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Filters and Annihilators in Implication Algebras

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Abstract

The concept of filter in implication algebra is characterized in term operations and also in lattice operation. The set of all filters of an implication algebra forms a complete lattice whose boolean elements are annihilators. The set of all annihilators forms a Boolean algebra.

Key words: Implication algebra, filter, annihilator, pseudocomplement.

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The concept of implication algebra was introduced by J. C. Abbott [1], see also W. C. Nemitz [4]. A groupoid $(A, \cdot)$ is an *implication algebra* if it satisfies the following identities:

(i)	$(ab)\,a = a$	(contraction)
(ii)	$(ab)\,b = (ba)\,a$	(quasi-commutativity)
(iii)	$a\,(bc) = b\,(ac)$	(exchange).

Hence, the class of all implication algebras forms a variety.

The study of implication algebras was motivated by the fact that for every Boolean algebra $\mathcal{B} = (B;\ \vee, \wedge, ', 0, 1)$, the groupoid $(B, \rightarrow)$ where $a \rightarrow b = a' \vee b$ is an implication algebra. Hence, implication algebras describe properties of the connective implication in logic (not necessary in a classical logic).

The following concepts was introduced by J. C. Abbott [1]:

Definition 1 A nonvoid subset I of an implication algebra $\mathcal{A} = (A, \cdot)$ is called a *filter* if for each $b_1, b_2, b \in I$ and every $x \in A$ we have

(a) $xb \in I$ and

(b) whenever $b_1 \wedge b_2$ exists in $\mathcal{A}$ then $b_1 \wedge b_2 \in I$.

We must explain the symbol $\wedge$ in Definition 1. For this, let us repeat some necessary results of [1]:

Lemma 1 *Let $\mathcal{A} = (A, \cdot)$ be an implication algebra. Then*

(i) *for any $a, b \in A$, $aa = bb$, i.e. there exists a nullary term denoted by* 1 *such that $a \cdot a = 1$ is the identity of $\mathcal{A}$;*

(ii) *for each $a \in A$, $1 \cdot a = a$, $a \cdot 1 = 1$.*

Let us introduce the relation $\leq$ by setting $a \leq b$ if and only if $a \cdot b = 1$.

Lemma 2 *Let $\mathcal{A} = (A, \cdot)$ be an implication algebra. Then $(A, \leq)$ is a $\vee$–semilattice with respect to $\leq$ with the greatest element 1, where*

$$a \vee b = (ab)\, b.$$

If for $a, b \in A$ there exists $p \in A$ such that $p \leq a$, $p \leq b$ then there exists an infimum $a \wedge b$ (w.r.t. $\leq$) and $a \wedge b = (a\,(b \cdot p))\,p$.

For the proof, see Theorems 3, 4 and 5 in [1].

From this point of view, the definition of filter of implication algebra need not be suitable in all cases since the condition (a) is formulated in term operation of $\mathcal{A} = (A, \cdot)$ but (b) contains a partial operation $\wedge$ which is not a term operation of $\mathcal{A}$. To avoid this discrepancy, we prove the following:

Theorem 1 *A nonvoid subset I of an implication algebra $\mathcal{A} = (A, \cdot)$ is a filter if and only if for each $a \in A$ and every $b, b_1, b_2 \in I$ we have*

$$(a) \quad ab \in I \qquad \text{and} \qquad (c) \quad (b_1\,(b_2 a))\, a \in I.$$

Proof Let $\emptyset \neq I \subseteq A$. Suppose that (a), (c) hold. Let $b_1, b_2 \in I$ and $b_1 \wedge b_2$ exist. Denote by $p = b_1 \wedge b_2$. By Lemma 2, $b_1 \wedge b_2 = (b_1(b_2 p))\,p$ which belongs to I by (c). Hence, $b_1 \wedge b_2 \in I$ proving (b), i.e. I is a filter of $\mathcal{A}$.

Conversely, let I be a filter of $\mathcal{A}$. By Theorem 10 [1], I is a kernel of some congruence θ_I on $\mathcal{A}$, i.e. $b \in I$ if and only if $\langle b, 1\rangle \in \theta_I$. Suppose $a \in A$, $b_1, b_2 \in I$. Then $\langle a, a\rangle \in \theta_I$ and $\langle b_1, 1\rangle \in \theta_I$, $\langle b_2, 1\rangle \in \theta_I$ whence by Lemma 1:

$$\langle (b_1\,(b_2 a))\, a, 1\rangle = \langle (b_1\,(b_2 a))\, a, (1\,(1a))\, a\rangle \in \theta_I$$

i.e. $(b_1\,(b_2 a))\, a \in I$. Thus I satisfies (a) and (c) of Theorem 1. □

Corollary 1 *The set* $Fil\,\mathcal{A}$ *of all filters on an implication algebra* $\mathcal{A}$ *forms a complete lattice with respect to set inclusion. The least element is* $\{1\}$*, the greatest element is* A *and the operation meet in* $Fil\,\mathcal{A}$ *coincides with set-theoretical intersection.*

Proof It is almost trivial to show that if $\mathcal{S}$ is system of filters of $\mathcal{A}$ then also its intersection satisfies (a), (c) of Theorem 1. □

Applying the foregoing Corollary 1 we see that for any subset M of $\mathcal{A} = (A, \cdot)$ there exists the least filter of $\mathcal{A}$ containing M, the so called *filter generated by* M. It will be denoted by $F(M)$. If $M = \{bt\}$ (a singleton), $F(\{b\})$ will be denoted briefly by $F(b)$ and it will be called a *principal filter generated by* b.

Theorem 2 *Let* $\mathcal{A} = (A, \cdot)$ *be an implication algebra and* $b \in A$. *Then*

$$F(bt) = \{x \in A;\ b \leq x\}.$$

Proof Let $a \in A$, $b_0, b_1, b_2 \in F(b)$. Since $b \leq b_0 \leq ab_0$ then also $ab_0 \in F(b)$. Moreover, $b \leq b_1$, $b \leq b_2$ imply by Lemma 2 that $b_1 \wedge b_2$ exists. Since $b \leq b_1 \wedge b_2$ we conclude $b_1 \wedge b_2 \in F(b)$, i.e. $F(b)$ is a filter of $\mathcal{A}$ containing b.

Conversely, let $F \in Fil\ \mathcal{A}$ and $b \in F$. Let $c \in A$ and $b \leq c$. Then $b \cdot c = 1$ and, by Lemma 1, $b(bc) = b \cdot 1 = 1$. Applying Theorem 1 we conclude $c = 1 \cdot c = (b(bc))\,c \in F$, i.e. $F(b) \subseteq F$. □

Denote by $\vee_F$ the operation of join in the lattice $Fil\ \mathcal{A}$.

Theorem 3 *Let* $\mathcal{A} = (A, \cdot)$ *be an implication algebra and* $M \subseteq A$. *Then*

$$F(M) = \vee_F\{F(b); b \in M\}.$$

Proof Of course, for each $b \in M$ we have $F(b) \subseteq F(M)$ whence $\vee_F\{F(b); b \in M\} \subseteq F(M)$. On the other hand, $\vee_F\{F(b); b \in M\}$ is a filter of $\mathcal{A}$ containing each $b \in M$ and $F(M)$ is the least filter of this property thus also the converse inclusion holds. □

Let M be a nonvoid subset of an implication algebra $\mathcal{A}$. Introduce the following operator which assigns to M all meets of elements of M provided they exist. Set $M_0 = M$ and for $k = 0, 1, 2, \ldots$

$$M_{k+1} = \{p \wedge q;\ p, q \in M_k \text{ and } p \wedge q \text{ exists}\}.$$

Since $p \wedge q$ exists for $p = q$, the sets $M_0, M_1, M_2, \ldots$ form a sequence

$$M_0 \subseteq M_1 \subseteq M_2 \subseteq \ldots$$

Now put

$$\bar{M} = \cup\{M_k;\ k = 0, 1, 2, \ldots\}.$$

The following results are easy consequences:

Corollary 2 *Let* $\mathcal{A} = (A, \cdot)$ *be an implication algebra and* $M \subseteq A$. *Then*

$$F(M) = \{x \in A;\ m \leq x \text{ for some } m \in \bar{M}\}.$$

Corollary 3 *Let $\mathcal{A} = (A, \cdot)$ be an implication algebra and $I, J \in Fil\,\mathcal{A}$. Then*

$$I \vee_F J = F(I \cup J).$$

Surprisingly the previous results enable us to characterize filters of implication algebras in purely lattice terms:

Theorem 4 *Let $\mathcal{A} = (A, \cdot)$ be an implication algebra and $I \subset A$ be nonvoid. Then I is a filter of $\mathcal{A}$ if and only if for each $b, b_1, b_2 \in I$ and each $a \in A$ we have:*

(b) if $b_1 \wedge b_2$ exists then $b_1 \wedge b_2 \in I$;
(d) if $b \leq a$ then $a \in I$.

Proof Let I satisfies (b), (d) of Theorem 4. For $b \in I$ and $a \in A$ we have $b \leq ab$ thus (d) implies $ab \in I$ proving (a) of Definition 1. Hence, I is a filter of $\mathcal{A}$. The converse follows directly by Corollary 2 in account of $I = F(I)$ (for $b = b_1 = b_2$). □

Introduce one more concept connected with filters in implication algebra, which was investigated for lattices by B.A. Davey and J. Nieminen in [2], [3]:

Definition 2 Let $\mathcal{A} = (A, \cdot)$ be an implication algebra and $\emptyset \neq M \subseteq A$. By an *annihilator induced by M* is meant the set

$$M^a = \{x \in A;\ x \vee y = 1 \text{ for each } y \in M\}.$$

Remark 1 It is easy to see that for each $\emptyset \neq M \subseteq A$ we have $M^a = F(M)^a$. Hence, we will investigate only annihilators induced by filters in the sequel.

Theorem 5 *For every filter I of an implication algebra , the induced annihilator I^a is a filter of $\mathcal{A}$.*

Proof Let $b, b_1, b_2 \in I^a$ and $x \in A$. Then $b \vee y = 1$ for each $y \in I$ and, since $b \leq xb$, we have $1 = b \vee y \leq xb \vee y$, whence $xb \vee y = 1$ proving $xb \in I^a$. If $b_1 \wedge b_2$ exists in A then, by Theorem 9 [1]:

$$(b_1 \wedge b_2) \vee y = (b_1 \vee y) \wedge (b_2 \vee y) = 1 \wedge 1 = 1$$

proving $b_1 \wedge b_2 \in I^a$. Hence, I^a is a filter of $\mathcal{A}$. □

Theorem 6 *For each filter I of an implication algebra $\mathcal{A}$, the induced annihilator I^a is a pseudocomplement in the lattice $Fil\mathcal{A}$.*

Proof If $z \in I \cap I^a$ then $z \vee z = z = 1$ whence $I \cap I^a = \{1\}$. Conversely, let F be such a filter that $I \cap F = \{1\}$. Then for every $i \in I$ and $z \in F$ we have $i \vee z \in I \cap F = \{1\}$, i.e. $z \vee i = 1$. Hence, $F \subseteq I^a$. □

Theorem 7 *The set $Ann(\mathcal{A})$ of al annihilators induced by all filters of $\mathcal{A}$ forms a Boolean algebra with respect to set inclusion. $\{1\}$ is the least and A the greatest element of $Ann(\mathcal{A})$, its complement is $B' = B^a$. For $I_\gamma \in Fil\mathcal{A}$ $(\gamma \in \Gamma)$ we have*

$$(\vee\{I_\gamma; \gamma \in \Gamma\})^a = \cap\{I_\gamma^a; \gamma \in \Gamma\}$$

which is the operation meet in $Ann\,(\mathcal{A})$.

Proof An element I of the pseudocomplemented lattice $Fil\,\mathcal{A}$ (see Theorem 6) is called boolean if $(I^a)^a = I$. It is clear that every annihilator is a boolean element of $Fil\,\mathcal{A}$. Conversely, if $G \in Fil\,\mathcal{A}$ is a boolean element then $G = (G^a)^a$ is an annihilator. Hence, the set of all boolean elements of $Fil\,\mathcal{A}$ is exactly the set $Ann(\mathcal{A})$. Of course, by Glivenko theorem, $Ann(\mathcal{A})$ is a Boolean algebra whose induced order is set inclusion.

Further, it is evident that for $I_\gamma \in Fil\,\mathcal{A}$ $(\gamma \in \Gamma)$ we have

$$(\vee\{I_\gamma; \gamma \in \Gamma\})^a \subseteq \cap\{I_\gamma^a; \gamma \in \Gamma\}.$$

Conversely, suppose $z \in \cap\{I_\gamma^a; \gamma \in \Gamma\}$. Hence $z \vee y = 1$ for every $y \in \cup\{I_\gamma; \gamma \in \Gamma\}$. By Corollary 2, $\vee\{I_\gamma; \gamma \in \Gamma\} = \{x \in A;\ m \leq x \text{ for some } m \in \bar{M}\}$ where

$M = M_0 = \cup\{I_\gamma; \gamma \in \Gamma\}$

$M_{k+1} = \{p \wedge q;\ p, q \in M_k \text{ and } p \wedge q \text{ exists}\}$

$\bar{M} = \cup\{M_k; k = 0, 1, 2, \ldots\}$.

We prove by induction that $z \vee x = 1$ for each $x \in \vee\{I_\gamma; \gamma \in \Gamma\}$.

If $k = 0$ and $x \geq m$ for some $m \in M$ then $m \in I_\gamma$ for some $\gamma \in \Gamma$ and hence $z \vee x \geq z \vee m = 1$ proving $z \vee x = 1$.

Suppose now that for each $x \in \{a \in A; m \leq a \text{ for some } m \in M_k\}$ we have $z \vee x = 1$. Let $x' \geq p \wedge q$ for $p, q \in M_k$. Then $z \vee p = z \vee q = 1$ which yields $x' \vee z \geq (p \wedge q) \vee z = (p \vee z) \wedge (q \vee z) = 1 \wedge 1 = 1$ by Theorem 9 {1}, i.e. $x' \vee z = 1$. □

Concluding remark Let b be an element of an implication algebra $\mathcal{A} = (A, \cdot)$. As it was shown, for an annihilator induced by the principal filter $I(b)$ we have $I(b)^a = \{b\}^a$ whence $I(b)^a = \{x \in A; x \vee b = 1\}$. It is the annihilator denoted by $\langle b, 1\rangle$ in the sense of [2], [3].

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