Jiří Rachůnek

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Connections between Ideals of Non-Commutative Generalizations of MV-algebras and Ideals of their Underlying Lattices *

JIŘÍ RACHŮNEK

Department of Algebra and Geometry, Faculty of Science,
Palacký University, Tomkova 40, 779 00 Olomouc, Czech Republic
e-mail: rachunek@risc.upol.cz

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Abstract

GMV-algebras are a non-commutative generalization of MV-algebras. In the paper we study connections between ideals of any GMV-algebra $\mathcal{A}$ and those of the corresponding underlying lattice $L(\mathcal{A})$.

Key words: GMV-algebra, ideal, Stonean ideal.

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1 Introduction

As is well-known, MV-algebras were introduced by C. C. Chang in [2] as an algebraic counterpart of the Łukasiewicz infinite valued propositional logic. GMV-algebras introduced recently by G. Georgescu and A. Iorgulescu in [6] and [7], and by the author in [8], are a non-commutative generalization of MV-algebras. Recall that by a fundamental result of A. Dvurečenskij in [4], GMV-algebras are in a close connection with unital lattice ordered groups (ℓ-groups).

If $\mathcal{A}$ is a GMV-algebra then one can define by a standard method the lattice $L(\mathcal{A})$ on the same underlying set. In the paper we study connections between

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ideals of any GMV-algebra $\mathcal{A}$ and those of the corresponding lattice $L(\mathcal{A})$. In particular, we deal with the cases of prime ideals. Further we characterize GMV-algebras $\mathcal{A}$ with the property that each ideal of $\mathcal{A}$ is a Stonean ideal of $L(\mathcal{A})$.

Necessary results concerning the theory of MV-algebras can be found e.g. in [3], the book [5] contains also the foundations of the theory of GMV-algebras.

2 Ideals and prime ideals of GMV-algebras and corresponding lattices

The following notion of a GMV-algebra has been introduced and studied by G. Georgescu and A. Iorgulescu in [6] and [7], and independently by the author in [8].

Definition Let $\mathcal{A} = (A, \oplus, \neg, \sim, 0, 1)$ be an algebra of type $\langle 2,1,1,0,0\rangle$. Set $x \odot y = \sim (\neg x \oplus \neg y)$ for any $x, y \in A$. Then $\mathcal{A}$ is called a *generalized MV-algebra* (in short: *GMV-algebra*) if for any $x, y, z \in A$ the following conditions are satisfied:

(A1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;

(A2) $x \oplus 0 = x = 0 \oplus x$;

(A3) $x \oplus 1 = 1 = 1 \oplus x$;

(A4) $\neg 1 = 0 = \sim 1$;

(A5) $\neg(\sim x \oplus \sim y) = \sim (\neg x \oplus \neg y)$;

(A6) $x \oplus (y \odot \sim x) = y \oplus (x \odot \sim y) = (\neg y \odot x) \oplus y = (\neg x \odot y) \oplus x$;

(A7) $(\neg x \oplus y) \odot x = y \odot (x \oplus \sim y)$;

(A8) $\sim \neg x = x$.

(If the operation $\oplus$ is commutative then the unary operations $\neg$ and $\sim$ coincide and $\mathcal{A}$ is an MV-algebra.)

If we put $x \leq y$ if and only if $\neg x \oplus y = 1$ then "$\leq$" is an order on A. Moreover, $(A, \leq)$ is a bounded distributive lattice in which $x \vee y = x \oplus (y \odot \sim x)$ and $x \wedge y = x \odot (y \oplus \sim x)$ for each $x, y \in A$, and 0 is the least and 1 is the greatest element in A, respectively. We set $L(\mathcal{A}) = (A, \vee, \wedge)$ for any GMV-algebra $\mathcal{A}$.

(The above definition is that introduced by Georgescu and Iorgulescu in [6] and [7], where they use the name a *pseudo-MV algebra*.)

GMV-algebras are in a close connection with unital ℓ-groups. (Recall that a *unital ℓ-group* is a pair (G, u) where G is an ℓ-group and u is a strong order unit of G.) If G is an ℓ-group and $0 \leq u \in G$ then $\Gamma(G, u) = ([0, u], \oplus, \neg, \sim, 0, 1)$, where $[0, u] = \{x \in G; 0 \leq x \leq u\}$, and for any $x, y \in [0, u]$, $x \oplus y = (x + y) \wedge u$, $\neg x = u - x$, $\sim x = -x + u$, is a GMV-algebra. Conversely, A. Dvurečenskij in [4] proved that every GMV-algebra is isomorphic to $\Gamma(G, u)$ for an appropriate unital ℓ-group (G, u).

Let us recall the notion of an ideal of a GMV-algebra. (See [7].) Let $\mathcal{A}$ be a GMV-algebra and $\emptyset \neq H \subseteq A$. Then H is called an *ideal* of $\mathcal{A}$ if

(i) $x \oplus y \in H$ for any $x, y \in H$;

(ii) $y \leq x$ implies $y \in H$ for any $x \in H$ and $y \in A$.

An ideal I of a GMV-algebra $\mathcal{A}$ is called *normal* if

(iii) $\neg x \odot y \in I$ if and only if $y \odot \sim x \in I$ for each $x, y \in A$.

If $\mathcal{A}$ is a GMV-algebra, denote by $\mathcal{C}(\mathcal{A})$ the set of ideals of $\mathcal{A}$. Then $\mathcal{C}(\mathcal{A})$ ordered by set inclusion is a complete lattice. An ideal H of a GMV-algebra $\mathcal{A}$ is called *prime* (see [7]) if H is a finitely meet-irreducible element in the lattice $\mathcal{C}(\mathcal{A})$.

Theorem 1 *If $\mathcal{A}$ is a GMV-algebra and $I \in \mathcal{C}(\mathcal{A})$ then I is an ideal of $L(\mathcal{A})$. Moreover, $I \in \mathcal{C}(\mathcal{A})$ is a prime ideal of $\mathcal{A}$ if and only if I is a prime ideal of the lattice $L(\mathcal{A})$.*

Proof If $I \in \mathcal{C}(\mathcal{A})$ and $x, y \in I$, then $x \vee y \leq x \oplus y \in I$, and thus $x \vee y \in I$, and hence I is an ideal of the lattice $L(\mathcal{A})$. At the same time, the prime ideals of $\mathcal{A}$ are characterized by [7], Theorem 2.17, as ideals satisfying the property

$$\forall x, y \in A; x \wedge y \in I \Longrightarrow x \in I \quad \text{or} \quad y \in I.$$

The same property also characterizes the prime ideals of the lattice $L(\mathcal{A})$, hence the second assertion. □

Remark 1 Note that an ideal of the lattice $L(\mathcal{A})$ need not be an ideal of $\mathcal{A}$. Obviously, if $x \in A$ is not additively idempotent, i.e. $x < x \oplus x$, then the principal ideal of the lattice $L(\mathcal{A})$ is not an ideal of $\mathcal{A}$.

Theorem 2 *Let $\mathcal{A}$ be a GMV-algebra and let I be a proper ideal of the lattice $L(\mathcal{A})$. Set $I_z = \{x \in A; \neg x \odot z \notin I\}$ for $z \in A$. Let $K = K_I = \bigcap(I_z; z \notin I)$. Then $K \subseteq I$ and K is an ideal of the GMV-algebra $\mathcal{A}$. Moreover, if I is a prime ideal of $L(\mathcal{A})$ then K is a prime ideal of $\mathcal{A}$.*

Proof Obviously $0 \in K$, hence $K \neq \emptyset$.

Let $x, y \in K$ and let $z \notin I$. Then $\neg y \odot z \notin I$, and thus also $\neg(x \oplus y) \odot z = \neg x \odot (\neg y \odot z) \notin I$. Therefore $x \oplus y \in K$. If $x \in K$, $v \in A$, $v \leq x$ and $z \notin I$, then $\neg x \odot z \leq \neg v \odot z$, hence $\neg v \odot z \notin I$ and so $v \in K$. That means $K \in \mathcal{C}(\mathcal{A})$.

Let $x, y, z \in A$ and let $x \wedge y \in I_z$. Then $(\neg x \odot z) \vee (\neg y \odot z) = (\neg x \vee \neg y) \odot z = \neg(x \wedge y) \odot z \notin I$, and since I is an ideal of $L(\mathcal{A})$, we get $\neg x \odot z \notin I$ or $\neg y \odot z \notin I$. Therefore, if $x \wedge y \in I_z$ then $x \in I_z$ or $y \in I_z$.

Now let us suppose that I is a prime ideal of $L(\mathcal{A})$. Let $x, y \notin K$. Then there are $u, v \in A \backslash I$ such that $x \notin I_u$ and $y \notin I_v$. Obviously $u \wedge v \notin I$. We want to prove that $x \wedge y \notin K$. Let us suppose that $x \wedge y \in K$. Then $x \wedge y \in I_{u \wedge v}$, and thus $x \in I_{u \wedge v}$ or $y \in I_{u \wedge v}$. If $x \in I_{u \wedge v}$ then $I_{u \wedge v} \subseteq I_u \cap I_v$ implies $x \in I_u$, a contradiction. Similarly $y \in I_{u \wedge v}$ gives $y \in I_v$, a contradiction again. Therefore $x \wedge y \notin K$, and hence K is a prime ideal of $\mathcal{A}$. □

Analogously we also obtain the following theorem.

Theorem 3 *Let $\mathcal{A}$ be a GMV-algebra and I be a proper ideal of the lattice $L(\mathcal{A})$. Set $J_z = \{x \in A; z \odot \sim x \notin I\}$. Let $L = L_I = \bigcap(J_z; z \notin I)$. Then $L \subseteq I$ and L is an ideal of the GMV-algebra $\mathcal{A}$. Moreover, if I is a prime ideal of $L(\mathcal{A})$ then L is a prime ideal of $\mathcal{A}$.*

The following assertion is a consequence of Theorem 2 and Theorem 3, respectively.

Theorem 4 *If $\mathcal{A}$ is a GMV-algebra then the minimal prime ideals of $\mathcal{A}$ coincide with the minimal prime ideals of $L(\mathcal{A})$.*

If $\mathcal{A}$ is a GMV-algebra then an ideal of the lattice $L(\mathcal{A})$ will be called *normal* if (analogously as in the case of a normal ideal of the GMV-algebra $\mathcal{A}$)

$$\forall x, y \in A; \neg x \odot y \in I \Leftrightarrow y \odot \sim x \in I.$$

Proposition 5 *Let I be a normal ideal of $L(\mathcal{A})$. Then $K_I = L_I$.*

Proof Let $x \in A$ and let $x \in K_I$. Then for any $z \notin I$ we have $x \in I_z$, and hence $\neg x \odot z \notin I$. The normality of I implies $z \odot \sim x \notin I$ for each $z \notin I$, thus $z \in J_z$ for each $z \notin I$. Therefore $I_z \subseteq J_z$ for each $z \notin I$. Similarly we show $J_z \subseteq I_z$, hence $I_z = J_z$, and so $K_I = \bigcap_{z \notin I} I_z = \bigcap_{z \notin I} J_z = J_I$. □

Remark 2 The converse implication is not valid. If I is a minimal prime ideal of the lattice $L(\mathcal{A})$, then by Theorem 4, I is also a minimal prime ideal of the GMV-algebra $\mathcal{A}$ and $I = K_I = L_I$. Let a GMV-algebra $\mathcal{A}$ be not representable. Then by [7], Proposition 3.13, $\mathcal{A}$ contains a minimal prime ideal H which is not normal. Hence H is an ideal of $L(\mathcal{A})$ satisfying $K_H = L_H$, but H is not normal.

Proposition 6 *Let I be a proper ideal of $L(\mathcal{A})$ satisfying the property*

$$\forall x \in A; x \in I \Longleftrightarrow \neg x \notin I. \tag{$*$}$$

If the ideal K_I is normal then I is normal too.

Proof Let K_I be normal. Then for every $z \notin I$, $\neg x \odot y \in I_z$ if and only if $y \odot \sim x \in I_z$. Since $1 \notin I$, we have $\neg(\neg x \odot y) \notin I$ if and only if $\neg(y \odot \sim x) \notin I$, and hence by $(*)$, $\neg x \odot y \in I$ if and only if $y \odot \sim x \in I$. Therefore I is normal. □

3 Stonean ideals of GMV-algebras

If $\mathcal{A}$ is a GMV-algebra, denote by $B(\mathcal{A})$ the set of additive idempotents of $\mathcal{A}$, i.e. $B(\mathcal{A}) = \{x \in A; x \oplus x = x\}$. By [7], Corollary 4.5, or [8], Corollary 18, $B(\mathcal{A})$ is a subalgebra of $\mathcal{A}$ which is a Boolean algebra and $x \oplus y = x \vee y$ for any $x, y \in B(\mathcal{A})$. Let us recall that if $x \in B(\mathcal{A})$, then for the complement x' of x in $B(\mathcal{A})$ we have $x' = \neg x = \sim x$.

Further, let $\mathcal{A}$ be an GMV-algebra and $x \in A$. Put $n \cdot x = x \oplus \ldots \oplus x$ (n times). If $\mathcal{A}$ is an MV-algebra then $x \in A$ is called *archimedean* ([3], Definition 6.2.3) if there is an $n \in \mathbb{N}$ such that $n \cdot x \in B(\mathcal{A})$. An MV-algebra is said to be *hyperarchimedean* if every its element is archimedean. ([3], Definition 6.3.1.)

Let now $\mathcal{A}$ be a GMV-algebra and let I be an ideal of the lattice $L(\mathcal{A})$. Then I will be called *Stonean* if for any $x \in I$ there exists $z \in I \cap B(\mathcal{A})$ such that $x \leq z$. (For MV-algebras see [3].)

We will show some connections between Stonean ideals of $L(\mathcal{A})$ and ideals of $\mathcal{A}$.

Theorem 7 *If $\mathcal{A}$ is a GMV-algebra then every Stonean ideal of $L(\mathcal{A})$ is an ideal of $\mathcal{A}$.*

Proof Let I be a Stonean ideal of $L(\mathcal{A})$ and let $x, y \in I$. Then there are $u, v \in I \cap B(\mathcal{A})$ such that $x \leq u$, $y \leq v$, thus $x \oplus y \leq u \oplus v = u \vee v \in I \cap B(\mathcal{A})$, and hence $x \oplus y \in I$. □

Now we will characterize the GMV-algebras $\mathcal{A}$ having the property that every ideal of $\mathcal{A}$ is a Stonean ideal of $L(\mathcal{A})$.

Theorem 8 *If $\mathcal{A}$ is a GMV-algebra then the following conditions are equivalent.*

1. *For every $x \in A$ there is an $n \in \mathbb{N}$ such that $\neg x \vee n \cdot x = 1$.*
2. *For every $x \in A$ there is an $n \in \mathbb{N}$ such that $\sim x \vee n \cdot x = 1$.*
3. *For every $x \in A$ there is an $n \in \mathbb{N}$ such that $n \cdot x \in B(\mathcal{A})$.*
4. *Any ideal of $\mathcal{A}$ is a Stonean ideal of $L(\mathcal{A})$.*
5. *Any prime ideal of $\mathcal{A}$ is maximal.*
6. *Any prime ideal of $\mathcal{A}$ is minimal.*
7. *$\mathcal{A}$ is a hyperarchimedean MV-algebra.*

Proof The equivalence of conditions 1–3 is proved in [7], Proposition 4.6.

$3 \Rightarrow 4$: Let I be an ideal of $\mathcal{A}$ and let $x \in I$. Then there exists $n \in \mathbb{N}$ such that $n \cdot x \in B(\mathcal{A})$. Since $x \leq n \cdot x$, we get I is Stonean.

$4 \Rightarrow 5$: Let P be a prime ideal of $\mathcal{A}$ and let $J \in \mathcal{C}(\mathcal{A})$ be such that $P \subset J$. If $x \in J \setminus P$ then by the assumption there exists $z \in J \cap B(\mathcal{A})$ such that $x \leq z$. Since $z \notin P$, we have $P \cap B(\mathcal{A}) \subset J \cap B(\mathcal{A})$. If $u, v \in P \cap B(\mathcal{A})$ then (by [8], Theorem 10, or [7], Proposition 4.3) $u \oplus v = u \vee v \in P \cap B(\mathcal{A})$. For $w \in B(\mathcal{A})$ and $u \in P \cap B(\mathcal{A})$ it is obvious that $w \leq u$ implies $w \in P \cap B(\mathcal{A})$. Let $s, t \in B(\mathcal{A})$ and $s \wedge t \in P \cap B(\mathcal{A})$. Then, by [7], Theorem 2.17, $s \in P$ or $t \in P$, hence $s \in P \cap B(\mathcal{A})$ or $t \in P \cap B(\mathcal{A})$. Thus $P \cap B(\mathcal{A})$ is a maximal ideal of the Boolean algebra $B(\mathcal{A})$.

Therefore we get $1 \in J \cap B(\mathcal{A})$, hence $J = A$, and therefore P is a maximal ideal of $\mathcal{A}$.

$5 \Leftrightarrow 6$: Obvious.

$5 \Rightarrow 7$: Recall that by Theorem 3.9 in [4], we can suppose that $\mathcal{A} = \Gamma(G, u)$, where G is an ℓ-group and u is a strong unit in G. By [9], Theorem 2, the ordered sets of prime ideals of $\mathcal{A}$ and prime subgroups of G are isomorphic. Hence every prime subgroup of G is maximal, therefore by [1], Theorem 55.1, G is hyperarchimedean. Thus G is abelian and this implies that $\mathcal{A}$ is an MV-algebra. Therefore, by Theorem 6.3.2 in [3], $\mathcal{A}$ is a hyperarchimedean MV-algebra.

$7 \Rightarrow 1$: Follows from [3], Corollary 6.2.4. □

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