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Filters and Annihilators in Implication Algebras

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Abstract

The concept of filter in implication algebra is characterized in terms of operations and also in lattice operation. The set of all filters of an implication algebra forms a complete lattice whose boolean elements are annihilators. The set of all annihilators forms a Boolean algebra.

Key words: Implication algebra, filter, annihilator, pseudocomplement.

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The concept of implication algebra was introduced by J. C. Abbott [1], see also W. C. Nemitz [4]. A groupoid \((A, \cdot)\) is an implication algebra if it satisfies the following identities:

\[
\begin{align*}
(i) & \quad (ab)a = a \\
(ii) & \quad (ab)b = (ba)a \\
(iii) & \quad a(bc) = b(ac)
\end{align*}
\]

Hence, the class of all implication algebras forms a variety.
The study of implication algebras was motivated by the fact that for every Boolean algebra $B = (B; \lor, \land, '0, 1)$, the groupoid $(B, \rightarrow)$ where $a \rightarrow b = a' \lor b$ is an implication algebra. Hence, implication algebras describe properties of the connective implication in logic (not necessary in a classical logic).

The following concepts was introduced by J. C. Abbott [1]:

**Definition 1** A nonvoid subset $I$ of an implication algebra $A = (A, \cdot)$ is called a filter if for each $b_1, b_2, b \in I$ and every $x \in A$ we have

(a) $xb \in I$  
(b) whenever $b_1 \land b_2$ exists in $A$ then $b_1 \land b_2 \in I$.

We must explain the symbol $\land$ in Definition 1. For this, let us repeat some necessary results of [1]:

**Lemma 1** Let $A = (A, \cdot)$ be an implication algebra. Then

(i) for any $a, b \in A$, $aa = bb$, i.e. there exists a nullary term denoted by 1 such that $a \cdot a = 1$ is the identity of $A$;

(ii) for each $a \in A$, $a \cdot 1 = 1$. $1 \cdot a = a$.

Let us introduce the relation $\leq$ by setting $a \leq b$ if and only if $a \cdot b = 1$.

**Lemma 2** Let $A = (A, \cdot)$ be an implication algebra. Then $(A, \leq)$ is a $\lor$-semilattice with respect to $\leq$ with the greatest element 1, where

$$a \lor b = (ab)b.$$ 

If for $a, b \in A$ there exists $p \in A$ such that $p \leq a, p \leq b$ then there exists an infimum $a \land b$ (w.r.t. $\leq$) and $a \land b = (a (b \cdot p))p$.

For the proof, see Theorems 3, 4 and 5 in [1].

From this point of view, the definition of filter of implication algebra need not be suitable in all cases since the condition (a) is formulated in term operation of $A = (A, \cdot)$ but (b) contains a partial operation $\land$ which is not a term operation of $A$. To avoid this discrepancy, we prove the following:

**Theorem 1** A nonvoid subset $I$ of an implication algebra $A = (A, \cdot)$ is a filter if and only if for each $a \in A$ and every $b, b_1, b_2 \in I$ we have

(a) $ab \in I$  
(c) $(b_1 (b_2 a)) a \in I$.

**Proof** Let $\emptyset \neq I \subseteq A$. Suppose that (a), (c) hold. Let $b_1, b_2 \in I$ and $b_1 \land b_2$ exist. Denote by $p = b_1 \land b_2$. By Lemma 2, $b_1 \land b_2 = (b_1 (b_2 p))p$ which belongs to $I$ by (c). Hence, $b_1 \land b_2 \in I$ proving (b), i.e. $I$ is a filter of $A$.

Conversely, let $I$ be a filter of $A$. By Theorem 10 [1], $I$ is a kernel of some congruence $\theta_I$ on $A$, i.e. $b \in I$ if and only if $(b, 1) \in \theta_I$. Suppose $a \in A$, $b_1, b_2 \in I$. Then $(a, a) \in \theta_I$ and $(b_1, 1) \in \theta_I$, $(b_2, 1) \in \theta_I$ whence by Lemma 1:

$$(b_1 (b_2 a)) a = (b_1 (b_2 a)) a, (1 (1 a)) a) \in \theta_I$$

i.e. $(b_1 (b_2 a)) a \in I$. Thus $I$ satisfies (a) and (c) of Theorem 1. \qed
Corollary 1 The set Fil\(A\) of all filters on an implication algebra \(A\) forms a complete lattice with respect to set inclusion. The least element is \(\{1\}\), the greatest element is \(A\) and the operation meet in \(\text{Fil}A\) coincides with set-theoretical intersection.

**Proof** It is almost trivial to show that if \(S\) is system of filters of \(A\) then also its intersection satisfies (a), (c) of Theorem 1.

Applying the foregoing Corollary 1 we see that for any subset \(M\) of \(A = (A, \cdot)\) there exists the least filter of \(A\) containing \(M\), the so called filter generated by \(M\). It will be denoted by \(F(M)\). If \(M = \{b\}\) (a singleton), \(F(\{b\})\) will be denoted briefly by \(F(b)\) and it will be called a principal filter generated by \(b\).

**Theorem 2** Let \(A = (A, \cdot)\) be an implication algebra and \(b \in A\). Then

\[F(bt) = \{x \in A; b \leq x\}.
\]

**Proof** Let \(a \in A, b_0, b_1, b_2 \in F(b)\). Since \(b \leq b_0 \leq ab_0\) then also \(ab_0 \in F(b)\). Moreover, \(b \leq b_1, b \leq b_2\) imply by Lemma 2 that \(b_1 \wedge b_2\) exists. Since \(b \leq b_1 \wedge b_2\) we conclude \(b_1 \wedge b_2 \in F(b)\), i.e. \(F(b)\) is a filter of \(A\) containing \(b\).

Conversely, let \(F \in \text{Fil}A\) and \(b \in F\). Let \(c \in A\) and \(b \leq c\). Then \(b \cdot c = 1\) and, by Lemma 1, \(b(bc) = b \cdot 1 = 1\). Applying Theorem 1 we conclude \(c = 1 \cdot c = (b(bc))c \in F\), i.e. \(F(b) \subseteq F\). 

Denote by \(\lor F\) the operation of join in the lattice \(\text{Fil}A\).

**Theorem 3** Let \(A = (A, \cdot)\) be an implication algebra and \(M \subseteq A\). Then

\[F(M) = \lor F\{F(b); b \in M\}.
\]

**Proof** Of course, for each \(b \in M\) we have \(F(b) \subseteq F(M)\) whence \(\lor F\{F(b); b \in M\} \subseteq F(M)\). On the other hand, \(\lor F\{F(b); b \in M\}\) is a filter of \(A\) containing each \(b \in M\) and \(F(M)\) is the least filter of this property thus also the converse inclusion holds.

Let \(M\) be a nonvoid subset of an implication algebra \(A\). Introduce the following operator which assigns to \(M\) all meets of elements of \(M\) provided they exist. Set \(M_0 = M\) and for \(k = 0, 1, 2, \ldots\)

\[M_{k+1} = \{p \land q; p, q \in M_k \text{ and } p \land q \text{ exists}\}.
\]

Since \(p \land q\) exists for \(p = q\), the sets \(M_0, M_1, M_2, \ldots\) form a sequence

\[M_0 \subseteq M_1 \subseteq M_2 \subseteq \ldots
\]

Now put

\[\bar{M} = \cup\{M_k; k = 0, 1, 2, \ldots\}.
\]

The following results are easy consequences:

**Corollary 2** Let \(A = (A, \cdot)\) be an implication algebra and \(M \subseteq A\). Then

\[F(M) = \{x \in A; m \leq x \text{ for some } m \in \bar{M}\}.
\]
Corollary 3 Let \( A = (A, \cdot) \) be an implication algebra and \( I, J \in \text{Fil}A \). Then
\[
I \vee_F J = F(I \cup J).
\]

Surprisingly the previous results enable us to characterize filters of implication algebras in purely lattice terms:

Theorem 4 Let \( A = (A, \cdot) \) be an implication algebra and \( I \subset A \) be nonvoid. Then \( I \) is a filter of \( A \) if and only if for each \( b, b_1, b_2 \in I \) and each \( a \in A \) we have:

(b) if \( b_1 \land b_2 \) exists then \( b_1 \land b_2 \in I \);

(d) if \( b \leq a \) then \( a \in I \).

Proof Let \( I \) satisfies (b), (d) of Theorem 4. For \( b \in I \) and \( a \in A \) we have
\( b \leq ab \) thus (d) implies \( ab \in I \) proving (a) of Definition 1. Hence, \( I \) is a filter of \( A \). The converse follows directly by Corollary 2 in account of \( I = F(I) \) (for \( b = b_1 = b_2 \)). \( \square \)

Introduce one more concept connected with filters in implication algebra, which was investigated for lattices by B.A. Davey and J. Nieminen in [2], [3]:

Definition 2 Let \( A = (A, \cdot) \) be an implication algebra and \( \emptyset \neq M \subseteq A \). By an annihilator induced by \( M \) is meant the set
\[
M^a = \{ x \in A; x \lor y = 1 \text{ for each } y \in M \}.
\]

Remark 1 It is easy to see that for each \( \emptyset \neq M \subseteq A \) we have \( M^a = F(M)^a \). Hence, we will investigate only annihilators induced by filters in the sequel.

Theorem 5 For every filter \( I \) of an implication algebra \( A \), the induced annihilator \( I^a \) is a filter of \( A \).

Proof Let \( b, b_1, b_2 \in I^a \) and \( x \in A \). Then \( b \lor y = 1 \) for each \( y \in I \) and, since \( b \leq xb \), we have \( 1 = b \lor y \leq xb \lor y \), whence \( xb \lor y = 1 \) proving \( xb \in I^a \). If \( b_1 \land b_2 \) exists in \( A \) then, by Theorem 9 [1]:
\[
(b_1 \land b_2) \lor y = (b_1 \lor y) \land (b_2 \lor y) = 1 \land 1 = 1
\]
proving \( b_1 \land b_2 \in I^a \). Hence, \( I^a \) is a filter of \( A \). \( \square \)

Theorem 6 For each filter \( I \) of an implication algebra \( A \), the induced annihilator \( I^a \) is a pseudocomplement in the lattice \( \text{Fil}A \).

Proof If \( z \in I \cap I^a \) then \( z \lor z = z = 1 \) whence \( I \cap I^a = \{1\} \). Conversely, let \( F \) be such a filter that \( I \cap F = \{1\} \). Then for every \( i \in I \) and \( z \in F \) we have \( i \lor z \in I \cap F = \{1\} \), i.e. \( z \lor i = 1 \). Hence, \( F \subseteq I^a \). \( \square \)

Theorem 7 The set \( \text{Ann}(A) \) of all annihilators induced by all filters of \( A \) forms a Boolean algebra with respect to set inclusion. \( \{1\} \) is the least and \( A \) the greatest element of \( \text{Ann}(A) \), its complement is \( B' = B^a \). For \( I_\gamma \in \text{Fil}A \) (\( \gamma \in \Gamma \)) we have
\[
(V\{I_\gamma; \gamma \in \Gamma\})^a = \cap\{I_\gamma^a; \gamma \in \Gamma\}
\]
which is the operation meet in \( \text{Ann}(A) \).
Proof  An element $I$ of the pseudocomplemented lattice $\text{Fil} A$ (see Theorem 6) is called boolean if $(I^a)^a = I$. It is clear that every annihilator is a boolean element of $\text{Fil} A$. Conversely, if $G \in \text{Fil} A$ is a boolean element then $G = (G^a)^a$ is an annihilator. Hence, the set of all boolean elements of $\text{Fil} A$ is exactly the set $\text{Ann}(A)$. Of course, by Glivenko theorem, $\text{Ann}(A)$ is a Boolean algebra whose induced order is set inclusion.

Further, it is evident that for $I_\gamma \in \text{Fil} A$ ($\gamma \in \Gamma$) we have

$$(\vee \{I_\gamma; \gamma \in \Gamma\})^a \subseteq \cap \{I_\gamma^a; \gamma \in \Gamma\}.$$ 

Conversely, suppose $z \in \cap \{I_\gamma^a; \gamma \in \Gamma\}$. Hence $z \vee y = 1$ for every $y \in \cup \{I_\gamma; \gamma \in \Gamma\}$. By Corollary 2, $\vee \{I_\gamma; \gamma \in \Gamma\} = \{x \in A; m \leq x \text{ for some } m \in M\}$ where

$\begin{align*}
M &= M_0 = \cup \{I_\gamma; \gamma \in \Gamma\} \\
M_{k+1} &= \{p \wedge q; p, q \in M_k \text{ and } p \wedge q \text{ exists}\} \\
M &= \cup \{M_k; k = 0, 1, 2, \ldots\}.
\end{align*}$

We prove by induction that $z \vee x = 1$ for each $x \in \vee \{I_\gamma; \gamma \in \Gamma\}$.

If $k = 0$ and $a \geq m$ for some $m \in M$ then $m \in I_\gamma$ for some $\gamma \in \Gamma$ and hence $z \vee x \geq z \vee m = 1$ proving $z \vee x = 1$.

Suppose now that for each $x \in \{a \in A; m \leq a \text{ for some } m \in M_k\}$ we have $z \vee x = 1$. Let $x' \geq p \wedge q$ for $p, q \in M_k$. Then $z \vee p = z \vee q = 1$ which yields $x' \vee z \geq (p \wedge q) \vee z = (p \vee z) \wedge (q \vee z) = 1 \wedge 1 = 1$ by Theorem 9 (1), i.e. $x' \vee z = 1$. □

Concluding remark  Let $b$ be an element of an implication algebra $A = (A, \cdot)$. As it was shown, for an annihilator induced by the principal filter $I(b)$ we have $I(b)^a = \{b\}^a$ whence $I(b)^a = \{x \in A; x \vee b = 1\}$. It is the annihilator denoted by $(b, 1)$ in the sense of [2], [3].

References