Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica

Ivan Chajda; Helmut Länger

A simple basis of ideal terms of Brouwerian semilattices

Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica, Vol. 40 (2001), No. 1, 37--42

Persistent URL: http://dml.cz/dmlcz/120437

Acta Univ. Palacki. Olomuc., Fac. rer. nat.,
Mathematica **40** (2001) 37–42

A Simple Basis of Ideal Terms of Brouwerian Semilattices *

IVAN CHAJDA [1], HELMUT LÄNGER [2]

[1]*Department of Algebra and Geometry, Faculty of Science,*
Palacký University, Tomkova 40, 779 00 Olomouc, Czech Republic
e-mail: chajda@risc.upol.cz
[2]*Technische Universität,*
Institut für Algebra und Computermathematik,
Wiedner Hauptstraße 8–10, A - 1040 Wien, Österreich
e-mail: h.laenger@tuwien.ac.at

(Received June 28, 2000)

Abstract

A list of four terms is given such that a subset of a Brouwerian semilattice **S** containing 1 is a kernel (i.e. 1-class) of some congruence on **S** if and only if it is closed with respect to these four terms.

Key words: Ideal term, ideal, congruence kernel, Brouwerian semilattice.

2000 Mathematics Subject Classification: 06A12, 06B10, 08A30, 08B05

Definition 1 By an algebra with 1 we mean an algebra with a distinguished element 1. By a variety with 1 we mean a variety with an equationally definable constant 1. Let $\mathbf{V}$ be a variety of type τ with 1 and $\mathbf{A} = (A, F) \in \mathbf{V}$. A term $t(x_1, \ldots, x_n)$ of type τ is called an *ideal term* of $\mathbf{V}$ in $x_{i_1}, \ldots, x_{i_k}$ $(i_1, \ldots, i_k \in \{1, \ldots, n\})$ if $t(x_1, \ldots, x_n) = 1$ holds in $\mathbf{V}$ provided $x_{i_1} = \ldots = x_{i_k} = 1$. $I \subseteq A$ is said to be *closed under the ideal term* $t(x_1, \ldots, x_n)$ of $\mathbf{V}$ in $x_{i_1}, \ldots, x_{i_k}$ if

*This paper is a result of the collaboration of the authors within the framework of the "Aktion Österreich – Tschechische Republik" (grant No. 26p2 "Local and global congruence properties").

$t(a_1, \ldots, a_n) \in I$ for all $a_1, \ldots, a_n \in A$ satisfying $a_{i_1}, \ldots, a_{i_k} \in I$. A subset I of A is called an *ideal* of $\mathbf{A}$ if it is closed with respect to all ideal terms of $\mathbf{V}$. (Observe that ideals are non-empty since 1 is an ideal term.) A set B of ideal terms of $\mathbf{V}$ is called a *basis of ideal terms* of $\mathbf{V}$ if a subset I of the base set of some algebra $\mathbf{A}$ belonging to $\mathbf{V}$ is an ideal of $\mathbf{A}$ if and only if it is closed with respect to all terms belonging to B. For every $\Theta \in \operatorname{Con}\mathbf{A}$, $[1]\Theta$ is called the *kernel* of Θ. $I \subseteq A$ is called a *congruence kernel* of $\mathbf{A}$ if there exists a congruence $\Theta \in \operatorname{Con}\mathbf{A}$ with $[1]\Theta = I$.

Remark 1 Obviously, every congruence kernel is an ideal. The converse is true only in certain varieties.

Definition 2 An *algebra* with 1 is called *permutable at* 1 if

$$[1](\Theta \circ \Phi) = [1](\Phi \circ \Theta)$$

for any two of its congruences Θ, Φ. A *class* of algebras with 1 is called *permutable at* 1 if each of its members has this property.

Permutable at 1 varieties can be characterized by the following Mal'cev condition:

Proposition 1 (cf. [1] and [6]) *A variety with* 1 *is permutable at* 1 *if and only if there exists a binary term s with* $s(x,1) = x$ *and* $s(x,x) = 1$.

Now we formulate the mentioned result:

Proposition 2 (cf. [1] and [6]) *In permutable at* 1 *varieties the notions of ideal and congruence kernel coincide.*

In some cases the congruences corresponding to congruence kernels are unique.

Definition 3 An *algebra* with 1 is called *weakly regular* if any two of its congruences having the same 1-class, coincide. A *class* of algebras with 1 is called *weakly regular* if each of its members has this property.

Also weakly regular varieties can be characterized by a Mal'cev condition as follows:

Proposition 3 (cf. [5]) *A variety with* 1 *is weakly regular if and only if there exist positive integers n and k, binary terms* $d_1, \ldots, d_n$ *and* $(n+2)$*-ary terms* $t_1, \ldots, t_k$ *satisfying the following identities:*

$$\begin{aligned}
&d_1(x,x) = \ldots = d_n(x,x) = 1, \\
&t_1(1, \ldots, 1, x, y) = x, \\
&t_i(d_1(x,y), \ldots, d_n(x,y), x, y) = t_{i+1}(1, \ldots, 1, x, y) \quad \textit{for } i = 1, \ldots, k-1, \\
&t_k(d_1(x,y), \ldots, d_n(x,y), x, y) = y.
\end{aligned}$$

Definition 4 An *algebra* with 1 is called *ideal determined* if every of its ideals is the kernel of a unique one of its congruences. A *class* of algebras with 1 is called *ideal determined* if each of its members has this property.

Proposition 4 (cf. [6]) *A variety with* 1 *is ideal determined if and only if it is weakly regular and permutable at* 1.

Proposition 5 (cf. [3]) *Every ideal determined variety has a finite basis of ideal terms.*

In fact, in [3] an explicit construction of such a basis was given.

Definition 5 A *Brouwerian semilattice* is an algebra $(S, \wedge, *)$ of type $(2, 2)$ such that $(S, \wedge)$ is a meet-semilattice and for any $x, y \in S$, $x*y$ is the greatest element z of S satisfying $x \wedge z \leq y$, i.e. $x * y$ is the so-called *relative pseudocomplement* of x with respect to y (where $\leq$ denotes the induced partial ordering on S).

It is well-known that Brouwerian semilattices form a variety.

In the sequel we often use the statements of the following lemma holding in every Brouwerian semilattice (see e.g. [7]):

Lemma 1 *For elements a, b, c of a Brouwerian semilattice the following statements are true:*

(i) $a * a = b * b =: 1$,
(ii) $a \leq b \Rightarrow a * c \geq b * c$,
(iii) $b \leq c \Rightarrow a * b \leq a * c$,
(iv) $a \leq (a * b) * b$,
(v) $a \wedge (a * b) = a \wedge b$,
(vi) $a \leq b \Leftrightarrow a * b = 1$,
(vii) $a * 1 = 1$,
(viii) $1 * a = a$,
(ix) $a * b \geq b$.

Theorem 1 *In the variety* **V** *of Brouwerian semilattices (i)–(iii) hold:*

(a) *The term $s(x, y) := y * x$ satisfies the identities of Proposition 1.*
(b) *The terms $d_1(x, y) := x * y$, $d_2(x, y) := y * x$, $t_1(x, y, z, u) := x \wedge z$ and $t_2(x, y, z, u) := (y * z) \wedge u$ satisfy the identities of Proposition 3.*
(c) **V** *is ideal determined.*

Proof (a) follows from (viii) and (i) of Lemma 1, (b) follows from (i), (v), (viii) and (iv) of Lemma 1 and (c) follows from (i), (ii) and Propositions 1, 3 and 4. □

Though we could now construct a finite basis of ideal terms of Brouwerian semilattices using the method described in [3] this basis would be rather complicated. The aim of this paper is to provide a simple basis and to give a direct proof of the corresponding result.

Lemma 2 *Let $(S, \wedge, *)$ be a Brouwerian semilattice and assume $I \subseteq S$ to contain 1 and to be closed under the ideal term $(y_1 * (y_2 * x)) * x$ (in y_1, y_2). If $a \in I$, $b \in S$ and $a * b \in I$ then $b \in I$. Especially, if $a \in I$, $b \in S$ and $a \leq b$ then $b \in I$.*

Proof If $t(x, y_1, y_2)$ denotes the ideal term mentioned in the lemma then $b = 1 * b = ((a * b) * (a * b)) * b = t(b, a * b, a) \in I$ by (viii) and (i) of Lemma 1. If $a \leq b$ then $a * b = 1 \in I$ by (vi) of Lemma 1. □

Lemma 3 *Let $(S, \wedge, *)$ be a Brouwerian semilattice, let q be a binary term and assume $I \subseteq S$ to contain 1 and to be closed under the ideal terms*

$$(y_1 * (y_2 * x)) * x,$$
$$(x_1 * q((y * x_2) \wedge x_3, x_4)) * (x_1 * q(x_2 \wedge x_3, x_4)),$$
$$(x_1 * q(y \wedge x_2, x_3)) * (x_1 * q(x_2, x_3))$$

(in y_1, y_2 resp. y). If $a, b, c \in S$ and $a * b, b * a \in I$ then $q(a, c) * q(b, c) \in I$.

Proof Let $t(x, y_1, y_2), t'(x_1, x_2, x_3, x_4, y)$ and $t''(x_1, x_2, x_3, y)$ denote the ideal terms just mentioned. Since

$$q(a, c) * q(((a * b) * b) \wedge a, c) = q(a, c) * q(a, c) = 1 \in I$$

by (iv) and (i) of Lemma 1 and

$$(q(a, c) * q(((a * b) * b) \wedge a, c)) * (q(a, c) * q(b \wedge a, c)) = t'(q(a, c), b, a, c, a * b) \in I,$$

we have $q(a, c) * q(b \wedge a, c) \in I$ according to Lemma 2. Since

$$q(a, c) * q((b * a) \wedge b, c) = q(a, c) * q(b \wedge a, c) \in I$$

by (v) of Lemma 1 and

$$(q(a, c) * q((b * a) \wedge b, c)) * (q(a, c) * q(b, c)) = t''(q(a, c), b, c, b * a) \in I$$

we have $q(a, c) * q(b, c) \in I$ according to Lemma 2. □

Now we can prove our main theorem:

Theorem 2 *For a Brouwerian semilattice $\mathbf{S} = (S, \wedge, *)$ and a subset I of S containig 1 the following are equivalent:*

(i) *I is an ideal of $\mathbf{S}$.*

(ii) *I is closed with respect to the following ideal terms (in y_1, y_2 resp. y):*
$t_1(x, y_1, y_2) := (y_1 * (y_2 * x)) * x$,
$t_2(x_1, x_2, x_3, y) := (x_1 * ((y * x_2) \wedge x_3)) * (x_1 * (x_2 \wedge x_3))$,
$t_3(x_1, x_2, x_3, y) := (x_1 * ((y \wedge x_2) * x_3)) * (x_1 * (x_2 * x_3))$ *and*
$t_4(x_1, x_2, x_3, x_4, y) := (x_1 * (x_2 * ((y * x_3) \wedge x_4))) * (x_1 * (x_2 * (x_3 \wedge x_4)))$.

(iii) *There exists a congruence $\Theta \in Con\,\mathbf{S}$ with $[1]\Theta = I$.*

(iv) *There exists exactly one congruence $\Theta \in Con\,\mathbf{S}$ with $[1]\Theta = I$.*

Proof (i) $\Rightarrow$ (ii): This is trivial.

(ii) $\Rightarrow$ (iii): Put

$$\Theta := \{(a, b) \in S^2 \mid a * b \text{ and } b * a \in I\}.$$

(1) Θ is reflexive. This follows from (i) of Lemma 1 and from $1 \in I$.
(2) Θ is symmetric. This is obvious.
(3) Θ is transitive. Assume $a, b, c \in S$ and $a \,\Theta\, b \,\Theta\, c$. Then $a*b, b*a, b*c, c*b \in I$. Now

$$c * (((b * a) * a) \wedge b) = c * b \in I$$

by (iv) of Lemma 1 and since

$$(c * (((b * a) * a) \wedge b)) * (c * (a \wedge b)) = t_2(c, a, b, b * a) \in I$$

we have $c * (a \wedge b) \in I$ according to Lemma 2. Since $c * a \geq c * (a \wedge b)$ by (iii) of Lemma 1 it follows $c * a \in I$ again by Lemma 2. By a symmetry argument it follows $a * c \in I$. This shows $a \,\Theta\, c$.
(4) $a \,\Theta\, b \Rightarrow a \wedge c \,\Theta\, b \wedge c$. Let $a, b, c \in S$ and $i \in I$. Then

$$(a * ((i * b) \wedge c)) * (a * (b \wedge c)) = t_2(a, b, c, i) \in I.$$

Moreover, $i \wedge b \leq b$ implies $a * (i \wedge b) \leq a * b$ by (iii) of Lemma 1 and hence

$$(a * (i \wedge b)) * (a * b) = 1 \in I$$

by (vi) of Lemma 1. The rest follows from Lemma 3 with $q(x_1, x_2) := x_1 \wedge x_2$.
(5) $a \,\Theta\, b \Rightarrow a * c \,\Theta\, b * c$. Let $a, b, c, d \in S$ and $i \in I$. Then $i * b \geq b$ by (ix) of Lemma 1 and hence

$$a * (((i * b) \wedge c) * d) \leq a * ((b \wedge c) * d)$$

by (ii) and (iii) of Lemma 1 whence

$$(a * (((i * b) \wedge c) * d)) * (a * ((b \wedge c) * d)) = 1 \in I$$

by (vi) of Lemma 1. Moreover,

$$(a * ((i \wedge b) * c)) * (a * (b * c)) = t_3(a, b, c, i) \in I.$$

The rest follows from Lemma 3 with $q(x_1, x_2) := x_1 * x_2$.
(6) $a \,\Theta\, b \Rightarrow c * a \,\Theta\, c * b$. Let $a, b, c, d \in S$ and $i \in I$. Then

$$(a * (d * ((i * b) \wedge c))) * (a * (d * (b \wedge c))) = t_4(a, d, b, c, i) \in I.$$

Moreover, $i \wedge b \leq b$ implies $a * (c * (i \wedge b)) \leq a * (c * b)$ by (iii) of Lemma 1 and hence

$$(a * (c * (i \wedge b))) * (a * (c * b)) = 1 \in I$$

by (vi) of Lemma 1. The rest follows from Lemma 3 with $q(x_1, x_2) := x_2 * x_1$. Hence $\Theta \in \operatorname{Con} \mathbf{S}$. Obviously, $[1]\Theta = I$.

(iii) $\Rightarrow$ (iv): This follows from Remark 1 and Theorem 1.

(iv) $\Rightarrow$ (i): This follows from Remark 1. □

Finally, we characterize congruence classes in Brouwerian semilattices:

Theorem 3 *A non-empty subset C of a Brouwerian semilattice $\mathbf{S} = (S, \wedge, *)$ is a class of some congruence on $\mathbf{S}$ if and only if there exists an ideal I of $\mathbf{S}$ with*

$$C = \{a \in S \mid a * c \textit{ and } c * a \in I \textit{ for some } c \in C\}.$$

Proof First assume C to be a class of some $\Theta \in \operatorname{Con}\mathbf{S}$. Then $I := [1]\Theta$ is an ideal of $\mathbf{S}$. Let $c \in C$. If $a \in C$ then $a * c, c * a \in [a * a]\Theta = [1]\Theta = I$ by (i) of Lemma 1. If, conversely, $a \in S$ and $a * c, c * a \in I$ then

$$a = ((a * c) * c) \wedge a \,\Theta\, (1 * c) \wedge a = c \wedge a = c \wedge (c * a) \,\Theta\, c \wedge 1 = c$$

by (iv), (viii) and (v) of Lemma 1 and hence $a \in C$.

If, conversely, I is an ideal of $\mathbf{S}$ and

$$C = \{a \in S \mid a * c, c * a \in I \text{ for some } c \in C\}$$

then

$$\Phi := \{(a, b) \in S^2 \mid a * b \text{ and } b * a \in I\} \in \operatorname{Con}\mathbf{S}$$

according to the proof of Theorem 2 and obviously $C = [c]\Phi$. □

References

[1] Chajda, I.: *A localization of some congruence conditions in varieties with nullary operations.* Ann. Univ. Sci. Budapest., Math. **30** (1987), 17–23.

[2] Chajda, I.: *A characterization of congruence kernels in pseudocomplemented semilattices.* (Preprint).

[3] Chajda, I., Halaš, R.: *Finite basis of ideal terms in ideal determined varieties.* Algebra Universalis **37** (1997), 243–252.

[4] Chajda, I., Zedník, J.: *Determined congruence classes in Brouwerian semilattices.* East-West J. Math. (to appear).

[5] Fichtner, K.: *Eine Bemerkung über Mannigfaltigkeiten universeller Algebren mit Idealen.* Monatsh. Dt. Akad. Wiss. (Berlin) **12** (1970), 21–45.

[6] Gumm, H. P., Ursini, A.: *Ideals in universal algebras.* Algebra Universalis **19** (1984), 45–54.

[7] Köhler, P.: *Brouwerian semilattices, the lattice of total subalgebras.* Universal Algebra Appl. **8** (1982), Banach Centre Publ., 47–56.