direction different from \((V)\). Thus we have two bijections \(M \rightarrow (V) \setminus \{n\}, \ a \mapsto [a]_S\) and \(M \rightarrow n \setminus (V), \ u \mapsto (u)_S\), where \((u)_S\) denotes also the corresponding ideal point of the direction considered. We conclude that

\[
[m, b]_S = \{(x, y)_S \in A \mid y = t(x, m, b)\} \cup \{(m)_S\}
\]

for all \(m, b \in M\).

Remark that in the case of an \(\text{APTR } (M, t)\) we have a distinguished vertical line \(v = [0]_S\). Now let \([m, b]_S, [\bar{m}, \bar{b}]_S\) be distinct skew lines and denote by \(c, \bar{c}\) the elements such that \(c^* = b, \ \bar{c}^* = \bar{b}\). Assuming \([m, b]_S, [\bar{m}, \bar{b}]_S\) have a common point on \(v\) we get \(c = t(0, m, b) = \bar{c} = t(0, \bar{m}, \bar{b})\) and consequently \(b = \bar{b}\). Conversely, if \(b = \bar{b}\) then \(c = \bar{c}\) and \(t(0, m, b) = c = \bar{c} = t(0, \bar{m}, \bar{b})\) so that \((0, c)_S\) is a common point of both lines. We can formulate the result as

**Theorem 1** Two distinct skew lines \([m, b]_S, [\bar{m}, \bar{b}]_S\) have a common point on the vertical axis iff \(b = \bar{b}\).
3 Transitivities

First recall some important notions and results concerning transitivities of central collineations groups. Let Q be a point and q a line of a given projective plane \( P(V,n) \). Denote by \( G(Q,q) \) the group consisting of all collineations of \( P(V,n) \) which fix every line through Q and every point of q. (Q is the centre and q the axis of the collineation under consideration). If \( Q \notin q \) we have a homology and if \( Q \in q \) we have an elation. A projective plane is said to be \((Q,q)\)-transitive if for all lines \( l \neq q \), \( Q \in l \) \( G(Q,q) \) operates transitively on \( l \backslash \{Q, l \wedge q\} \). Necessary and sufficient for \( P(V,n) \) to be \((Q,q)\)-transitive is the existence of a line \( l \neq q \), \( Q \in l \) and a point \( P \in l \), \( P \neq Q \), \( P \notin q \) such that every point \( P' \in l , P' \neq Q, P' \notin q \) there is an \( \kappa \in G(Q,q) \) with \( \kappa : P \mapsto P' \).

If \( q \) is a line of \( P(V,n) \) we say that \( P(V,n) \) is \( q\)-transitive if it is \((Q,q)\)-transitive for any \( Q \in q \). If we denote by \( G(q) \) the group of all collineations fixing all points of \( q \), then \( P(V,n) \) is \( q\)-transitive iff the group \( G(q) \) operates transitively on the set \( U \backslash q \) (\( U \) is the set of all points of \( P(V,n) \)). \( P(V,n) \) is \( q\)-transitive iff it is \((Q,q)\)-transitive and \((Q,q)\)-transitive for distinct points \( R, Q \in q \). In the case \( G(q) = G(Q,q) \oplus G(R,q) \), the group \( G(q) \) is abelian. Dually, let \( Q \) be a point of \( P(V,n) \). We say that \( P(V,n) \) is \( Q\)-transitive if it is \((Q,q)\)-transitive and \((Q,q)\)-transitive for distinct points \( R, Q \in q \). The elation (homology) of \( P(V,n) \) whose axis is the line \( n \) is said to be a translation (a homology) of \( P(V,n) \). The \((n,q)\)-transitive plane is called vertically transitive plane, the \( n\)-transitive plane called also translation plane. The translation plane \( P(V,n) \) is desarguesian iff there exists an affine point \( P \) such \( P(V,n) \) is also \((P,n)\)-transitive. The desarguesian plane \( P(V,n) \) is pappian if for all lines \( q \) and all points \( Q \notin q \) the group \( G(Q,q) \) is abelian. If there exists for a \( q\)-transitive plane \( P(V,n) \) a point \( Q \notin q \) such that \( P(V,n) \) is \((Q,q)\)-transitive and the group \( G(Q,q) \) is abelian then \( P(V,n) \) is pappian. Especially a translation plane \( P(V,n) \) is pappian iff there exists an affine point \( P \) such that \( G(V,n) \) is \((P,n)\)-transitive and the group \( G(P,n) \) is abelian.

4 APTR's of vertically transitive planes and of translation planes

Here we recall some results concerning the APTR's coordinatizing a \((V,n)\)-transitive or an \( n\)-transitive projective plane \( P(V,n) \). In what follows we assume that the given projective plane \( P(V,n) \) is coordinatized by an APTR \((M,t)\).

Theorem 2 A \( P(V,n) \) is vertically transitive iff

(a) \( \forall a,b,c \in M : \ a + (b + c) = (a + b) + c \ and \)
(b) \( \forall x,m,b \in M : \ t(x,m,b^*) = x \cdot m + b^* \quad ((M,t) \ is \ linear) \).

Remark: If \( P(V,n) \) is vertically transitive then \((M,+)\) is a group.
Theorem 3  A vertically transitive plane $P(V, n)$ is a translation plane iff for any $a, b, c \in M, b \neq 0$ the equation
\[ c \cdot m - b \cdot m - a \cdot m = c \cdot n_b - a \cdot n_b \] (1)
has either just one solution $m = n_b$ or is fulfilled identically.

Remark: If $P(V, n)$ is a translation plane then the group $(M, +)$ is abelian.

5 APTR’s of V-transitive planes

Suppose $P(V, n)$ is a vertically transitive plane. Then $P(V, n)$ is $V$-transitive iff it is $(V, v)$-transitive ($v$ is the vertical axis $[0]$). Any $(V, n)$-transitive plane $P(V, n)$ is $(V, v)$-transitive iff for any $d, a \in M$ there exists an elation $\epsilon \in G(V, v)$ such that $\epsilon : (d)_S \mapsto (a)_S$.

Theorem 4 A vertically transitive plane $P(V, n)$ is $V$-transitive iff for any $a, b, c, d \in M$ the equation
\[ m \cdot a - m \cdot d = m \cdot c - m \cdot b \] (1)
has only trivial solution ($m = 0$) or is fulfilled identically.

Proof  Assume that the given $(V, n)$-transitive plane $P(V, n)$ is $(V, v)$-transitive and that for given $a, b, c, d, \tilde{m} \in M, \tilde{m} \neq 0$ the equality
\[ \tilde{m} \cdot a - \tilde{m} \cdot d = \tilde{m} \cdot c - \tilde{m} \cdot b \] (2)
holds. Then there exists an $\epsilon \in G(V, v)$ such that $\epsilon((a)_S) = ((d)_S)$. Let $(c')_S = \epsilon((b)_S)$. If $m$ is an arbitrary non left-quasizero element of $M$ then $\epsilon$ maps $[d, 0]_S$ onto $[a, 0]_S$ and $\epsilon : (m, m \cdot d)_S \mapsto (m, m \cdot a)_S$. As $(m, m \cdot d)_S \in [b, (-m \cdot b + m \cdot d)^*_S]$ we have $(m, m \cdot a)_S \in [c', (-m \cdot b + m \cdot d)^*_S]$. Therefore
\[ m \cdot a - m \cdot d = m \cdot c' - m \cdot b \] (3)
(for any $m \in M \setminus \{0\}_S$). Especially for $m = \tilde{m}$ we have
\[ \tilde{m} \cdot a - \tilde{m} \cdot d = \tilde{m} \cdot c' - \tilde{m} \cdot b. \] (4)
Comparing (2) with (4) we obtain $c = c'$ and consequently for all $m \in M$ (1) is satisfied.

Let $P(V, n)$ be $(V, n)$-transitive plane and let the coordinatizing APTR $(M, t)$ satisfy the condition of the theorem. If $\tilde{m} \in M \setminus \{0\}$ and $d, a \in M$ then define a mapping $U : M \rightarrow M, u \mapsto u'$ by
\[ u' = U(u) \iff \tilde{m} \cdot a - \tilde{m} \cdot d = \tilde{m} \cdot u' - \tilde{m} \cdot u, \] (5)
\( \mathcal{U} \) is a permutation of \( \mathcal{M} \). Now define the map \( \epsilon \) of \( \mathbb{P}(V, n) \) onto itself by

\[
\forall (x, y)_s \in \mathcal{A} \quad \epsilon((x, y)_s) = (x \cdot a - x \cdot d + y)_s; \\
(u)_s \in \mathcal{M} \setminus V \quad \epsilon((u)_s) = (u')_s, \ u' = \epsilon(u); \\
\epsilon(V) = V.
\]

\( \epsilon \) is map of \( \mathbb{P}(V, n) \) onto itself carrying every affine point onto an affine point and fixing all vertical lines and all points of the vertical axis. In addition there holds \( \epsilon((d)_s) = (a)_s \) (as \( \mathcal{U}(d) = a \)). Let us have skew lines \( 1 = [u, q^*]_s, 1' = [u', q^*]_s \) and let \( (x, y)_s \) be an affine point. According to our supposition we get from (5) also

\[
x \cdot a - x \cdot d = x \cdot u' - c \cdot u.
\]

As \( (x, y)_s \in 1 \iff y = x \cdot u + q \iff x \cdot a - x \cdot d + y = x \cdot a - x \cdot d + q \iff x \cdot a - x \cdot d + y = x \cdot u' + q \iff (x, x \cdot a - x \cdot d + y)_s \in 1' \iff \epsilon((x, y)_s) \in 1', \ epsilon \) is a collineation.

6 APTR’s of desarguesian planes

**Theorem 5** If a translation plane \( \mathbb{P}(V, n) \) is also \( V \)-transitive then it is desarguesian iff the corresponding \((\mathcal{M}, \mathcal{T})\) satisfies the condition \((P)\) for all \( u, u, x, x \in \mathcal{M}\setminus \{0\}:

\[
x \setminus (x \cdot m - u \cdot m + u \cdot r) = \bar{x} \setminus (\bar{x} \cdot m - \bar{u} \cdot m + \bar{u} \cdot r) \tag{1}
\]

either admits just one solution \( m = r \) or is fulfilled for all \( m, r \in \mathcal{M} \).

**Proof** (i) Let \( \mathbb{P}(V, n) \) be desarguesian. For given \( u, \bar{u}, x, x \in \mathcal{M}\setminus \{0\} \) let there exist different \( \bar{m}, \bar{r} \) satisfying

\[
x \setminus (x \cdot \bar{m} - u \cdot \bar{m} + u \cdot \bar{r}) = \bar{x} \setminus (\bar{x} \cdot \bar{m} - \bar{u} \cdot \bar{m} + \bar{u} \cdot \bar{r}). \tag{2}
\]

Investigate a homology \( \kappa \in G(\mathbb{P}, n), P = (0, 0)_s \) carrying \([u]_s\) onto \([\bar{u}]_s\) and \([x]_s\) onto \([x']_s\). Let \( m, r \) be distinct elements of \( \mathcal{M} \). Since the line \([m, 0]_s\) is fixed under \( \kappa \), it follows that

\[
\kappa((u, u \cdot m)_s) = (\bar{u}, \bar{u} \cdot m)_s, \quad \kappa((x, x \cdot m)_s) = (x', x' \cdot m)_s. \tag{3}
\]

The lines \([r, (u \cdot m - u \cdot r)^*]_s, [r, (\bar{u} \cdot m - \bar{u} \cdot r)^*]_s\) belong to the same direction \((r)_s\) and contain the points \((u, u \cdot m)_s\) and \((\bar{u}, \bar{u} \cdot m)_s\), respectively. Hence \( \kappa([r, (u \cdot m - u \cdot r)^*]_s) = [r, (\bar{u} \cdot m - \bar{u} \cdot r)^*]_s\) and consequently

\[
\kappa: (0, (u \cdot m - u \cdot r)^*)_s = (0, (\bar{u} \cdot m - \bar{u} \cdot r)^*)_s. \tag{4}
\]

Assume \( \kappa \in \mathcal{M} \) to be such that \([k, (u \cdot m - u \cdot r)^*]_s\) contains the point \((x, x \cdot m)_s\). We get \( \kappa([k, (u \cdot m - u \cdot r)^*]_s) = [k, (\bar{u} \cdot m - \bar{u} \cdot r)^*]_s\) so that \((x', x' \cdot m)_s \in [k, (\bar{u} \cdot m - \bar{u} \cdot r)^*]_s\). This means that

\[
x \cdot m = x \cdot k + u \cdot m - u \cdot r, \\
x' \cdot m = x' \cdot k + \bar{u} \cdot m - \bar{u} \cdot r.
\]
Eliminating $k$ we get
\[ x \cdot (x \cdot m - u \cdot m + u \cdot r) = x' \cdot (x' \cdot m - u' \cdot m + u' \cdot r). \]  
(5)

Since (5) is true especially for $m = \bar{m}$, $r = \bar{r}$, we obtain
\[ x \cdot (x \cdot \bar{m} - u \cdot \bar{m} + u \cdot \bar{r}) = x' \cdot (x' \cdot \bar{m} - u' \cdot \bar{m} + u' \cdot \bar{r}). \]  
(6)

Rewriting (2) and (6) as
\[ \bar{x} \cdot (x \cdot (x \cdot m - u \cdot m + u \cdot r)) = \bar{x} \cdot (x' \cdot m - u' \cdot m + u' \cdot r), \]
\[ x' \cdot (x \cdot (x \cdot m - u \cdot m + u \cdot r)) = x' \cdot (x' \cdot m - u' \cdot m + u' \cdot r) \]
and using $x \cdot (x \cdot \bar{m} - u \cdot \bar{m} + u \cdot \bar{r}) \neq \bar{m}$ we reach $\bar{x} = x'$. Hence (1) is true for all $m, r \in M$.

(ii) Let $P(V, n)$ be a $V$-transitive translation plane and let its $\text{APTR}(M, t)$ have the property $(P)$. For given vertical lines $[u]_s, [\bar{u}]_s$ different from vertical axis $u, \bar{u}$ are non-zero elements. Choosing different elements $\bar{m}, \bar{r} \in M$ we may define a map $\mathcal{U}$ as follows:
\[ \forall x, \bar{x} \in M \setminus \{0\} : \bar{x} = \mathcal{U}(x) \iff \bar{x} \cdot (x \cdot (x \cdot \bar{m} - u \cdot \bar{m} + u \cdot \bar{r})) = \bar{x} \cdot (x \cdot m - u \cdot m + u \cdot r), \quad \mathcal{U}(0) = 0. \]  
(7)

According to $(P)$ it follows that
\[ \bar{x} \cdot (x \cdot (x \cdot m - u \cdot m + u \cdot r)) = \bar{x} \cdot (x \cdot m - u \cdot m + u \cdot r). \]  
(8)

for all $m, r \in M$.

Take an $\bar{s} \in M$ and define a further map $\mathcal{V}$ of $M$ onto $M$ with help of
\[ \forall q, \bar{q} \in M : \bar{q} = \mathcal{V}(q) \iff \bar{q} = \bar{u} \cdot \bar{s} - \bar{u} \cdot (u \cdot (u \cdot s - q)). \]  
(9)

Here we have $\mathcal{V}(0) = 0$ and if $s$ is an arbitrary element of $M$ then for
\[ a = u \cdot (u \cdot s - q), \quad b = u \cdot (u \cdot \bar{s} - q) \]  
(10)
we obtain
\[ u \cdot a = u \cdot s - q, \quad u \cdot b = u \cdot \bar{s} - q \]
and consequently
\[ u \cdot a - u \cdot s = u \cdot b - u \cdot \bar{s}. \]  
(11)

According to theorem 3, we obtain
\[ \bar{u} \cdot a - \bar{u} \cdot s = \bar{u} \cdot b - \bar{u} \cdot \bar{s} \]
and consequently
\[ \bar{u} \cdot b = u \cdot a - u \cdot s + \bar{u} \cdot \bar{s}. \]  
(12)
Using (9), (10), (12) and (9) we get

\[
\bar{q} = \bar{u} \cdot \bar{s} - \bar{u} \cdot (u \backslash (u \cdot \bar{s} - q)) = \bar{u} \cdot \bar{s} - \bar{u} \cdot \bar{b} = \\
\bar{u} \cdot \bar{s} - \bar{u} \cdot a + \bar{u} \cdot s - \bar{u} \cdot s = \bar{u} \cdot s - \bar{u} \cdot a = \bar{u} \cdot s - \bar{u} \cdot (u \backslash (u \cdot s - q)).
\]

Thus if there is an \( \bar{s} \in M \) such that \( \bar{q} = \bar{u} \cdot \bar{s} - \bar{u} \cdot (u \backslash (u \cdot \bar{s} - q)) \) then for any \( s \in M \)

\[
\bar{q} = \bar{u} \cdot s - \bar{u} \cdot (u \backslash (u \cdot s - q)) \tag{13}
\]

is true.

Now if \( \bar{x} = U(x) \), \( x \neq 0 \) and \( c = u \backslash (u \cdot s - q) \) then \( u \cdot c = u \cdot s - q \) and \( \bar{q} = \bar{u} \cdot s - \bar{u} \cdot c \). Using (P) and (8) we obtain for \( m = s \) and \( r = c \) that

\[
\bar{x} \cdot (x \backslash (x \cdot s - u \cdot s + u \cdot c)) = \bar{x} \cdot s - \bar{u} \cdot s + \bar{u} \cdot c,
\]
\[
\bar{x} \cdot (x \backslash (x \cdot s - q)) = \bar{x} \cdot s - \bar{q}
\]

and finally

\[
\bar{q} = \bar{x} \cdot s - \bar{x} \cdot (x \backslash (x \cdot s - q)). \tag{14}
\]

We obtain a result: (13) and \( \bar{x} = U(x) \) imply (14).

Take an \( \bar{t} \in M \) and define third map \( \mathcal{W} \) of \( M \) onto \( M \) by

\[
\forall y, y^x \in M: \quad y^x = \mathcal{W}(y) \iff y^x = \bar{u} \cdot \bar{t} + \bar{x} \cdot \bar{t} - \bar{u} \cdot (u \backslash (u \cdot \bar{t} + x \cdot \bar{t} - y)). \tag{15}
\]

We will prove that for all \( t \in M \) there holds

\[
y^x = \bar{u} \cdot t + \bar{x} \cdot t - \bar{u} \cdot (u \backslash (u \cdot t + x \cdot t - y)). \tag{16}
\]

If \( x = 0 \) then also \( \bar{x} = 0 \) and \( y^x = \mathcal{V}(y) \). Then we can state that for all \( t \in M \)

\[
y^x = \bar{u} \cdot t - \bar{u} \cdot (u \backslash (u \cdot t - y))
\]

holds true.

Now let \( x \neq 0 \) and \( p, q \) be elements of \( M \) satisfying

\[
x \cdot \bar{t} + p = y; \quad x \cdot t + q = y. \tag{17}
\]

Denoting \( \bar{p} = \mathcal{V}(p), \bar{q} = \mathcal{V}(q) \) we obtain

\[
\bar{p} = \bar{x} \cdot s - \bar{x} \cdot (x \backslash (x \cdot s - p)), \tag{18}
\]
\[
\bar{q} = \bar{x} \cdot s - \bar{x} \cdot (x \backslash (x \cdot s - q)) \tag{19}
\]

for some \( s \in M \) and consequently for all \( s \in M \). Putting \( \alpha = x \backslash p, \beta = x \backslash q \) and replacing \( s \) by \( \alpha \) in (18) as well as in (19) we get

\[
\bar{p} = \bar{x} \cdot \alpha - \bar{x} \cdot n_x, \quad \bar{q} = \bar{x} \cdot \beta - \bar{x} \cdot n_x. \tag{20}
\]

As \( p = x \cdot \alpha \) and \( q = x \cdot \beta \), we obtain by (17)

\[
x \cdot \bar{t} + x \cdot \alpha = x \cdot t + x \cdot \beta.
\]
Hence
\[\bar{x} \cdot \bar{t} + \bar{x} \cdot \alpha = \bar{x} \cdot t + \bar{x} \cdot \beta\]
and consequently
\[\bar{x} \cdot \bar{t} + \bar{x} \cdot \alpha - \bar{x} \cdot n_x = \bar{x} \cdot t + \bar{x} \cdot \beta - \bar{x} \cdot n_x. \tag{21}\]
According to (20) we have
\[\bar{x} \cdot \bar{t} + \bar{p} = \bar{x} \cdot t + \bar{q}. \tag{22}\]
Using (Q) we obtain
\[\bar{p} = \bar{u} \cdot \bar{t} - \bar{u} \cdot (u \backslash (u \cdot \bar{t} - p)) \quad \text{and} \quad \bar{q} = \bar{u} \cdot t - \bar{u} \cdot (u \backslash (u \cdot t - q)). \tag{23}\]
Now it follows from (15), (22) and (23) that
\[y^x = \bar{u} \cdot \bar{t} + \bar{x} \cdot \bar{t} - \bar{u} \cdot (u \backslash (u \cdot \bar{t} + x \cdot \bar{t} - y)) = \bar{u} \cdot \bar{t} + \bar{x} \cdot \bar{t} - \bar{u} \cdot (u \backslash (u \cdot \bar{t} - p)) = \bar{x} \cdot \bar{t} + (\bar{u} \cdot \bar{t} - \bar{u} \cdot (u \backslash (u \cdot \bar{t} - p))) = \bar{x} \cdot \bar{t} + \bar{q} = \bar{x} \cdot t + (\bar{u} \cdot t - \bar{u} \cdot (u \backslash (u \cdot t - q))) = \bar{u} \cdot t + \bar{x} \cdot t - \bar{u} \cdot (u \backslash (t + x \cdot t - y)).\]
Hence (16) is true.

Further let us define a map \(\kappa\) of \(P(V, n)\) onto itself by
[a] \(\forall (x, y) \in M \times M, x \neq 0 \quad \kappa((x, y)_S) = (\bar{x}, y^x)_S, \) where \(\bar{x} = U(x), y^x = V(y),\)
[b] \(\forall y \in M \quad \kappa((0, y)_S) = (0, \bar{y})_S, \) where \(\bar{y} = V(y),\)
[c] \(\forall u \in M \quad \kappa((u)_S) = (u)_S, \) and
[d] \(\kappa(V) = V.\)
Evidently \(\kappa\) is bijective and all ideal points together with \(P = (0, 0)_S\) are fixed under \(\kappa.\) Moreover any vertical line \([z]_S\) is carried onto the vertical line \([\bar{z}]_S,\) where \(\bar{z} = U(z).\) Especially we have \(\kappa([0]_S) = [0]_S, \kappa([u]_S) = [\bar{u}]_S.\) It remains to prove that the image of every skew line is a skew line of the same direction. Thus consider a skew line \(l = [h, q^*]_S\) and denote \(l' = [h, \bar{q}^*]_S \quad (\bar{q} = V(q)).\) Evidently \(\kappa((0, q)_S) = (0, \bar{q})_S\) so that the image of \((0, q)_S \in l\) is the point \((0, \bar{q})_S \in l'.\) Now let \((x, y)_S\) be an affine point lying not on the vertical axis. If \((x, y)_S \in l,\) then \(y = x \cdot h + q.\) We know that
\[y^x = \bar{x} \cdot h + \bar{u} \cdot h - \bar{u} \cdot (u \backslash (u \cdot h + x \cdot h - y)). \tag{24}\]
Thus \(y^x = \bar{x} \cdot h + \bar{u} \cdot h - \bar{u} \cdot (u \backslash (u \cdot h - q)) = \bar{x} \cdot h + (\bar{u} \cdot h - \bar{u} \cdot (u \backslash (u \cdot h - q))) = \bar{x} \cdot h + \bar{q} \implies (\bar{x}, y^x)_S \in l'.\]
Conversely, let \((\bar{x}, y^x)_S \in l', \bar{x} \neq 0.\) As \(y^x = \bar{x} \cdot h + \bar{q},\) we have
\[y^x = \bar{x} \cdot h + (\bar{u} \cdot h - \bar{u} \cdot (u \backslash (u \cdot h - q))). \tag{25}\]
On the other side, we have
\[y^x = \bar{u} \cdot h + \bar{x} \cdot h - \bar{u} \cdot (u \backslash (u \cdot h + x \cdot h - y)). \tag{26}\]
Comparing (25) with (26) yields
\[u \cdot h - q = u \cdot h + x \cdot h - y \quad \text{and} \quad y = x \cdot h + q,\]
which means that \((x, y)_S \in l.\) Therefore we have proved that \(\kappa \in G(P, n).\)
7 APTR's of pappian planes

Theorem 6 A desarguesian plane $P(V, n)$ is pappian iff its APTR $(M, t)$ satisfies the condition

$$\forall a, b, c, d \in M, \quad b \neq 0:\ 
a \cdot n_b - a \cdot (b \langle -c \cdot n_b + c \cdot d \rangle) = c \cdot n_b - c \cdot (b \langle -a \cdot n_b + a \cdot d \rangle).$$

(1)

Proof Consider the group $G(P, n)$ where $P = (0, 0)_S$. Then $G(P, n)$ is abelian iff for any two homologies $\kappa, \rho \in G(P, n)$ there exists an affine point $Y = (0, y)_S$, $y \neq 0$, such that $(\rho \circ \kappa)(Y) = (\kappa \circ \rho)(Y)$. Let $a, b, c, d$ be given elements of $M, b \neq 0$. We may assume that $a \neq 0, c \neq 0, d \neq n_b$.

I. Let $P(V, n)$ be pappian and $\kappa, \rho$ homologies from $G(P, n)$ carrying the vertical line $[b]_S$ onto $[a]_S$ or $[c]_S$, respectively. Consider an arbitrary point $(0, y)_S, y \neq 0$. If $(0, y_1) = \kappa((0, y)_S)$ and $(0, y_2) = \kappa((0, y)_S)$ then

$$y_1 = a \cdot s - a \cdot (b \langle b \cdot s - y \rangle), \quad y_2 = c \cdot t - c \cdot (b \langle b \cdot t - y \rangle).$$

(2)

We know that if (2) is true for some $s \in M$ (for some $t \in M$) then it is true for all $s \in M$ (for all $t \in M$). Thus putting $s = t = n_b$, we have

$$y_1 = c \cdot n_b - a \cdot (b \langle -y \rangle), \quad y_2 = c \cdot n_b - c \cdot (b \langle -y \rangle).$$

(3)

Similarly, denoting $(0, y_3)_S = \rho((0, y_1)_S)$ and $(0, y_4)_S = \kappa((0, y_2)_S)$, we obtain

$$y_3 = c \cdot n_b - c \cdot (b \langle -y_1 \rangle), \quad y_4 = a \cdot n_b - c \cdot (b \langle -y_2 \rangle).$$

(4)

As $\rho \circ \kappa = \kappa \circ \rho$, we have

$$y_3 = y_4.$$ 

(5)

Now choose $y = -(b \cdot d)$. Then $y_1 = a \cdot n_b - a \cdot d$, $y_2 = c \cdot n_b - c \cdot d$ and furthermore

$$y_3 = c \cdot n_b - c \cdot (b \langle -a \cdot n_b + a \cdot d \rangle), \quad y_4 = a \cdot n_b - a \cdot (b \langle -c \cdot n_b + c \cdot d \rangle).$$

(6)

Thus (5) and (6) imply (1).

II. Conversely let (1) be true. Let us take two homologies $\kappa, \rho \in G(P, n)$ and suppose that $\kappa([b]_S) = [a]_S, \rho([b]_S) = [c]_S$. As in the first part we find that

$$(\rho \circ \kappa)((0, -(b \cdot d))_S) = (0, c \cdot n_b - c \cdot (b \langle -a \cdot n_b + a \cdot d \rangle))_S,$$

$$(\kappa \circ \rho)((0, -(b \cdot d))_S) = (0, a \cdot n_b - a \cdot (b \langle -c \cdot n_b + c \cdot d \rangle))_S.$$ (1) implies that both points are equal so that $\rho \circ \kappa = \kappa \circ \rho$.

References


