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# Independence in l-Groups

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**Abstract**

Four kinds of an independence on l-groups are studied in connection with order properties of l-groups and notions of generators and a direct product of subgroups.

**Key words:** l-group, independence, generator, direct product of subgroups.

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The linear independence of vectors is a basic notion in the theory of vector spaces. A general notion of independence (called "algebraic independence"), which contains as special cases majority of independence notions used in various branches of mathematics, was introduced by E. Marczewski [3] in 1958. However, there are independence notions which are not covered by this scheme, although they have much in common with it, such as linear independence in Abelian groups (see [2]).

In this paper we shall pay attention to properties of independence in lattice ordered groups (briefly l-groups). The methods of the research of the independence in l-groups come from ideas about the independence in vector spaces and Abelian groups and take advantage of the lattice order. The theory of l-groups useful in this paper is contained in the book of L. Fuchs [1].

**Notation** 1. Let $G$ be an l-group and $M \subseteq G$ be a set. Then $[M]$, ($<M>$, resp.) denotes the normal subgroup (the l-ideal, resp.) in $G$ generated by $M$. We shall write $[g]$ instead of $[\{g\}]$ and $<g>$ instead of $<\{g\}>$, for $g \in G$.

2. If $g \in G$ then $g^c$ denotes a conjugated element with $g$, i.e., $g^c = -a+g+a$, for some $a \in G$. Let us remark that $|-a+g+a| = -a+|g|+a$ holds.

3. Let us remark thet a sum is equal to zero in the case that the superscript is less than the subscript.

**Definition 1** A subset $M$ of an l-group $G$ is called *independent* (*l-independent*, resp.), when $M$ is a non-empty set and $N \subset M \Rightarrow [N] \subset [M]$ ($N \subset M \Rightarrow\ <N> \subset <M>$, resp.) holds.

On the other hand, i.e., if a subset $N$ of $M$ exists such that $N \neq M$ and $[N] = [M]$ ($<N> = <M>$, resp.)- then $M$ is called *dependent* (*l-dependent*, resp.).

**Remarks** 1. This definition is an analogy of the linear independence in vector spaces, where $<M>$ is the vector subspace generated by $M$.

2. If $0 \in M \subseteq G$ then $M$ is dependent and l-dependent. Namely, $[M] = [M \backslash \{0\}]$ and $<M> = <M\backslash\{0\}>$ hold.

**Proposition 2** *Every non-empty subset $N$ in an independent (l-independent, resp.) set $M$ in an l-group $G$ is independent (l-independent,resp.).*

**Proof** Let $N$ be l-dependent, i.e., there exists $N' \subset N$ such that $<N'> = <N>$. Then $N' \cup (M \backslash N) \subset M$ and we shall prove that $<N' \cup (M - N)> = <M>$. If $x \in <M>$ then

$$|x| \leq \sum_{i=1}^{n} p_i |m_i^c|,$$

for $m_1, \ldots, m_j \in M \backslash N$, $m_{j+1}, \ldots, m_n \in N$ and natural numbers $p_1, \ldots, p_n$ (see Notation 2). The fact $<N'> = <N>$ implies an existence of $h_{l_1}, \ldots, h_{l_{n_l}} \in N'$ and natural numbers $q_{l_1}, \ldots, q_{l_{n_l}}$ such that

$$|m_l| \leq \sum_{k=1}^{n_l} q_{l_k} |h_{l_k}^c|$$

holds for all $l \in \{j+1, \ldots, n\}$. Finally, we have

$$|x| \leq \sum_{i=1}^{j} p_i |m_i^c| + \sum_{i=j+1}^{n} p_i \Big(\sum_{k=1}^{n_l} q_{l_k} |h_{l_k}^c|\Big) \in\ <N' \cup (M \backslash N)>$$

and $<M> = <N' \cup (M \backslash N)>$, which is a contradiction.

The proposition for the independence can be proved similarly. □

**Proposition 3** *A non-empty set $M$ in an l-group $G$ is independent (l-independent, resp.) if and only if every non-empty finite subset of $M$ is independent (l-independent, resp.).*

**Proof** The proposition will be shown for the l-independence.

$\Rightarrow$: It follows from 2.

$\Leftarrow$: If $M$ is l-dependent then a set $N$ exists such that $N \subset M$, $<N> = <M>$. It means that $|x| \leq \sum_{i=1}^{n} p_i |m_i^c|$ holds for $x \in M \backslash N$, where $m_1, \ldots, m_n \in N$ and $p_1, \ldots, p_n$ are natural numbers (see Notation 2). This facts imply $\{m_1, \ldots, m_n\} \subset \{m_1, \ldots, m_n, x\} \subseteq M$ and $<m_1, \ldots, m_n, x> \subseteq\ <m_1, \ldots, m_n>$, i.e., $\{m_1, \ldots, m_n, x\}$ is an l-dependent set, a contradiction. □

**Proposition 4** *Let $M$ be a non-empty set in an l-group $G$. Then $M$ is independent (l-independent, resp.) if and only if it holds $m$ non $\in [M\setminus\{m\}]$ ($m$ non $\in <M\setminus\{m\}>$, resp.), for all $m \in M$.*

**Proof** $\Rightarrow$: If $m \in M$ exists such that $m \in <M\setminus\{m\}>$ then $<M> = <M\setminus\{m\}>$, a contradiction.

$\Leftarrow$: If $M$ is l-dependent then $N \subset M$ exists such that $<N> = <M>$, and thus $m \in M\setminus N$ exists with the property $m \in <M> = <N> \subsetneq <M\setminus\{m\}>$, a contradiction.

The proposition for the independence can be proved similarly. □

**Remark** If 0 non $\in M$ is a non-empty subset in an l-group $G$ and $[m] \cap [M\setminus\{m\}] = \{0\}$ holds for all $m \in M$, then $M$ is independent. The linear independence on vector spaces with scalar products is equivalent with the upper property. From these reasons the independence on l-groups is not equivalent with the upper property.

**Definition 5** A set $M$ in an l-group $G$ is called *linearly independent (linearly l-independent*, resp.) when 0 non $\in M$, $M$ is a non-empty set and $[m] \cap [M\setminus\{m\}] = \{0\}$ ($<m> \cap <M\setminus\{m\}> = \{0\}$) holds for all $m \in M$.

If an element $m \in M$ exists with the property $[m] \cap [M\setminus\{m\}] \neq \{0\}$ ($<m> \cap <M\setminus\{m\}> \neq \{0\}$, resp.) then $M$ is called *linearly dependent (linearly l-dependent, resp.)*.

**Proposition 6** *Let $M = \{m_1, \dots, m_n\}$ be a finite subset of an l-group $G$. Then it holds:*

*a) $M$ is linearly dependent if and only if integer numbers $p_1, \dots, p_n$ exist such that*

$$\sum_{j=1}^{n} p_j m_j^c = 0 \quad \text{and} \quad \sum_{j=1}^{n} p_j \neq 0.$$

*b) $M$ is dependent if and only if $i \in \{1, \dots, n\}$ and integer numbers $p_1, \dots, p_{j-1}, p_{j+1}, \dots, p_n$ exist such that*

$$m_i = \sum_{j=1}^{i-1} p_j m_j^c + \sum_{j=i+1}^{n} p_j m_j^c.$$

*c) $M$ is linearly l-dependent if and only if $0 \neq x \in G$, $i \in \{1, \dots, n\}$ and natural numbers $l_i, p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n$ exist such that*

$$|x| \leq l_i |m_i^c| \wedge \Big( \sum_{j=1}^{i-1} p_j |m_j^c| + \sum_{j=i+1}^{n} p_j |m_j^c| \Big).$$

d) *M is l-dependent if and only if* $i \in \{1, \dots, n\}$ *and natural numbers* $p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n$ *exist such that*

$$|m_i| \leq \sum_{j=1}^{i-1} p_j |m_j^c| + \sum_{j=i+1}^{n} p_j |m_j^c|.$$

**Proof** We can prove all propositions a)–d) similarly. Let us prove the prop. a): $M$ is linearly dependent if and only if an index $i \in \{1, \dots, n\}$ exists such that $[m_i] \cap [M \backslash \{m_i\}] \neq \{0\}$. This fact is equivalent with the existence of an index $i \in \{1, \dots, n\}$ and non-zero integer number $l_i$ and integer numbers $p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n$ such that $l_i m_i^c = \sum_{j=1}^{i-1} p_j m_j^c + \sum_{j=i+1}^{n} p_j m_j^c$ (see Notation 2). □

**Corollary 7** *a) All non-empty subsets in a linearly independent (linearly l-independent, resp.) set M of an l-group G are linearly independent (linearly l-independent, resp.).*

*b) A non-empty set M of an l-group G is linearly independent (linearly l-independent, resp.) if and only if all non-empty finite subsets in M are linearly independent (linearly l-independent, resp.).*

**Proof** a) If $N$ is a linearly l-dependent non-empty subset in $M$ then $n \in N$ exists such that $<n> \cap <N \backslash \{n\}> \neq 0$ and thus $<n> \cap <M \backslash \{n\}> \neq 0$, a contradiction.

b) $\Rightarrow$: It follows from a).

$\Leftarrow$: If $M$ is linearly l-dependent then $m_1 \in M$ exists such that $<m_1> \cap <M \backslash \{m_1\}> \neq \{0\}$ holds. It means, that $l_1 |m_1^c| \wedge \sum_{j=2}^{n} p_j |m_j^c| \neq 0$ holds for suitable natural numbers $l_1, p_2, \dots, p_n$ and elements $m_2, \dots, m_n \in M \backslash \{m_1\}$ (see Notation 2).

Finally, $<m_1> \cap <m_2, \dots, m_n> \neq 0$ and therefore the set $\{m_1, m_2, \dots, m_n\}$ is linearly l-dependent, a contradiction.

The corollary for linear independence can be proved similarly. □

**Remark** If $G$ is an l-group and a subset $\{a, b\}$ in $G$ is dependent (independent, resp.) in the sense of upper definitions that we can say that elements $a, b$ are dependent (independent, resp.) of the corresponding type.

**Corollary 8** *Let G be an l-group and* $a, b \in G$, $a \neq 0 \neq b$. *Then it holds:*

1. *Elements* $a, b$ *are independent if and only if normal subgroups* $[a], [b]$ *are incomparable.*
2. *Elements* $a, b$ *are l-independent if and only if l-ideals* $<a>, <b>$ *are incomparable.*
3. *Elements* $a, b$ *are linearly independent if and only if* $[a] \cap [b] = \{0\}$ *holds.*

*4. The following propositions are equivalent:*

*a) Elements $a, b$ are linearly l-independent.*

*b) $\langle a\rangle \cap \langle b\rangle = \{0\}$.*

*c) Elements $a^c, b^c$ are orthogonal (i.e., $|-g+a+g| \wedge |-h+b+h| = 0$ for all $g, h \in G$.*

**Proof** It follows from 6. □

**Proposition 9** *A non-empty set $M$ in an l-group $G$ is linearly l-independent if and only if $|-g+a+g| \wedge |-h+b+h| = 0$ holds for all $a, b \in M, g, h \in G, a \neq b$.*

**Proof** $\Rightarrow$: It follows from 7, b and 8, 4 a) $\Rightarrow$ c).

$\Leftarrow$: If $\{m_1, \ldots, m_n\}$ is a finite subset in $M$ then it is sufficient to prove that $\{m_1, \ldots, m_n\}$ is linearly l-independent (see 7, b). Namely, if $\{m_1, \ldots, m_n\}$ is linearly l-dependent then $0 \neq x \in G, i \in \{1, \ldots, n\}$, a natural number $l_i$ and natural numbers $p_1, \ldots, p_{i-1}, p_{i+1}, \ldots, p_n$ exist (see 6, c)) with the property $|x| \leq l_i|m_i^c| \wedge (\sum_{j=1}^{i-1} p_j|m_j^c| + \sum_{j=i+1}^{n} p_j|m_j^c|)$. Finally, the fact $|m_i^c| \wedge |m_j^c| = 0$ , for $j = 1, \ldots, i-1, i+1, \ldots, n$ implies $l_i|m_i^c| \wedge (\sum_{j=1}^{i-1} p_j|m_j^c| + \sum_{j=i+1}^{n} p_j|m_j^c|) = 0$ (see Notation 2), a contradiction. □

**Theorem 10** *Let $G$ be an l-group and $M$ be a subset of $G$. Then $1^o \Rightarrow 2^o \Rightarrow 3^o \Rightarrow 4^o$ hold for the following assertions:*

*$1^o$ $M$ is linearly l-independent.*

*$2^o$ $M$ is l-independent.*

*$3^o$ $M$ is linearly independent.*

*$4^o$ $M$ is independent.*

**Proof** Let (i) ((ii) resp., (iii) resp., (iv) resp.) means that the set $M$ is dependent (linearly dependent resp., l-dependent resp., linearly l-dependent resp.). Let us prove that $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv)$ hold:

$(i) \Rightarrow (ii)$: There exists $N \subset M$ such that $[N] = [M]$. Therefore an element $0 \neq m \in M \backslash N$ exists with the property $m \in [N] \subseteq [M\backslash\{m\}]$. Clearly, $[m] \cap [M\backslash\{m\}] = [m] \cap [M] = [m] \neq \{0\}$ holds.

$(ii) \Rightarrow (iii)$: An element $0 \neq m \in M$ exists such that $[m] \cap [M\backslash\{m\}] \neq \{0\}$. This fact implies

$$x = km^c = \sum_{i=1}^{n} l_i m_i^c$$

for suitable elements $0 \neq x \in G$, $m_1, \ldots, m_n \in M\backslash\{m\}$ and integer numbers $k, l_1, \ldots, l_n$. Therefore

$$|m^c| \leq |k||m^c| = |km^c| = |\sum_{i=1}^{n} l_i m_i^c|$$

holds (see Notation 2). It is clear that $m \in \langle M\backslash\{m\}\rangle$ and $\langle m\rangle \subseteq \langle M\backslash\{m\}\rangle$. Finally, we have $\langle M\rangle \subseteq \langle M\backslash\{m\}\rangle$.

$(iii) \Rightarrow (iv)$: We can prove similarly as $(i) \Rightarrow (ii)$ for l-ideals. □

**Remark** If $a, b$ are linearly dependent elements then $[a] \cap [b] \neq \{0\}$ and $<a> \subseteq <b>$ follows from the previous proof $((ii) \Rightarrow (iii))$. Similarly $<b> \subseteq <a>$ holds and thus $<a> = <b>$.

**Corollary 11** *Let $a, b$ be non-zero elements of an abelian l-group $G$. Then it holds:*

1. *$[a] = [b]$ if and only if $a = b$ or $a = -b$.*
2. *$a, b$ are dependent if and only if an integer number $k$ exists such that $a = kb$ or an integer number $l$ exists such that $b = la$.*
3. *$a, b$ are linearly dependent if and only if non-zero integer numbers $k, l$ exist such that $ka = lb$.*
4. *$<a> = <b>$ if and only if natural numbers k,l exist such that $|a| \leq k|b|$ and $|b| \leq l|a|$.*
5. *$a, b$ are l-dependent if and only if a natural number $k$ exists such that $|a| \leq k|b|$ or a natural number $l$ exists such that $|b| \leq l|a|$.*
6. *$a, b$ are linearly l-dependent if and only if $|a| \wedge |b| \neq 0$.*

**Proof** It follows from 6, 8 and the previous remark. Clearly, $[a] = [b]$ if and only if integer numbers k,l exist such that $a = kb, b = la$ and therefore $a = kb = kla$ and $a(kl - 1) = 0$ hold. The fact that $G$ is torsion free implies $kl = 1$ and $k = l = 1$ or $k = l = -1$. □

**Proposition 12** *A union of an increasing chain of independent (l-independent resp., linearly independent resp., linearly l-independent resp.) subsets of an l-group $G$ is again independent (l-independent resp., linearly independent resp., linearly l-independent resp.) in $G$.*

**Proof** We shall prove only for the independence. Let $\{M_i : i \in I\}$ be an increasing chain of independent subsets in $G$, $M = \cup_{i \in I} M_i$ and let us prove that all non-empty finite subset $K$ in $M$ is also independent. If $K = \{k_1, \ldots, k_n\}$ then subsets $M_j$ exist such that $k_j \in M_j$ for $j = 1, \ldots, n$. Therefore $K \subseteq \cup_{j=1}^{n} M_j \subseteq M_l$ for some $l \in I$ and $K$ is independent. □

**Corollary 13** *Every independent (l-independent resp., linearly independent resp., linearly l-independent resp.) set of an l-group $G$ is contained in a maximal independent (l-independent resp., linearly independent resp., linearly l-independent resp.) set in $G$.*

**Proof** Follows from 12 and the Zorn's lemma. □

**Definition 14** *We say that a subset $M$ of an l-group $G$ generates (l-generates, resp.) the l-group $G$ when $[M] = G(<M> = G$,resp.) holds. The elements from $M$ are called generators (l-generators, resp.) of the l-group $G$.*

**Proposition 15** *All maximal independent sets of an l-group $G$ are systems of l-generators in $G$.*

**Proof** Let $L$ be a maximal independent set in $G$ and let $g \in G \setminus L$. Then $L \cup \{g\}$ is dependent and thus a finite dependent subset $F = \{f_1, \dots, f_n\}$ exists in $L \cup \{g\}$. Clearly $g \in F$ and we can choose $g = f_1$. Proposition 6 implies the existence of $i \in \{1, \dots, n\}$ and integer numbers $p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n$ such that

$$f_i = \sum_{j=1}^{i-1} p_j f_j^c + \sum_{j=i+1}^{n} p_j f_j^c.$$

If $i = 1$ then

$$g = \sum_{j=1}^{i-1} p_j f_j^c + \sum_{j=i+1}^{n} p_j f_j^c$$

and thus $g$ is l-generated by elements from $L$. If $i \neq 1$ then

$$f_i = \sum_{j=2}^{i-1} p_j f_j^c + \sum_{j=i+1}^{n} p_j f_j^c + p_1 g^c$$

and thus

$$|g^c| \leq |p_1||g^c| = |p_1 g^c| = |f_i - \sum_{j=2}^{i-1} p_j f_j^c - \sum_{j=i+1}^{n} p_j {f_j}^c|$$

(see Notation 2). Finally, $g \in \langle F \rangle \subseteq \langle L \rangle$ and $L$ is a system of l-generators of $G$. □

**Theorem 16** *Let $G$ be an l-group, $I$ be a non-empty set and $M = \{m_i : i \in I\}$ be a subset of $G$. Then $G$ is a direct product of subgroups $\langle m_i \rangle$ if and only if $M$ is linearly l-independent system of l-generators of $G$.*

**Proof** Recall that $G$ is a direct product of subgroups $H_i (i \in I)$ if and only if all $H_i$ are normal subgroups in $G$, $G$ is generated by subgroups $H_i$ and

$$H_i \cap \sum_{j \in I \setminus \{i\}} H_j = \{0\}$$

holds for all $i \in I$.

$\Rightarrow$: For all $g \in G$ it holds $|g| = \sum_{l=1}^{n} h_l$, where $h_l \in \langle m_l \rangle$, i.e.,

$$|g| \leq \sum_{l=1}^{n} \sum_{j=1}^{n_l} p_j |m_j^c|,$$

where $p_j, n, n_l$ are natural numbers (see Notation 2). Finally, $M$ is a system of l-generators in $G$.

Further, $|m_i^c| \wedge |m_j^c| \in \langle m_i \rangle \cap \langle m_j \rangle = \{0\}$ for all $m_i, m_j \in M$, $m_i \neq m_j$ and Proposition 9 follows that $M$ is a linearly l-independent set.

$\Leftarrow$: l-ideals $<m_i>$ are normal subgroups (for $i \in I$) and $g^+ \leq \sum_{j=1}^{n} p_j|m_j^c|$, $-g^- \leq \sum_{j=1}^{n'} p_j'|m_j'^c|$ hold for suitable elements $m_j, m_j' \in M$ and natural numbers $p_j, p_j', n, n'$. Corollary 2, p. 105, [1] follows the existence of elements $h_j, h_j' \in G$ such that $0 \leq h_j \leq p_j|m_j^c|$, $0 \leq h_j' \leq p_j'|m_j'^c|$ and $g^+ = \sum_{j=1}^{n} h_j$, $-g^- = \sum_{j=1}^{n'} h_j'$. Therefore

$$g = g^+ + g^- = \sum_{j=1}^{n} h_j - \sum_{j=1}^{n'} h_j'$$

and $G$ is generated by subgroups $<m_i>, i \in I$. Finally, $g = h_1 + \cdots + h_l$ holds for all $g \in <m_i> \cap \sum_{j \in I\setminus\{i\}} <m_j>$, where $h_j \in <m_j>$, $j = 1, \ldots, l$, $j \neq i$. Thus

$$|g| \wedge |h_j| \leq \sum_{k=1}^{u} p_k|a_k| \wedge \sum_{h=1}^{v} q_h|b_h|,$$

where $p_k, q_h, u, v$ are natural numbers and $a_k$ ($b_h$, resp.) is conjugated with $m_i$ ($m_j$, resp.). Proposition 9 follows that $|a_k| \wedge |b_h| = 0$ (for $k = 1, \ldots, u$, $l = 1, \ldots, v$) and thus $|g| \wedge |h_j| = 0$ ($j = 1, \ldots, l$). We have $|g| = |g| \wedge |h_1 + \cdots + h_l| = 0$, $g = 0$ and together $G$ is a direct product of subgroups $<m_i>$. □

**Remark** Let $M = \{m_i : i \in I\}$ be a non-empty subset of an l-group $G$. Then we can prove similarly, that $G$ is a direct product of subgroups $[m_i] (i \in I)$ if and only if $M$ is a linearly independent system of generators in $G$.

Let us investigate a linearly l-independent system of l-generators of an l-group $G$ in the last part of this paper. We could take this system for an l-basis of $G$ but the following propositions show that an introduction of that notion is not acceptable.

**Corollary 17** *A linearly l-independent system $S$ of l-generators of an l-group $G$ is a maximal linearly l-independent set and a minimal system of l-generators in $G$.*

**Proof** If $H$ is a linearly l-independent subset in $G$, $S \subsetneq H$, then $|h| \leq \sum_{j=1}^{n} p_j|s_j^c|$ for $h \in H\setminus S$, where $p_j$ are natural numbers, $s_j \in S$ for $j = 1, \ldots, n$ (Notation 2). Proposition 9 follows $|h| = |h| \wedge \sum_{j=1}^{n} p_j|s_j^c| = 0$, i.e., $h = 0$, a contradiction. Thus $S$ is a maximal linearly l-independent set in $G$.

If $L$ is a system of l-generators of an l-group $G$, $L \subsetneq S$, then $|s| \leq \sum_{j=1}^{n} p_j|l_j^c|$ for $s \in S\setminus L$, where $p_j$ are natural numbers and $l_j \in L$ for $j = 1, \ldots, n$. Proposition 9 follows $|s| = |s| \wedge \sum_{j=1}^{n} p_j|l_j^c| = 0$, i.e., $s = 0$, a contradiction. □

**Remark** A minimal system of l-generators, which is l-independent but is not linearly l-independent, exists in the additive l-group of all real functions on $[0, 1]$.

**Proposition 18** *Let $G$ be an l-group and $M$ be a system of l-generators in $G$. Then the following assertions are equivalent:*
*(i) $M$ is a minimal system of l-generators in $G$.*
*(ii) $M$ is an l-independent set in $G$.*
*(iii) $M$ is a maximal l-independent set in $G$.*

**Proof** It follows from definitions immediately. □

**Example 19** Let $G$ be a linearly ordered group and $0 \leq f \in G, f \neq 0$. Then $|g| \leq f$ or $f \leq |g|$ for all $g \in G$, i.e., $\{f, g\}$ is l-dependent. $\{f\}$ is a maximal l-independent set and a maximal linearly l-independent set in $G$. If $\{f\}$ is a system of l-generators in $G$ then $|g| \leq nf$ holds for all $g \in G$ and suitable natural number $n$. Finally, if $G$ is totally non-archimedean, in the sense that for all $a \in G$ there exists $b \in G$ such that $n|a| \leq b$ holds for all natural numbers n, then $G$ has no l-independent system of l-generators. If a finite system of l-generators exists in an l-group $G$ then a minimal system of l-generators exists in $G$.

**Example 20** The additive group $K$ of complex numbers is an archimedean l-group with the positive cone $K^+ = \{a+bi : 0 \leq a, 0 \leq b\}$. Then $K = <1+i> = <1> + <i>$ and $\{1+i\}, \{1, i\}$ are linearly l-independent systems of l-generators in $K$ (see Theorem 16). Thus l-groups exist with finite (linearly) l-independent systems of l-generators which have different numbers of elements.

**Proposition 21** *Let $M, N$ be infinite l-independent systems of l-generators of an l-group $G$. Then sets $M$ and $N$ have the same cardinality.*

**Proof** A finite subsets $N_a$ in $N$ exist such that $a \in <N_a>$ for all $a \in M$. It means that $G = <M> \subseteq < \cup_{a \in M} N_a>$ and $N \subseteq \cup_{a \in M} N_a \subseteq N$ holds from 18. Finally, $N = \cup_{a \in M} N_a$ and $|N| \leq \sum_{a \in M} |N_a| \leq \aleph_0 |M| \leq |M|$ hold. We have $|N| \leq |M|$ and similarly $|M| \leq |N|$. □

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