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Boundary Value Problems for Ordinary Linear Differential Equations in the Colombeau Algebra

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Abstract

It is shown that from the fact that the unique solution of homogeneous problem is the trivial one it follows the existence of solution of nonhomogeneous problem in the Colombeau algebra.

Key words: Generalized ordinary linear differential equations, boundary value problems, Colombeau algebra.

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1 Introduction

We consider the following problem

$$x_k'(t) = \sum_{j=1}^{n} A_{kj}(t)x_j(t) + f_k(t) \tag{1.0}$$

$$L_k(x_k) = d_k, \quad d_k \in \overline{\mathbb{R}}, \quad k = 1, \dots, n; \tag{1.1}$$

where A_{kj}, f_k and x_k are elements of the Colombeau algebra $\mathcal{G}(\mathbb{R})$; $d_1, \dots, d_n$ are known elements of the Colombeau algebra $\overline{\mathbb{R}}$ od generalized real numbers and L_k are operations on $\mathcal{G}(\mathbb{R})$ (see [1], [2]), the multiplication, the sum, the derivative and the equality is meant in the Colombeau algebra sense. We prove theorems on existence and uniqueness of solutions of problem (1.0)–(1.1). Our theorems generalize some results given in [14], [15] and [17]–[18].

2 Notation

Let $\mathcal{D}(\mathbb{R})$ be the set of all C^∞ functions $\mathbb{R} \to \mathbb{R}$ with compact support. For $q = 1, 2, \ldots$ we denote by $\mathcal{A}_q$ the set of all functions $\varphi \in \mathcal{D}(\mathbb{R})$ such that the relations

$$\int_{-\infty}^{\infty} \varphi(t)dt = 1, \quad \int_{-\infty}^{\infty} t^k \varphi(t)dt = 0, \quad 1 \leq k \leq q \tag{2.1}$$

hold.

Next, $\mathcal{E}[\mathbb{R}]$ is the set of all functions $R : \mathcal{A}_1 \times \mathbb{R} \to \mathbb{R}$ such that $R(\varphi, t) \in C^\infty(\mathbb{R})$ for each fixed $\varphi \in \mathcal{A}_1$.

If $R \in \mathcal{E}[\mathbb{R}]$, then $D_k R(\varphi, t)$ for any fixed φ denotes a differential operator in t (i.e. $D_k R(\varphi, t) = \frac{d^k}{dt^k}(R(\varphi, t))$ for $k \geq 1$ and $D_0 R(\varphi, t) = R(\varphi, t)$).

For given $\varphi \in \mathcal{D}(\mathbb{R})$ and $\varepsilon > 0$, we define φ_ε by

$$\varphi_\varepsilon(t) = \frac{1}{\varepsilon}\varphi\left(\frac{t}{\varepsilon}\right). \tag{2.2}$$

An element R of $\mathcal{E}[\mathbb{R}]$ is moderate if for every compact set K of $\mathbb{R}$ and every differential operator D_k there is $N \in \mathbb{N}$ such that there are $c > 0$ and $\varepsilon_0 > 0$ such that

$$\sup_{t \in K} |D_k R(\varphi_\varepsilon, t)| \leq c\varepsilon^{-N} \quad \text{for} \quad 0 < \varepsilon < \varepsilon_0. \tag{2.3}$$

We denote by $\mathcal{E}_M[\mathbb{R}]$ the set of all moderate elements of $\mathcal{E}[\mathbb{R}]$.

By Γ we denote the set of all increasing functions α from $\mathbb{N}$ into $\mathbb{R}^+$ such that $\alpha(q) \to \infty$ if $q \to \infty$.

We define an ideal $\mathcal{N}[\mathbb{R}]$ in $\mathcal{E}_M[\mathbb{R}]$ as follows: $R \in \mathcal{N}[\mathbb{R}]$ if for every compact set K of $\mathbb{R}$ and every differential operator D_k there are $N \in \mathbb{N}$ and $\alpha \in \Gamma$ such that the following conditions holds: for every $q \geq N$ and $\varphi \in \mathcal{A}_q$ there are $c > 0$ and $\varepsilon_0 > 0$ such that

$$\sup_{t \in K} |D_k R(\varphi_\varepsilon, t)| \leq c\varepsilon^{\alpha(q) - N} \qquad \text{if} \quad 0 < \varepsilon < \varepsilon_0. \tag{2.4}$$

The algebra $\mathcal{G}(\mathbb{R})$ (the Colombeau algebra) is defined as quotient algebra of $\mathcal{E}_M[\mathbb{R}]$ with respect to $\mathcal{N}[\mathbb{R}]$ (see [1]).

We denote by $\mathcal{E}_0$ the set of all functions from $\mathcal{A}_1$ into $\mathbb{R}$. Next, we denote by $\mathcal{E}_M$ the set of all the so-called moderate elements of $\mathcal{E}_0$ defined by

(2.5) $\mathcal{E}_M = \{R \in \mathcal{E}_0$: there is $N \in \mathbb{N}$ such that for every $\varphi \in \mathcal{A}_N$ there are $c > 0$ and η_0 such that $|R(\varphi_\varepsilon)| \leq c\varepsilon^{-N}$ if $0 < \varepsilon < \eta_0\}$.

Further, we define an ideal $\mathcal{N}$ of $\mathcal{E}_M$ by

(2.6) $\mathcal{N} = \{R \in \mathcal{E}_0$: there are $N \in \mathbb{N}$ and $\alpha \in \Gamma$ such that for every $q \geq N$ and $\varphi \in \mathcal{A}_q$ there are $c > 0$, $\eta_0 > 0$ such that $|R(\varphi_\varepsilon)| \leq c\varepsilon^{\alpha(q) - N}$ if $0 < \varepsilon < \eta_0\}$.

We define an algebra $\overline{\mathbb{R}}$ by setting

$$\overline{\mathbb{R}} = \frac{\mathcal{E}_M}{\mathcal{N}} \qquad \text{(see [1])}.$$

It is known that $\overline{\mathbb{R}}$ is not a field.

If $R \in \mathcal{E}_M[\mathbb{R}]$ is a representative of $G \in \mathcal{G}(\mathbb{R})$, then for a fixed t the map $Y : \varphi \to R(\varphi, t) \in \mathbb{R}$ is defined on $\mathcal{A}_1$ and $Y \in \mathcal{E}_M$. The class of Y in $\mathbb{R}$ depends only on G and t. This class is denoted by $G(t)$ and is called the value of generalized function G at the point t (see [1]).

We say that $G \in \mathcal{G}(\mathbb{R})$ is a constant generalized function on $\mathbb{R}$ if it admits a representative $R(\varphi, t)$ which is independent of $t \in \mathbb{R}$. With any $Z \in \overline{\mathbb{R}}$ we associate a constant generalized function which admits $R(\varphi, t) = Z(\varphi)$ as its representation, provided we denote by Z a representative of Z (see [1]).

(Throughout in the paper K denotes a compact interval in $\mathbb{R}$ containing zero.) We denote by

$$R_{A_{kj}}(\varphi, t), \; R_{fk}(\varphi, t), \; R_{x_{0j}}(\varphi), \; R_{x_j(t_0)}(\varphi), \; R_{x_j}(\varphi, t) \text{ and } R_{x'_j}(\varphi, t)$$

representative of elements $A_{k_j}, f_k, x_{0j}, x_{0j}(t_0), x_j$ and x'_j for $k, j = 1, \dots, n$. Let

$$A(t) = (A_{kj}(t)), \;\; f(t) = (f_1(t), \dots, f_n(t))^T, \;\; x(t) = (x_1(t), \dots, x_r(t))^T,$$
$$x'(t) = (x'_1(t), \dots, x'_n(t))^T, \quad x_0 = (x_{10}, \dots, x_{n0})^T,$$

where T denotes the transpose. We put

$$R_A(\varphi, t) = (R_{A_{kj}}(\varphi, t)), \;\; R_f(\varphi, t) = (R_{f_1}(\varphi, t), \dots, R_{f_n}(\varphi, t))^T,$$
$$R_x(\varphi, t) = (R_{x_1}(\varphi, t), \dots, R_{x_n}(\varphi, t))^T,$$
$$R_{x'}(\varphi, t) = (R_{x'_1}(\varphi, t), \dots, , R_{x'_n}(\varphi, t))^T,$$
$$R_{x_0}(\varphi) = (R_{x_{10}}(\varphi), \dots, R_{x_{n_0}}(\varphi))^T,$$
$$R_{x(t_0)}(\varphi) = (R_{x_1(t_0)}(\varphi), \dots, R_{x_n(t_0)}(\varphi))^T,$$
$$\int_{t_0}^{t} R_A(\varphi, s)ds = \left(\int_{t_0}^{t} R_{A_{kj}}(\varphi, s)ds \right),$$
$$\int_{t_0}^{t} R_f(\varphi, s)ds = \left(\int_{t_0}^{t} R_{f_1}(\varphi, s)ds, \dots, \int_{t_0}^{t} R_{f_n}(\varphi, s)ds \right)^T,$$
$$||R_A(\varphi, t)|| = \sum_{k,j=1}^{n} |R_{A_{kj}}(\varphi, t)|, \qquad ||R_A(\varphi, t)||_K = \sum_{k,j=1}^{n} \sup_{t \in K} |R_{A_{k_j}}(\varphi, t)|,$$
$$||R_f(\varphi, t)||_K = \sum_{j=1}^{n} \sup_{t \in K} |R_{f_j}(\varphi, t)|.$$

If
$$A_{kj}, f_j \in \mathcal{G}(\mathbb{R}), \quad u = (u_1, \dots, u_n) \in \mathbb{R}^n, \quad v = (v_1, \dots, v_n) \in \mathbb{R}^n,$$
$$a_{kj}, b_j \in \mathcal{N}[\mathbb{R}]; \quad m_{kj}, p_j \in \mathcal{N}, \; q_j \in \mathbb{R}, \; r_j \in \overline{\mathbb{R}}$$
for $k, j = 1, \dots, n$, then we write -

$$\begin{array}{ll} A = (A_{kj}) \in \mathcal{G}^{n\times n}(\mathbb{R}), & f = (f_1, \dots, f_n)^T \in \mathcal{G}^n(\mathbb{R}), \\ b = (b_1, \dots, b_n)^T \in \mathcal{N}^n[\mathbb{R}], & a = (a_{kj}) \in \mathcal{N}^{n\times n}[\mathbb{R}], \\ m = (m_{kj}) \in \mathcal{N}^{n\times n}, & p = (p_1, \dots, p_n)^T \in \mathcal{N}^n, \\ q = (q_1, \dots, q_n)^T \in \mathbb{R}^n, & r = (r_1, \dots, r_n)^T \in \overline{\mathbb{R}}^n, \\ R_A(\varphi, t) \in \mathcal{E}_M^{n\times n}[\mathbb{R}], & R_x(\varphi, t) \in \mathcal{E}_M^n[\mathbb{R}] \end{array}$$

and $(u, v) = \sum_{j=1}^{n} u_j v_j$.

We say that $x = (x_1, \dots, x_n)^T \in \mathcal{G}^n(\mathbb{R})$ is a solution of system (1.0) if there is $\eta \in \mathcal{N}^n[\mathbb{R}]$ such that it holds

$$R_{x'}(\varphi, t) = R_A(\varphi, t) R_x(\varphi, t) + R_f(\varphi, t) + \eta(\varphi, t)$$

for all $\varphi \in \mathcal{A}_1$ and $t \in \mathbb{R}$, where R_x denotes an arbitrary representative of x.

3 The main results

First we shall introduce four hypotheses.

Hypothesis H_1

(3.1) $$A \in \mathcal{G}^{n\times n}(\mathbb{R}), \quad f \in \mathcal{G}(\mathbb{R}),$$

the matrix A admits a representative $R_A(\varphi, t) = (R_{A_{kj}}(\varphi, t))$ with the following property: for every compact interval K there is $N \in \mathbb{N}$ such that for every $\varphi \in \mathcal{A}_N$ there are constants $c > 0$, $\varepsilon_0 > 0$ and $\gamma_0 > 0$ satisfying at least one of the following four conditions:

(3.2) $$\left\| \int_0^t |R_{A_{kj}}(\varphi_\varepsilon, s)| ds \right\|_K \le c \quad \text{for } 0 < \varepsilon < \varepsilon_0 \text{ and } k, j = 1, \dots, n;$$

(3.3) $$(u^T, R_A(\varphi_\varepsilon, t) u) \ge \gamma_0 (u, u) \quad \text{for } 0 < \varepsilon < \varepsilon_0, \; t \in K \text{ and } u \in \mathbb{R}^n,$$

(3.4) $$R_{A_{kj}}(\varphi_\varepsilon, t) = -R_{A_{jk}}(\varphi_\varepsilon, t) \quad \text{for } k = 2, \dots, n, \; t \in K \text{ and } 0 < \varepsilon < \varepsilon_0;$$

(3.5) $$R_{A_{jj}}(\varphi_\varepsilon, t) \ge \gamma_0 \quad \text{for } 0 < \varepsilon < \varepsilon_0, \; t \in K \text{ and } j = 1, \dots, n;$$

the matrix A admits a representative $R_A(\varphi,t) = (R_{A_{kj}}(\varphi,t))$ with the following property: for a fixed compact interval $[a,b]$ there is $N \in \mathbb{N}$ such that for every $\varphi \in \mathcal{A}_N$ there are constants $\varepsilon_0 > 0$ and $\gamma > 0$ such that

$$(3.6) \qquad \beta_\varepsilon := \sum_{k,j=1}^{n} \int_a^b |R_{A_{kj}}(\varphi_\varepsilon,t)|dt \leq \frac{1}{2} - \gamma \quad \text{for } 0 < \varepsilon < \varepsilon_0.$$

Hypothesis H_2

$$(3.7) \qquad p, q, r \in L^1_{loc}(\mathbb{R}),$$

$$(3.8) \qquad p \in L^1_{loc}(\mathbb{R}), \quad \int_a^b |p(t)|dt \leq \frac{4}{b-a}, \quad a, b \in \mathbb{R},\ a < b,$$

$$(3.9) \quad p, q \in L^1_{loc}(\mathbb{R}), \quad \int_a^b (|p(t)| + |q(t)|)dt < \frac{4}{b-a+4}, \quad a, b \in \mathbb{R},\ a < b,$$

$(3.10) \qquad p \in L^1_{loc}(\mathbb{R}),\quad p$ is an ω-periodic function such that

$$p(t) \not\equiv 0, \quad \int_0^\omega p(t)dt \geq 0, \quad \int_0^\omega |p(t)|dt \leq \frac{16}{\omega},$$

the elements $p, q \in \mathcal{G}(\mathbb{R})$ admit representatives $R_p(\varphi,t)$ and $R_q(\varphi,t)$ with the following properties: for every compact interval K there is $N \in \mathbb{N}$ such that for every $\varphi \in \mathcal{A}_N$ there are constants $c > 0$ and $\varepsilon_0 > 0$ such that

$$(3.2)' \qquad \left\| \int_0^t |R_p(\varphi_\varepsilon,s)|ds \right\|_K \leq c, \qquad \left\| \int_0^t |R_q(\varphi_\varepsilon,s)|ds \right\|_K \leq c$$

for $0 < \varepsilon < \varepsilon_0$, the elements $p, q \in \mathcal{G}(\mathbb{R})$ admit representatives $R_p(\varphi,t)$ and $R_q(\varphi,t)$ with the following properties: for a fixed compact interval $[a,b]$ there is $N \in \mathbb{N}$ such that for every $\varphi \in \mathcal{A}_N$ there are constants $\gamma > 0$ and $\varepsilon_0 > 0$ satisfying at least one of the following two conditions:

$$(3.11) \qquad \int_a^b |R_p(\varphi_\varepsilon,t)|dt \leq \frac{4}{b-a} - \gamma \quad \text{for } 0 < \varepsilon < \varepsilon_0,$$

$$(3.12) \quad \int_a^b |R_p(\varphi_\varepsilon,t)|dt + \int_a^b |R_q(\varphi_\varepsilon,t)|dt \leq \frac{4}{b-a+4} - \gamma \quad \text{for } 0 < \varepsilon < \varepsilon_0,$$

the element $p \in \mathcal{G}(\mathbb{R})$ admits an ω-periodic representative $R_p(\varphi,t)$ with the following property: there is $N \in \mathbb{N}$ such that for every $\varphi \in \mathcal{A}_N$ there are constants $\varepsilon_0 > 0$, $\gamma > 0$ satisfying at least one of the following four conditions:

$$(3.13) \qquad R_p(\varphi_\varepsilon,t) \leq -\gamma \quad \text{for } 0 < \varepsilon < \varepsilon_0 \text{ and } t \in \mathbb{R};$$

(3.13)′ $$R_p(\varepsilon_\varepsilon, t) \le 0 \quad \text{for } 0 < \varepsilon < \varepsilon_0 \text{ and } t \in \mathbb{R};$$

(3.14) $$|R_p(\varphi_\varepsilon, t)| \ge \gamma \quad \text{for } 0 < \varepsilon < \varepsilon_0 \text{ and } t \in \mathbb{R};$$

(3.15) $$\int_0^\omega |R_p(\varphi_\varepsilon, t)| dt \le \frac{16}{\omega} - \gamma, \quad \text{for } 0 < \varepsilon < \varepsilon_0.$$

Hypothesis H_3

L_i $(i = 1, \dots, n)$ are operations such that:

(3.16) $$L_i(y) \in \overline{\mathbb{R}} \quad \text{for } y \in \mathcal{G}(\mathbb{R}) \quad \text{and} \quad L_i(y) \in \mathbb{R} \quad \text{for } y \in C^\infty(\mathbb{R}),$$

(3.17) $$L_i(\lambda_1 y_1 + \lambda_2 y_2) = \lambda_1 L_i(y_1) + \lambda_2 L_i(y_2),$$

where $y_1, y_2 \in \mathcal{G}(\mathbb{R})$ and λ_1, λ_2 are constant generalized functions on $\mathbb{R}$,

(3.18) $$h_i(\varphi) := L_i(R_y(\varphi, t)) \in \mathcal{E}_M$$

for all $\varphi \in \mathcal{A}_1$ and $y \in \mathcal{G}(\mathbb{R})$ such that $R_y(\varphi, t) \in \mathcal{E}_M[\mathbb{R}]$,

(3.19) $$L_i[R_y(\varphi, t)] = [L_i R_y(\varphi, t)] \quad \text{for all } y \in \mathcal{G}(\mathbb{R}),$$

(3.20) $$L_i[1] = 1,$$

if the matrix $A \in \mathcal{G}^{n \times n}(\mathbb{R})$ has property (3.2) and if $x \in \mathcal{G}^n(\mathbb{R})$, then there is a compact interval $[a, b]$ and $N \in \mathbb{N}$ such that for every $\varphi \in \mathcal{A}_N$ there are $\varepsilon_0 > 0$ and $\gamma > 0$ such that the relations (3.6) and

(3.21)
$$\left| L_i\Big(\int_a^t R_{A_{kj}}(\varphi_\varepsilon, s) R_{x_j}(\varphi_\varepsilon, s) ds\Big) \right| \le \Big(\int_a^b |R_{A_{kj}}(\varphi_\varepsilon, t)| dt\Big) \|R_{x_j}(\varphi_\varepsilon, t)\|_{[a,b]}$$

are valid for all $i, j, k = 1, 2, \dots, n$ and $\varepsilon \in (0, \varepsilon_0)$.

Hypothesis H_4

L_i have properties (3.16)–(3.19) for $i = 1, \dots, n$;
$\tilde{L}_i$ $(i = 1, \dots, n)$ are operations such that:

(3.16)′ $$\tilde{L}_i(y) \in \mathbb{R} \quad \text{for } y \in C(\mathbb{R}),$$

(3.17)′ $$\tilde{L}_i(\lambda_1 y_1 + \lambda_2 y_2) = \lambda_1 \tilde{L}_i(y_1) + \lambda_2 \tilde{L}_i(y_2),$$

where $\lambda_1, \lambda_2 \in \mathbb{R}$ and $y_1, y_2 \in C(\mathbb{R})$,

(3.18)′ $$\tilde{L}_i(y) = L_i(y) \quad \text{for } y \in C^\infty(\mathbb{R}),$$

if $y_\varepsilon \in C^\infty(\mathbb{R})$ and $y_\varepsilon \rightrightarrows y$ as $\varepsilon \to 0$ (almost uniformly), then

$$(3.19)' \qquad \tilde{L_i}(y_\varepsilon) \to \tilde{L_i}(y).$$

Now we shall give theorems on the existence of the solution of problem (1.0)–(1.1). Apart from problem (1.0)–(1.1) we shall consider the homogeneous problem

$$(3.22) \qquad x_k'(t) = \sum_{j=1}^{n} A_{kj}(t)x_j(t),$$

$$(3.23) \qquad L_k(x_k) = 0, \quad k = 1, \dots, n.$$

Theorem 3.1 *We assume conditions* (3.1)–(3.2), (3.16)–(3.18). *Moreover, we assume that the trivial solution is the unique solution of problem* (3.22)–(3.23) *in* $\mathcal{G}^n(\mathbb{R})$. *Then problem* (1.0)–(1.1) *has exactly one solution in* $\mathcal{G}^n(\mathbb{R})$.

Remark 3.1 If A and f have properties (3.1)–(3.2), then the problem

$$(3.24) \qquad x'(t) = A(t)x(t) + f(t),$$

$$(3.25) \qquad x(t_0) = x_0, \quad t_0 \in \mathbb{R}, \quad x_0 \in \overline{\mathbb{R}}^n$$

has exactly one solution $x \in \mathcal{G}^n(\mathbb{R})$ (see [13]). Besides every solution $x \in \mathcal{G}^n(\mathbb{R})$ of equation (3.24) has a representation

$$(3.26) \qquad x(t) = Z(t)c + Q(t),$$

where $Z(t) = (z_{ij}(t))$ is a solution of the problem

$$(3.27) \qquad Z'(t) = A(t)Z(t), \quad Z(t_0) = I, \quad t_0 \in \mathbb{R},$$

$c = (c_1, \dots, c_n)^T$, c_i are generalized constants functions on $\mathbb{R}$ for $i = 1, \dots, n$, I denotes the identity matrix and Q is a particular solution of system (3.24). The solution x is the class of solutions of the problem

$$(3.28) \qquad x'(t) = R_A(\varphi, t)x(t) + R_f(\varphi, t)$$

$$(3.29) \qquad x(t_0) = R_{x_0}(\varphi), \quad \varphi \in \mathcal{A} \qquad \text{(see [13])}.$$

Remark 3.2 Let $x = (x_1, \dots, x_n)^T \in \mathcal{G}^n(\mathbb{R})$ and let

$$L_i^1(x_i) = x_i'(a), \quad L_i^2(x_i) = x_i(b) - x_i(a), \quad L_i^3(x_i) = x_i(t_i),$$

where $a, b, t_i \in \mathbb{R}$, $i = 1, \dots, n$. Then the operations L_i^1, L_i^2, L_i^3 have properties (3.16)–(3.19). The operations L_i^2 have not properties (3.20).

If $x_i \in C^1(\mathbb{R})$ and if $\tilde{L}_i^1(x_i) = x_i'(a)$, then $\tilde{L_i}^1$ have properties (3.16)′–(3.18)′, and $\tilde{L_i}$ have not property (3.19)′ in general.

Proof of Theorem 3.1 To this purpose we consider the following systems of equations

$$H \cdot c = b \tag{3.30}$$

and

$$H \cdot c = 0, \tag{3.31}$$

where

$$H = (H_{ij}), \quad H_{ij} = L_i(z_{ij}), \quad b = (b_1, \ldots, b_n)^T, \tag{3.32}$$

$$b_i = L_i(Q_i), \quad Q = (Q_1, \ldots, Q_n)^T; \quad i, j = 1, \ldots, n$$

and Z, Q have properties (3.26)–(3.27). From assumptions of Theorem 3.1 and from [16] we infer that $\det H$ is an invertible element of $\overline{\mathbb{R}}$. This proves the Theorem 3.1. □

Theorem 3.2 *We assume that*

(3.33) *all the assumptions of Theorem 3.1 are satisfied,*

$x(\varphi_\varepsilon, t)$ *is a solution of the problem*

$$x'(t) = R_A(\varphi_\varepsilon, t)x(t) + R_f(\varphi_\varepsilon, t), \tag{3.34}$$

$$L_i(x_i(\varphi_\varepsilon, t)) = R_{d_i}(\varphi_\varepsilon), \quad \varphi \in \mathcal{A}_N, \quad i = 1, \ldots, n \tag{3.35}$$

(for sufficiently large N and for small $\varepsilon > 0$).

Then

$$x(\varphi, t) \in \mathcal{E}_M^n[\mathbb{R}] \quad \text{and} \quad x = [x(\varphi, t)] \tag{3.36}$$

is a solution of problem (1.0)–(1.1) *(we put $x(\varphi_\varepsilon, t) = 0$ if $x(\varphi_\varepsilon, t)$ is not solution of problem* (3.34)–(3.35)*).*

Proof First we examine the problems

$$Z'(t) = R_A(\varphi_\varepsilon, t)Z(t), \quad Z(t_0) = I, \quad t_0 \in \mathbb{R}. \tag{3.37}$$

Let $R_Z(\varphi_\varepsilon, t)$ be a solution of problem (3.37). Then every solution $x(\varphi_\varepsilon, t)$ of equation (3.34) has the representation

$$x(\varphi_\varepsilon, t) = R_Z(\varphi_\varepsilon, t)c(\varphi_\varepsilon) + Q(\varphi_\varepsilon, t), \tag{3.38}$$

where

$$Q(\varphi_\varepsilon, t) = R_Z(\varphi_\varepsilon, t) \int_0^t (R_Z(\varphi_\varepsilon, s))^{-1} R_f(\varphi_\varepsilon, s) ds. \tag{3.39}$$

Now we consider equation (3.34) with the conditions

$$L_i(x, (\varphi_\varepsilon, t)) = R_{d_i}(\varphi_\varepsilon), \quad i = 1, \dots, n. \tag{3.40}$$

By (3.34), (3.38) and (3.40) we obtain the systems of equations

$$H(\varphi_\varepsilon)c(\varphi_\varepsilon) = b(\varphi_\varepsilon), \tag{3.41}$$

where

$$\begin{aligned} &H(\varphi_\varepsilon) = (H_{ij}(\varphi_\varepsilon)), \quad H_{ij}(\varphi_\varepsilon) = L_i(z_{ij}(\varphi_\varepsilon, t)), \\ &(z_{ij}(\varphi_\varepsilon, t)) = R_Z(\varphi_\varepsilon, t), \quad b_i(\varphi_\varepsilon) = R_{d_i}(\varphi_\varepsilon) - L_i(Q_i(\varphi_\varepsilon, t)), \\ &b(\varphi_\varepsilon) = (b_1(\varphi_\varepsilon), \dots, b_n(\varphi_\varepsilon))^T, \\ &Q(\varphi_\varepsilon, t) = (Q_1(\varphi_\varepsilon, t), \dots, Q_n(\varphi_\varepsilon, t))^T, \\ &c(\varphi_\varepsilon) = (c_1(\varphi_\varepsilon), \dots, c_n(\varphi_\varepsilon))^T; \quad i, j = 1, \dots, n. \end{aligned} \tag{3.42}$$

Taking into account relations (3.38)–(3.42), assumptions of Theorem 3.2 and Theorem from [16] we conclude that there is $N \in \mathbb{N}$ such that for every $\varphi \in \mathcal{A}_N$ there are $c > 0$, $\varepsilon_0 > 0$ such that

$$|\det H(\varphi_\varepsilon)| \geq c\varepsilon^N \quad \text{for } 0 < \varepsilon < \varepsilon_0. \tag{3.43}$$

Using (3.38)–(3.43) we deduce that problem (3.34)–(3.35) has exactly one solution $x(\varphi_\varepsilon, t)$ (for $\varphi \in \mathcal{A}_q$, $q \geq N$ and $0 < \varepsilon < \varepsilon_0$). By (3.41)–(3.43) we get

$$c(\varphi_\varepsilon) = H^{-1}(\varphi_\varepsilon)b(\varphi_\varepsilon) \tag{3.44}$$

(for $\varphi \in \mathcal{A}_N$ and $0 < \varepsilon < \varepsilon_0$). The last equalities, Remark 3.1 and relations (3.16)–(3.19) yield (we put $c_i(\varphi_\varepsilon) = 0$ and $x(\varphi_\varepsilon, t) = 0$ if $\det H(\varphi_\varepsilon) = 0$).

$$c_i(\varphi) \in \mathcal{E}_M \quad \text{for } i = 1, \dots, n. \tag{3.45}$$

Since

$$R_Z(\varphi, t) \in \mathcal{E}_M^{n \times n}[\mathbb{R}], \qquad (R_Z(\varphi, t))^{-1} \in \mathcal{E}_M^{n \times n}[\mathbb{R}], \tag{3.46}$$

therefore

$$x(\varphi, t) \in \mathcal{E}_M^n[\mathbb{R}] \tag{3.47}$$

which completes the proof of Theorem 3.2. □

Remark 3.3 We assume conditions (3.1)–(3.2) and $L_k(x_k) = x_k(a)$, $a \in \mathbb{R}$, $i = 1, \dots, n$. Then problem (3.22)–(3.23) has only the trivial solution in $\mathcal{G}^n(\mathbb{R})$ (see [13]).

Remark 3.4 We assume that the matrix A admits an ω-periodic representative $R_A(\varphi,t) = (R_{A_{kj}}(\varphi,t))$ satisfying conditions (3.1)–(3.3). Then system (3.22) has only the trivial ω-periodic solution in $\mathcal{G}^n(\mathbb{R})$ (see [17]).

Remark 3.5 We assume that the matrix A has an ω-periodic representative $R_A(\varphi,t) = (R_{A_{kj}}(\varphi,t))$ satisfying conditions (3.4)–(3.5). Then system (3.22) has only the trivial ω-periodic solution in $\mathcal{G}^n(\mathbb{R})$ (see [17]).

Remark 3.6 If the element p admits an ω-periodic representative $R_p(\varphi,t)$ satisfying conditions (3.2) and (3.13), then $x = 0$ is the unique ω-periodic solution of the equation

$$x''(t) + p(t)x(t) = 0 \tag{3.48}$$

in $\mathcal{G}(\mathbb{R})$ (see [15]).

Remark 3.7 If the element p admits an ω-periodic representative $R_p(\varphi,t)$ satisfying conditions (3.2)′ and (3.14)–(3.15), then $x = 0$ is the unique ω-periodic solution of equation (3.48) in $\mathcal{G}(\mathbb{R})$ (see [15]).

Remark 3.8 If conditions (3.10) are satisfied, then $x = 0$ is the unique ω-periodic solution of equation (3.48) in the Carathéodory sense (see [11]).

Remark 3.9 If conditions (3.8) are satisfied, then the problem

$$x''(t) + p(t)x(t) = 0, \quad x(a) = x(b) = 0, \ a \neq b, \ a, b \in \mathbb{R} \tag{3.49}$$

has only the trivial solution in the Carathéodory sense (see [3]).

If p and q have properties (3.9), then the problem

$$x''(t) + p(t)x'(t) + q(t)x(t) = 0, \quad x(a) = x(b) = 0, \ a \neq b, \ a, b \in \mathbb{R} \tag{3.50}$$

has only the trivial solution in the Carathéodory sense (see [4]).

Remark 3.10 If the element p admits a representative fulfilled conditions (3.2)′–(3.11), then problem (3.49) has only the trivial solution in $\mathcal{G}(\mathbb{R})$ (see [14]).

If elements p and q admit representatives $R_p(\varphi,t)$ and $R_q(\varphi,t)$ satisfying conditions (3.2)′; (3.12), then problem (3.5) has only the trivial solution in $\mathcal{G}(\mathbb{R})$ (see [14]).

Theorem 3.3 *We assume that the element $p \in \mathcal{G}(\mathbb{R})$ admits a representative $R_p(\varphi,t)$ satisfying conditions (3.2)′; (3.13)′. Then problem (3.49) has only the trivial solution in $\mathcal{G}(\mathbb{R})$.*

Proof Let x be a nontrivial solution of problem (3.49) in $\mathcal{G}(\mathbb{R})$. Then

$$R_{x''}(\varphi_\varepsilon,t) + R_p(\varphi_\varepsilon,t)R_x(\varphi_\varepsilon,t) = \eta(\varphi_\varepsilon,t), \tag{3.51}$$

where

$$(3.52)\qquad \eta(\varphi,t)\in\mathcal{N}[\mathbb{R}],\quad R_x(\varphi,a)\in\mathcal{N},\quad R_x(\varphi,b)\in\mathcal{N}.$$

Hence we get

$$(3.53)\qquad \int_a^b R_{x''}(\varphi_\varepsilon,t)R_x(\varphi_\varepsilon,t)dt+\int_a^b R_p(\varphi_\varepsilon,t)R_x^2(\varphi_\varepsilon,t)dt=\overline{\eta}(\varphi_\varepsilon),$$

where $\overline{\eta}(\varphi)=\int_a^b \eta(\varphi,t)R_x(\varphi,t)dt\in\mathcal{N}$.

Taking into account (3.31)–(3.53) we infer that

(3.54)

$$(R_{x'}(\varphi_\varepsilon,t)R_x(\varphi_\varepsilon,t))|_a^b-\int_a^b R_{x'}^2(\varphi_\varepsilon,t)dt+\int_a^b R_p(\varphi_\varepsilon,t)R_x^2(\varphi_\varepsilon,t)dt=\overline{\eta}(\varphi_\varepsilon).$$

Since

$$(3.55)\qquad R_{x'}(\varphi,b)R_x(\varphi,b)\in\mathcal{N},\qquad R_{x'}(\varphi,e)R_x(\varphi,a)\in\mathcal{N},$$

therefore

$$(3.56)\qquad \int_a^b (R_{x'}^2(\varphi_\varepsilon,t)-R_p(\varphi_\varepsilon,t)R_x^2(\varphi_\varepsilon,t))dt=\eta^*(\varphi_\varepsilon),$$

$$(3.57)\qquad \int_a^b R_{x'}^2(\varphi_\varepsilon,t)dt\in\mathcal{N}\qquad \text{(by (3.13)}')$$

and

$$(3.58)\qquad R_x(\varphi_\varepsilon,t)=\int_a^t R_{x'}(\varphi_\varepsilon,s)ds+R_x(\varphi_\varepsilon,a),$$

where $\eta^*(\varphi)\in\mathcal{N}$. Conditions (3.57)–(3.58) and the Schwarz inequality imply

$$(3.59)\qquad \|R_x(\varphi_\varepsilon,t)\|_{[a,b]}\le c\varepsilon^{\alpha(q)-N_0'}$$

where $\varphi\in\mathcal{A}_q$, $q\ge N_0'$ and $0<\varepsilon<\varepsilon_0'$. On the other hand we have

$$(3.60)\qquad \begin{aligned} R_x(\varphi_\varepsilon,t)=&-\int_a^t (t-s)(R_p(\varphi_\varepsilon,s)R_x(\varphi_\varepsilon,s)-\eta(\varphi_\varepsilon,s))ds\\ &+R_x(\varphi_\varepsilon,a)+R_{x'}(\varphi_\varepsilon,a)(t-a).\end{aligned}$$

Consequently (putting $t=b$)

$$(3.61)\qquad R_{x'}(\varphi_\varepsilon,a)\in\mathcal{N}$$

and (using the Gronwall inequality)

$$(3.62)\qquad \|D_rR_x(\varphi_\varepsilon,t)\|_K\le c_r\varepsilon^{\alpha(q)-N_r}$$

for $q\ge N_r$, $\varphi\in\mathcal{A}_q$ and $0<\varepsilon<\overline{\varepsilon}$. Thus

$$(3.63)\qquad R_x(\varphi,t)\in\mathcal{N}[\mathbb{R}]$$

which completes the proof of Theorem 3.3. □

Theorem 3.4 *We assume conditions* (3.1)–(3.2), (3.16)–(3.21). *Then* $x = (0,\dots,0)^T$ *is the unique solution of problem* (3.22)–(3.23) *in* $\mathcal{G}^n(\mathbb{R})$.

Proof If $x = (x_1,\dots,x_n)^T \in \mathcal{G}^n(\mathbb{R})$ is a nontrivial solution of problem (3.22)–(3.23), then

$$R_{x'}(\varphi,t) = R_A(\varphi,t)R_x(\varphi,t) + \eta(\varphi,t) \tag{3.64}$$

and

$$L_i(x_i(\varphi,t)) = r_i \in \mathcal{N}, \tag{3.65}$$

where $\eta \in \mathcal{N}^n[\mathbb{R}]$, $i = 1,\dots,n$ and R_x is a representative of x. By (3.64)–(3.65) and (3.16)–(3.21) we get

$$R_{x_k}(\varphi_\varepsilon,t) = \sum_{j=1}^{n}\Big(\int_a^t R_{A_{kj}}(\varphi_\varepsilon,s)R_{x_j}(\varphi_\varepsilon,s)\Big)\,ds \tag{3.66}$$

$$+\int_a^t \eta_k(\varphi_\varepsilon,s)ds + r_k - \sum_{j=1}^{n} L_i\Big(\int_a^t R_{A_{kj}}(\varphi_\varepsilon,s)R_{xj}(\varphi_\varepsilon,s)ds\Big) - L_i\Big(\int_a^t \eta_k(\varphi_\varepsilon,s)ds\Big).$$

Applying (3.20)–(3.21) and (3.66) we have

$$|R_{x_k}(\varphi_\varepsilon,t)| \le \sum_{j=1}^{n}\int_a^t |R_{A_{kj}}(\varphi_\varepsilon,t)R_{x_j}(\varphi_\varepsilon,t)|dt \tag{3.67}$$

$$+\int_a^t |\eta_k(\varphi_\varepsilon,t)|dt + |r_k| + \sum_{j=1}^{n} ||R_{x_j}(\varphi_\varepsilon,t)||_{[a,b]}\int_a^b |R_{A_{kj}}(\varphi_\varepsilon,t)|dt.$$

Hence

$$||R_x(\varphi_\varepsilon,t)||_{[a,b]} \le 2||R_x(\varphi_\varepsilon,t)||_{[a,b]}\beta_\varepsilon + \overline{\eta}(\varphi_\varepsilon), \tag{3.68}$$

where $\overline{\eta}(\varphi) \in \mathcal{N}$. Thus

$$||R_x(\varphi_\varepsilon,t)||_{[a,b]}(1-2\beta_\varepsilon) \le \overline{c}\varepsilon^{\alpha(q)-N} \tag{3.69}$$

Using (3.6) we obtain

$$||D_r R_x(\varphi_\varepsilon,t)||_{[a,b]} \le c_r\alpha^{(q)-N_r} \tag{3.70}$$

for $\varphi \in \mathcal{A}_q$, $q \ge N_r$ and $0 < \varepsilon < \overline{\varepsilon}$.

On the other hand

$$R_x(\varphi_\varepsilon,t) = R_x(\varphi_\varepsilon,t_0) + \int_{t_0}^t R_A(\varphi_\varepsilon,s)R_x(\varphi_\varepsilon,s) + \eta(\varphi_\varepsilon,s)ds, \tag{3.71}$$

where $t_0 \in (a,b)$.

By virtue of (3.70)–(3.71) and the Gronwall inequality we get

$$||D_r R_x(\varphi_\varepsilon, t)||_K \leq \bar{c}_r \varepsilon^{\alpha(q)-N_r'} \tag{3.72}$$

for $q \geq N_r'$, $\varphi \in \mathcal{A}_q$ and $0 < \varepsilon < \varepsilon'$ and consequently

$$R_x(\varphi, t) \in \mathcal{N}^n[\mathbb{R}]. \tag{3.73}$$

This proves the theorem. □

4 Final remarks

Remark 4.1 Let $p \in L^1_{loc}(\mathbb{R})$, then we put

$$R_p(\varphi, t) = \int_{-\infty}^{\infty} p(t+u)\varphi(u)du = (p * \varphi)(t), \tag{4.0}$$

where $\varphi \in \mathcal{A}_1$. Hence $p * \varphi_\varepsilon \to p$ in $L^1_{loc}(\mathbb{R})$ and $R_p(\varphi, t)$ has property (3.2). It is known that every distribution is moderate (see [1]). The problem (3.24) need not have a solution in $\mathcal{G}^n(\mathbb{R})$ (see [13]). Multiplication in $\mathcal{G}(\mathbb{R})$ does not coincide with usual multiplication of continuous function in general (see [1]). We denote the product in $\mathcal{G}(\mathbb{R})$ by $\circ$. If $p, x \in C^\infty(\mathbb{R})$, then the classical product px and the product $p \circ x$ in $\mathcal{G}(\mathbb{R})$ give rise to same element of $\mathcal{G}(\mathbb{R})$ (see [1]).

Theorem 4.1 *We assume that*

$$A_{kj}, f_k \in C^\infty(\mathbb{R}),\ d_k \in \mathbb{R} \quad \text{for } k, j = 1, \dots, n; \tag{4.1}$$

(4.2) $x = (0, \dots, 0)^T$ *is the unique solution of problem* (3.22)–(3.23) *in the classical sense,*

(4.3) $\bar{x} = (\bar{x_1}, \dots, \bar{x_n})^T$ *is the solution of problem* (1.0)–(1.1) *in the classical sense,*

$\tilde{x} = (\tilde{x_1}, \dots, \tilde{x_n})^T \in \mathcal{G}^n(\mathbb{R})$ *is the solution of the problem*

$$x_k'(t) = \sum_{j=1}^{n} A_{kj}(t) \circ x_j(t) + f_k(t), \tag{4.4}$$

$$L_k(x_k) = d_k, \quad k = 1, \dots, n; \tag{4.5}$$

(4.6) *the operations L_i have properties* (3.16)–(3.19).

Then $\bar{x}$ and $\tilde{x}$ give rise to the same element of $\mathcal{G}^n(\mathbb{R})$.

Proof Let $\tilde{x} = [R_{\tilde{x}}(\varphi, t)]$ be a solution of problem (4.4)–(4.5). Then

$$\overline{x}_k'(t) = \sum_{j=1}^{n} A_{kj}(t)\overline{x}_j(t) + f_k(t), \quad L_k(x_k) = d_k, \; k = 1, \ldots, n \tag{4.7}$$

and

$$R_{\tilde{x_k}'}(\varphi_\varepsilon, t) = \sum_{j=1}^{n} A_{kj}(t) R_{\tilde{x_j}}(\varphi_\varepsilon, t) + f_k(t) + \eta_k(\varphi_\varepsilon, t), \tag{4.8}$$

$$L_k(\tilde{x_k}(\varphi_\varepsilon, t)) = d_k + \overline{\eta}_k(\varphi_\varepsilon), \quad k = 1, \ldots, n, \tag{4.9}$$

where $\eta_k \in \mathcal{N}[\mathbb{R}]$, $\overline{\eta}_k \in \mathcal{N}$, $0 < \varepsilon < \varepsilon_0$, $\varphi \in \mathcal{A}_N$, N sufficiently large and $k = 1, \ldots, n$. Hence

$$R_{x'}(\varphi_\varepsilon, t) = A(t) R_x(\varphi_\varepsilon, t) - \eta(\varphi_\varepsilon, t) \tag{4.10}$$

$$L_k(R_x(\varphi_\varepsilon, t)) = -\overline{\eta}_k(\varphi_\varepsilon), \quad k = 1, \ldots, n. \tag{4.11}$$

where

$$R_x(\varphi_\varepsilon, t) = \overline{x}(t) - R_{\tilde{x}}(\varphi_\varepsilon, t), \qquad A(t) = (A_{kj}(t)). \tag{4.12}$$

On the other hand $R_x(\varphi, t)$ has the representation (3.38), where $R_A(\varphi, t) = A(t)$ and

$$Q(\varphi_\varepsilon, t) = -R_Z(\varphi_\varepsilon, t) \int_0^t (R_Z(\varphi_\varepsilon, s))^{-1} \eta(\varphi_\varepsilon, s) ds \in \mathcal{N}^n[\mathbb{R}]. \tag{4.13}$$

We consider system (3.41). The relations (4.6), (4.13), (3.38), (4.10)–(4.11), (3.42)–(3.44) and (3.46) yield

$$c(\varphi) \in \mathcal{N}^n \tag{4.14}$$

and consequently

$$\overline{x} - R_{\tilde{x}}(\varphi, t) \in \mathcal{N}^n[\mathbb{R}]. \tag{4.15}$$

This proves of Theorem 4.1.

"To repair" to consistency problem for multiplication we give the definition introduced by J. F. Colombeau in [1].

An element u of $\mathcal{G}(\mathbb{R})$ is said admit a number $W \in D'(\mathbb{R})$ as the associated distribution if it has a representative $R_u(\varphi, t)$ with the following property; for every $\psi \in \mathcal{D}(\mathbb{R})$ there is $N \in \mathbb{N}$ such that for every $\varphi \in \mathcal{A}_N$ we have

$$\lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} R_u(\varphi_\varepsilon, t)\psi(t) dt = W(\psi). \tag{4.16}$$

□

Theorem 4.2 *We assume that*

(4.17) $$A_{kj}, f_k \in L^1_{loc}(\mathbb{R}) \quad \text{for } k, j = 1, \dots, n;$$

$x^* = (0, \dots, 0)^T$ *is the unique solution of the problem*

(4.18) $$x'_k(t) = \sum_{j=1}^{n} A_{kj}(t) x_j(t), \quad \tilde{L}_k(x_k) = 0, \ k = 1, \dots, n$$

(in the Carathéodory sense),
x *is the solution of the problem*

(4.19) $$x'_k(t) = \sum_{j=1}^{n} A_{kj}(t) x_j(t) + f_k(t), \quad \tilde{L}_k(x_k) = d_k, \ d_k \in \mathbb{R}, \ k = 1, \dots, n$$

(in the Carathéodory sense),
$\overline{x} \in \mathcal{G}^n(\mathbb{R})$ *is the solution of the problem*

(4.20) $$x'_k(t) = \sum_{j=1}^{n} A_{kj}(t) \circ x_j(t) + f_j(t),$$

(4.21) $$L_k(x_k) = d_k, \quad k = 1, \dots, n'$$

(4.22) $$\tilde{L}_k, L_k \quad \text{have properties (3.16)–(3.19), (3.16)}'\text{–(3.19)}'.$$

Then $\overline{x}_k$ *admits associated distribution which equals* x_k *(*$k = 1, \dots, n$*).*

Proof follows from the facts that $R_{A_{kj}}(\varphi_\varepsilon, t) = (A_{kj} * \varphi_\varepsilon)(t) \to A_{kj}(t)$, $R_{f_k}(\varphi_\varepsilon, t) = (f_k * \varphi_\varepsilon)(t) \to f_k(t)$ in $L^1_{loc}(\mathbb{R})$ (for $k, j = 1, \dots, n$, $\varepsilon \to 0$) and the continuous dependence of x on coefficients A_{kj} and f_k. Indeed, let $R_Z(\varphi_\varepsilon, t) = (R_{z_{ij}}(\varphi_\varepsilon, t))$ be the solution of problem (3.37). Then we conclude that

(4.23) $$\lim_{\varepsilon \to 0} R_{z_{ij}}(\varphi_\varepsilon, t) = z_{ij}(t)$$

(almost uniformly for every fixed $\varphi \in \mathcal{A}_1$) and $i, j = 1, \dots, n$. By (3.19)′ and (4.18) we have

(4.24) $$\lim_{\varepsilon \to 0} \det H(\varphi_\varepsilon) = g \neq 0, \quad g \in \mathbb{R},$$

for every $\varphi \in \mathcal{A}_1$ ($\det H(\varphi_\varepsilon)$) is defined by (3.42). Let $R_x(\varphi_\varepsilon, t)$ be a solution of problem (3.34)–(3.35) (for small $\varepsilon > 0$, $\varphi \in \mathcal{A}_N$ and sufficiently large N). Relations (3.38)–(3.42), (3.34), (4.23)–(4 24), (3.16)–(3.19) yield

(4.25) $$\lim_{\varepsilon \to 0} R_{x_k}(\varphi_\varepsilon, t) = x_k(t), \quad k = 1, \dots, n.$$

(almost uniformly for every fixed $\varphi \in \mathcal{A}_N$) and $x = (x_1, \ldots, x_n)^T$ is a solution of problem (4.19) in the Caratheodory sense. On the other hand $\overline{x} = [R_x(\varphi, t)]$ is the solution of problem (4.20)–(4.21) (we put $R_x(\varphi_\varepsilon, t) = (0, \ldots, 0)^T$ if $\det H(\varphi_\varepsilon) = 0$). Proof of the fact is similar to the proof of Theorem 3.2. This proves of Theorem 4.2. □

Corollary 4.1 *We consider the following problems*

$$(4.26) \qquad \begin{cases} L(x) \equiv x^{(n)}(t) + p_1(t)x^{(n-1)}(t) + \ldots + p_n(t)x(t) = p_{n+1}(t) \\ L_i(x) = d_i, \quad d_i \in \overline{\mathbb{R}},\ i = 1, \ldots, n \end{cases}$$

and

$$(4.27) \qquad L(x) = 0, \quad L_i(x) = 0, \quad i = 1, \ldots, n;$$

where $p_j \in \mathcal{G}(\mathbb{R})$ ($j = 1, \ldots, n+1$) and L_i ($i = 1, \ldots, n$) have properties (3.16)–(3.19). *We assume that the matrix*

$$A = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ -p_n & -p_{n-1} & \ldots & \ldots & -p_1 \end{pmatrix}$$

satisfies condition (3.2) *and the trivial solution is the unique solution of problem* (4.27) *in $\mathcal{G}(\mathbb{R})$. Then problem* (4.26) *has exactly one solution x in $\mathcal{G}(\mathbb{R})$.*

The proof of the fact is similar to the proof of Theorem 3.1.

Corollary 4.2 *We assume that*

$$(4.28) \qquad p_j \in C^\infty(\mathbb{R}), \quad d_i \in \mathbb{R},\ j = 1, \ldots, n+1;\ i = 1, \ldots, n,$$

(4.29) *the zero function is the unique solution of problem* (4.27),

(4.30) *x is the solution of problem* (4.26) *in the classical sense,*

$\overline{x} \in \mathcal{G}(\mathbb{R})$ is the solution of the problem

$$(4.31) \qquad \begin{cases} \tilde{L}(x) \equiv x^{(n)} + p_1(t) \circ x^{(n-1)}(t) + \ldots + p_n(t) \circ x(t) = p_{n+1}(t) \\ L_i(x) = d_i, \end{cases}$$

(4.32) *the operations L_i ($i = 1, \ldots, n$) have properties* (3.16)–(3.19).

Then x and $\overline{x}$ give rise to the same element of $\mathcal{G}(\mathbb{R})$.

Corollary 4.3 *We assume that*

(4.33) $$p_j \in L^1_{loc}(\mathbb{R}), \quad d_i \in \mathbb{R}; \; j = 1, \ldots, n+1, \; i = 1, \ldots, n;$$

$\tilde{L}_i$ *(*$i = 1, \ldots, n$*) are operations such that:*

(3.16)* $$\tilde{L}_i(y) \in \mathbb{R} \quad \text{for } y \in C^{n-1}(\mathbb{R}),$$

(3.17)* $$\tilde{L}_i(\lambda_1 y_1 + \lambda_2 y_2) = \lambda_1 \tilde{L}_i(y_1) + \lambda_2 \tilde{L}_i(y_2),$$

$$\textit{where } \lambda_1, \lambda_2 \in \mathbb{R} \textit{ and } y_1, y_2 \in C^{n-1}(\mathbb{R}),$$

(3.19)* *if* $y_\varepsilon \in C^\infty(\mathbb{R})$, $y_\varepsilon^{(i)} \rightrightarrows y^{(i)}$ *as* $\varepsilon \to 0$ *(almost uniformly) for* $i = 0, 1, \ldots, n-1$, *then* $\tilde{L}_i(y_\varepsilon) \to \tilde{L}_i(y)$,

the operations $L_i, \tilde{L}_i$ *have properties* (3.16)–(3.19), (3.18)′,
the zero function is the unique solution of the problem

(4.34) $$L(x) = 0, \quad \tilde{L}_i(x) = 0, \quad i = 1, \ldots, n$$

in the Carathéodory sense,
x *is the solution of the problem*

(4.35) $$L(x) = p_{n+1}(t), \quad \tilde{L}_i(x) = d_i, \quad i = 1, \ldots, n$$

in the Carathéodory sense,

(4.36) $\overline{x} \in \mathcal{G}(\mathbb{R})$ *is the solution of problem (4.31).*

Then $\overline{x}$ *admits an associated distribution which equals* x.

Remark 4.2 Noncontinuous solutions of ordinary differential equations can be considered on the other way (for example: [1], [5]–[10], [12]–[13], [19]–[22]).

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