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A Note on the Example of J. Andres Concerning the Application of the Nielsen Fixed-Point Theory to Differential Systems

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Abstract

The goal of this note is two-fold: to give a slight improvement of the example due to J. Andres in [1], [2], concerning the application of the Nielsen number to differential equations, and a reprovement of the method, allowing us to avoid one condition (a connectedness of ANR-spaces).

Key words: Nielsen number, existence of two bounded solutions, nontrivial example.

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1 Introduction

As pointed out in [1], the main advantage of the so called Nielsen fixed point theory, first presented by a Danish mathematician Jakob Nielsen in 1927, is that unlike other fixed point theories this one gives us a lower bound of fixed points. The original Nielsen theory was formulated with respect to selfmaps of compact surfaces, i.e. in the form not suitable for analytic applications. But although since that time it has been systematically developed and is rather advanced today (see [3], [4], [8], [10] for a nice survey of this development and for exhaustive list of references) there are only several applications to differential equations (see again [3], [4], [8], [10]).

2 Method

At first let us recall some basic notions.

The nonempty subset A of a metric space X is called a *retract* of X if there is a retraction of X onto A, that is, a continuous map $r : X \to A$ such that $r(x) = x$ for all $x \in A$.

We say that a nonempty subset A of X is a *neighbourhood retract* of X if there exists an open subset U of X containing A, i.e. $A \subset U \subset X$, such that A is retract of U.

The metric space X is called an *absolute neighbourhood retract (ANR) space* if for any metric space Y, for any closed subset B of Y and any continuous map $f : B \to X$ there exists an open neighbourhood U of B in Y (i.e. open subset U such that $B \subset U \subset Y$) and an extension $\widetilde{f} : U \to X$ of f.

We say that a continuous map $f : X \to Y$ is *compact* if $f(X)$ is relatively compact, i.e. $\overline{f(X)}$ is compact, in Y. We call a compact map $H : X \times [0,1] \to Y$ a *compact homotopy.* For each $t \in [0,1]$ define $h_t(x) = H(t,x)$. When $H : X \times [0,1] \to Y$ is a compact homotopy, we say that h_0 and h_1 are *compactly homotopic.*

In the following we will give the definition of the Nielsen number based on [6].

Definition 1 Given X, a metric ANR, and a compact map $f : X \to X$, two fixed-points x and y are called *f-equivalent* if there is a path $C : [0,1] \to X$ homotopic to $f(C)$ with the property that $C(0) = f(C(0)) = x$ and $C(1) = f(C(1)) = y$ stayed fixed throughout the homotopy.

For a given map $f : X \to X$, we denote the set of all fixed points of f, i.e. the set $\{\widehat{x} \in X : f(\widehat{x}) = \widehat{x}\}$, as $Fix(f)$. It is easy to see that the relation given in Definition 1 is an equivalence on $Fix(f)$, so the next definition is correct.

Definition 2 The equivalence classes of $Fix(f)$, under f-equivalence, are called *fixed-point classes* of f.

It is known that f has a finite number of fixed-point classes (see [6]). Therefore we can give

Definition 3 Let $f : X \to X$ be a compact map on a metric ANR and let F be a fixed-point class of f. We call F to be *essential* if $i(F) \neq 0$, where $i(F)$ is the associated fixed-point index (see [11]). The *Nielsen number*, $N(f)$, of the map f is defined to be the number of essential fixed-point classes of f.

Theorem 1 ([6]) *Let X be a metric ANR and $f : X \to X$ be a compact map, then one can associate to f a non-negative integer $N(f)$ with the property that every map $g : X \to X$ which is compactly homotopic to f has at least $N(f)$ fixed-points.*

Theorem 2 ([9]) *Let J be a real interval, S a subset of $C(J, \Re^n)$ and $g : J \times \Re^n \times \Re^n \to \Re^n$ a Carathéodory function. Assume that, for a subset Q of $C(J, \Re^n)$, the following conditions are satisfied:*

(a) *for any $q \in Q$, the boundary value problem*

$$x'(t) = g(t, x(t), q(t)), \text{ for a.a. } t \in J,$$
$$x \in S$$

admits a unique solution $x = T(q)$,

(b) *$T(Q)$ is bounded in $C(J, \Re^n)$,*

(c) *there exists a locally integrable function $\alpha : J \to \Re$ such that*

$$|g(t, x(t), q(t))| \leq \alpha(t), \text{ a.e. in } J,$$

for any pair (q, x) in the graph of T.

Then $T(Q)$ is a relatively compact subset of $C(J, \Re^n)$. Moreover, under the above assumptions, the operator $T : Q \to S$ is continuous if and only if the following condition is verified:

(d) *given a sequence (q_k, x_k) in the graph of T, if $(q_k, x_k) \to (q, x)$ with $q \in Q$, then $x \in S$.*

In particular (d) is satisfied if the closure $\overline{T(Q)}$ of $T(Q)$ is contained in S.

Definition 4 ([8], [1]) We say that a mapping $T : Q \to S$ is *retractible onto* Q if there is a retraction $r : P \to Q$, where P is an open subset of $C(J, \Re^n)$ containing $Q \bigcup S$ and $p \in P \setminus Q$, $r(p) = q$ implies that $p \neq T(q)$.

It is easy to see from the definition above, that if $T : Q \to S$ is retractible onto Q then $\widehat{q} \in Q$ is fixed-point of $r \circ T : Q \to Q$ if and only if it is a fixed-point of T. Really, if $\widehat{q}$ is a fixed-point of T, i.e. $\widehat{q} = T(\widehat{q})$, then $(r \circ T)(\widehat{q}) = r(T(\widehat{q})) = r(\widehat{q}) = \widehat{q}$, because r is a retraction. On the other hand, let $(r \circ T)(\widehat{q}) = \widehat{q}$ and suppose that $T(\widehat{q}) \neq \widehat{q}$. Then $T(\widehat{q}) = p$, $p \in P \setminus Q$. So we have $p \in P \setminus Q$, $r(p) = \widehat{q}$ and $T(\widehat{q}) = p$, but it is contradiction with our assumption that T is retractible onto Q. Therefore $T(\widehat{q}) = \widehat{q}$.

Although the following theorem has been derived in [1] from a more general (multivalued) result in [4], we shall reprove it here alternatively, because the additional restriction imposed on Q to be connected can be avoided.

Theorem 3 (cf. [1]) *Let $g : J \times \Re^n \times \Re^n \to \Re^n$ be a Carathéodory mapping, where J is an arbitrary real interval. Assume, furthermore, that there exists a nonempty closed subset Q of $C(J, \Re^n)$ such that the problem*

$$x' = g(t, x(t), q(t)), \quad x \in S$$

has, for every $q \in Q$, a unique solution $x = T(q)$ with the property $\overline{T(Q)} \subset S$, where S is a nonempty bounded subset of $C(J, \Re^n)$, and $T : Q \to S$ is retractible onto Q with a retraction r in the sense of Definition 4.

At last, let there exist a locally Lebesgue-integrable function $\alpha : J \to \Re$ such that

$$|g(t, x(t), q(t))| \leq \alpha(t)$$

a.e. in J, for any pair (q, x) in the graph of T.

Then the Carathéodory system $x' = f(t, x)$ admits at least $N(r|_{T(Q)} \circ T(.))$ solutions belonging to Q, provided $g(t, c, c) = f(t, c)$ takes place a.e. in J, for any $c \in \Re^n$.

Proof Since S was supposed to be bounded and that $\overline{T(Q)} \subset S$, all assumptions of Theorem 2 are satisfied. So according to this theorem, $T(Q)$ is a relatively compact subset of S, i.e. $T : Q \to S$ is compact and (because of (d) in Theorem 2) continuous.

Now by the first Hanner theorem (see e.g. [5]), we obtain that, if X is a metrizable neighbourhood retract of a locally convex space, then X is an ANR-space. Since T is retractible onto Q in the sense of Definition 4, Q is a neighbourhood retract of $C(J, \Re^n)$. Since $C(J, \Re^n)$ is a Fréchet space (i.e. a completely metrizable locally convex topological vector space), Q is an ANR-space.

Denoting $r_Q = r|_Q$, then $r_Q \circ T : Q \to Q$ is a composition of compact mapping with continuous mapping, so $r_Q \circ T$ is also compact. Then according Theorem 1, the mapping $r_Q \circ T$ has at least $N(r_Q \circ T)$ fixed-points. All fixed-points of $r_Q \circ T$ are also fixed-points of T (see remarks after Definition 4) and it is clear that each fixed-point of T is a solution of the system $x' = g(t, x(t), q(t))$. Moreover, because of $g(t, c, c) = f(t, c)$, it is also a solution of the system $x' = f(t, x)$. This completes the proof. □

3 Application

In this section we give a slight generalization of an example in [1] (cf. also [2]). It consists in replacing the concrete nonlinearities $x^{\frac{1}{n}}$, $y^{\frac{1}{m}}$, where m, n are odd integers with $\min(m, n) \geq 3$, by monotone nonlinearities $u(y)$, $v(x)$ with similar properties (see bellow).

Consider the Carathéodory system (cf. [1])

$$\begin{aligned} x' + ax &= e(t, x, y)u(y) + g(t, x, y), \\ y' + by &= f(t, x, y)v(x) + h(t, x, y), \end{aligned} \tag{1}$$

with the following properties:

(i) a, b are positive numbers,

(ii) there exist positive constants E_0, F_0, G, H such that

$$|e(t, x, y)| \leq E_0, \quad |f(t, x, y)| \leq F_0, \tag{2}$$

$$|g(t, x, y)| \leq G, \quad |h(t, x, y)| \leq H, \tag{3}$$

for a.a. $t \in (-\infty, \infty)$ and all $(x, y) \in \Re^2$,

(iii) there are positive constants e_0, f_0, δ_1, δ_2 such that

$$0 < e_0 \le e(t, x, y) \tag{4}$$

for a.a. $t \in \Re$, all $x \in \Re$ and $|y| \ge \delta_2$ together with

$$0 < f_0 \le f(t, x, y) \tag{5}$$

for a.a. $t \in \Re$, $|x| \ge \delta_1$ and all $y \in \Re$,

(iv) functions $u(y)$, $v(x)$ are increasing for $\delta_1 \le |x| \le R$, $\delta_2 \le |y| \le R$, respectively and

$$\lim_{|y|\to\infty} \frac{|u(y)|}{|y|} = 0, \tag{6}$$

$$\lim_{|x|\to\infty} \frac{|v(x)|}{|x|} = 0, \tag{7}$$

$$u(\delta_2) > \frac{a}{e_0}\delta_1 + \frac{G}{e_0} \quad \text{and} \quad u(-\delta_2) < -\left(\frac{a}{e_0}\delta_1 + \frac{G}{e_0}\right), \tag{8}$$

$$v(\delta_1) > \frac{b}{f_0}\delta_2 + \frac{H}{f_0} \quad \text{and} \quad v(-\delta_1) < -\left(\frac{b}{f_0}\delta_2 + \frac{H}{f_0}\right). \tag{9}$$

Besides this, consider the periodic boundary condition S

$$(x(0), y(0)) = (x(\omega), y(\omega)). \tag{10}$$

More precisely, we take $S = Q = Q_1 \cap Q_2 \cap Q_3$, where

$$\begin{aligned} Q_1 &= \Big\{q(t) \in C([0,\omega], \Re^2) : \\ &\qquad \|q(t)\| = \max\Big(\max_{t\in[0,\infty]} |q_1(t)|, \max_{t\in[0,\infty]} |q_2(t)|\Big) \le R\Big\}, \\ Q_2 &= \Big\{q(t) \in C([0,\omega], \Re^2) : \\ &\qquad \Big(\min_{t\in[0,\infty]} |q_1(t)| \ge \delta_1 > 0\Big) \vee \Big(\min_{t\in[0,\infty]} |q_2(t)| \ge \delta_2 > 0\Big)\Big\}, \\ Q_3 &= \Big\{q(t) \in C([0,\omega], \Re^2) : q(0) = q(\omega)\Big\}, \end{aligned}$$

the constant R will be specified bellow.

Our aim is to prove that, under the assumptions specified above, system (1) has at least two solutions belonging to Q. Since we use the same approach as in [1] we go through it briefly and emphasize only the changes affected by the replacement of the nonlinearities $x^{\frac{1}{n}}$, $y^{\frac{1}{m}}$ with the monotone ones.

It is easy to see that Q is a (nonempty) closed subset of $C(\Re, \Re^2)$. Moreover Q is an ANR-space (cf. [1]). Furthermore, since $S = Q$ and Q is closed, in order to check that $\overline{T(Q)} \subset S = Q$, it is sufficient to prove only $T(Q) \subset Q$, and if $T(Q) \subset Q$ then T will be clearly retractible onto Q.

Besides (1), consider its embedding into the system

$$\begin{aligned} x' + ax &= [(1-\mu)e_0 + \mu e_t]u(y) + \mu g_t, \\ y' + by &= [(1-\mu)f_0 + \mu f_t]v(x) + \mu h_t, \end{aligned} \tag{11}$$

where $\mu \in [0,1]$ and

$$\begin{aligned} e_t &:= e(t, q_1(t), q_2(t)), \quad f_t := f(t, q_1(t), q_2(t)), \\ g_t &:= g(t, q_1(t), q_2(t)), \quad h_t := h(t, q_1(t), q_2(t)). \end{aligned}$$

For $\mu = 1$ (11) is reduced to (1).

Problem (11)–(10) has according to Green's formula a unique solution

$$X(t) = (x(t), y(t))$$

for each $q(t) \in Q$. We denote such a solution $T_\mu(q)$ (see [1]).

Now we would like to show that $T_\mu(Q) \subset Q$ for each $\mu \in [0,1]$. Since $X(0) = X(\omega)$ it is obvious that $T_\mu(Q) \subset Q_3$, so it is sufficient to prove that $T_\mu(Q) \subset Q_1$ and $T_\mu(Q) \subset Q_2$.

Consider the first inclusion $T_\mu(Q) \subset Q_1$. We obtain for the solution $X(t)$ that

$$\begin{aligned} \max_{t\in[0,\omega]} |x(t)| &= \max_{t\in[0,\omega]} \int_0^\omega G_1(t,s)[((1-\mu)e_0 + \mu e_s)u(q_2(s)) + \mu g_s]ds \\ &\le [(e_0 + E_0)U_R + G] \max_{t\in[0,\omega]} \int_0^\omega G_1(t,s)ds = \frac{1}{a}[(e_0 + E_0)U_R + G], \end{aligned}$$

where $U_R := \max\{U(R), -u(R)\}$. Similarly

$$\begin{aligned} \max_{t\in[0,\omega]} |y(t)| &= \max_{t\in[0,\omega]} \int_0^\omega G_2(t,s)[((1-\mu)f_0 + \mu f_s)v(q_1(s)) + \mu h_s]ds \\ &\le [(f_0 + F_0)V_R + H] \max_{t\in[0,\omega]} \int_0^\omega G_2(t,s)ds = \frac{1}{b}[(f_0 + F_0)V_R + H], \end{aligned}$$

where $V_R := \max\{v(R), -v(R)\}$. Since

$$\begin{aligned} \|X(t)\| &= \max\Big\{\max_{t\in[0,\omega]} |x(t)|, \max_{t\in[0,\omega]} |y(t)|\Big\} \\ &\le \max\Big\{\frac{1}{a}[(e_0 + E_0)U_R + G], \frac{1}{b}[(f_0 + F_0)V_R + H]\Big\}, \end{aligned}$$

then with respect to (6), (7) a sufficiently big constant R exists such that $\|X(t)\| \le R$. But it means that $T_\mu(Q) \subset Q_1$ for all $\mu \in [0,1]$.

Similarly we can verify the second inclusion $T_\mu(Q) \subset Q_2$. If we assume that $q(t) \in Q_2$, we have either

$$\min_{t\in[0,\omega]} |q_1(t)| \ge \delta_1 > 0 \quad \text{or} \quad \min_{t\in[0,\omega]} |q_2(t)| \ge \delta_2 > 0$$

Then according to (8) we have

$$\begin{aligned}\min_{t\in[0,\omega]}|x(t)| &= \min_{t\in[0,\omega]}\int_0^\omega G_1(t,s)[((1-\mu)e_0+\mu e_s)u(q_2(s))+\mu g_s]\,ds\\ &\geq |e_0u(\delta_2)-G|\int_0^\omega G_1(t,s)ds > \frac{1}{a}\left|e_0\left(\frac{a}{e_0}\delta_1+\frac{G}{e_0}\right)-G\right| = \delta_1,\end{aligned}$$

for $|q_2|\geq\delta_2$ or according to (9)

$$\begin{aligned}\min_{t\in[0,\omega]}|y(t)| &= \min_{t\in[0,\omega]}\int_0^\omega G_2(t,s)[((1-\mu)f_0+\mu f_s)v(q_1(s))+\mu f_s]\,ds\\ &\geq |f_0v(\delta_1)-H|\int_0^\omega G_2(t,s)ds > \frac{1}{b}\left|f_0\left(\frac{b}{f_0}\delta_2+\frac{H}{f_0}\right)-H\right| = \delta_2,\end{aligned}$$

for $|q_1|\geq\delta_1$. It means that $T_\mu(Q)\subset Q_2$, independently of $\mu\in[0,1]$.

If we put all together we can easily see that $T_\mu(Q)\subset Q$, independently of $\mu\in[0,1]$.

Now we can already see that all assumptions of Theorem 3 are satisfied, so the problem (11)–(10) has for every $\mu\in[0,1]$ at least $N(T_\mu(.))$ solutions in Q (in particular problem (1)-(10) has $N(T_1(.))$ solutions). But since it can be easily proved that the operator $(T_\mu,\mu): Q\times[0,1]\to Q$ is compact, T_1 and T_0 are compactly homotopic, and according to Theorem 1 $N(T_1(.)) = N(T_0(.))$. Thus we need to compute the Nielsen number $N(T_0(.))$.

It can be shown (see [1]) that $N(T_0(.))$ is equal to the Nielsen number of the operator $T^0(\overline{q}): Q\bigcap\Re^2\to Q\bigcap\Re^2$ where

$$T^0(\overline{q}) = \left(\frac{e_0}{a}u(\overline{q}_2), \frac{f_0}{b}v(\overline{q}_1)\right)$$

for $\overline{q} = (\overline{q}_1,\overline{q}_1) = (q_1(0),q_2(0))\in Q\bigcap\Re^2$. So we need to find two solutions of the system

$$\overline{q}_1 = \widetilde{u}(\overline{q}_2), \qquad \overline{q}_2 = \widetilde{v}(\overline{q}_1), \tag{12}$$

where $\widetilde{u} := \frac{e_0}{a}u$, $\widetilde{v} := \frac{f_0}{b}v$ and $(\overline{q}_1,\overline{q}_1)\in Q\bigcap\Re^2$, belonging to different fixed-point classes.

Because of $T^0(Q\bigcap\Re^2)\subset Q\bigcap\Re^2$ it is obvious that $\widetilde{u}(\overline{q}_2)\leq R$ for $\delta_2\leq\overline{q}_2\leq R$ and $\widetilde{v}(\overline{q}_1)\leq R$ for $\delta_1\leq\overline{q}_1\leq R$. On the other hand according to (8), (9) and the fact that $u(y)$, $v(x)$ are increasing we have

$$\begin{aligned} u(\overline{q}_2) &> \frac{a}{e_0}\delta_1 \quad \text{for} \quad \delta_2\leq\overline{q}_2\leq R,\\ v(\overline{q}_1) &> \frac{b}{f_0}\delta_2 \quad \text{for} \quad \delta_1\leq\overline{q}_1\leq R.\end{aligned}$$

So it is easy to see that $\widetilde{u}(\overline{q}_2) > \delta_1$ for $\delta_2\leq\overline{q}_2\leq R$ and $\widetilde{v}(\overline{q}_1) > \delta_2$ for $\delta_1\leq\overline{q}_1\leq R$. Now by the obvious geometrical reasons it is clear that system (12) has just one fixed point $\widehat{q}_+$ in $[\delta_1,R]\times[\delta_2,R]$. Quite analogously we can find another fixed point of (10) $\widehat{q}_-$ in $[-R,-\delta_1]\times[-R,-\delta_2]$.

So the operator $T^0(\overline{q})$ has just two fixed-points in $Q \bigcap \Re^2$, namely $\widehat{q}_+$, $\widehat{q}_-$. These fixed-points belong to different fixed-point classes of $T^0(\overline{q})$ because any path C that connects them in $Q \bigcap \Re^2$ is not homotopic with $T^0(C)$ in space $Q \bigcap \Re^2$. Therefore in accordance with Definition 3 $N(T^0(\overline{q})) = 2$. But $N(T_1(q)) = N(T^0(\overline{q}))$ and Theorem 1 assures that system (1) possesses at least two solutions belonging to Q.

Remark 1 When we transform (1) by replacing t with $t + \frac{\omega}{2}$, then for any $\omega > 0$ there are two solutions of (1) belonging to $Q_1 \bigcap Q_2$ and satisfying

$$\left(x\left(-\frac{\omega}{2}\right), y\left(-\frac{\omega}{2}\right)\right) = \left(x\left(\frac{\omega}{2}\right), y\left(\frac{\omega}{2}\right)\right).$$

and according to the Lemma 2.8.1 in [12] system (1) possesses two entirely bounded solutions in $Q_1 \bigcap Q_2$.

Remark 2 By replacing t with $-t$ in (1) we can obtain the same results for negative constants a, b as well.

Therefore we can finally give

Theorem 4 *Let suitable positive constants e_0, f_0, G, H, δ_1, δ_2 exist such that inequalities (2)–(9) are satisfied for constants a, b such that $ab > 0$. Then system (1) possesses two entirely bounded solutions.*

References

[1] Andres, J.: *A nontrivial example of application of the Nielsen fixed-point theory to differential systems: problem of Jean Leray.* Proceed. Amer. Math. Soc. **128**, 10 (2000), 2921–2931.

[2] Andres, J.: *Multiple bounded solutions of differential inclusions: the Nielsen theory approach.* J. Diff. Eqs. **155** (1999), 285–320.

[3] Andres, J., Górniewicz, L.: *From the Schauder fixed-point theorem to the applied multi-valued Nielsen Theory.* Topol. Meth. Nonlin. Anal. **14**, 2 (1999), 228–238.

[4] Andres, J., Górniewicz, L., Jezierski, J.: *A generalized Nielsen number and multiplicity results for differential inclusion.* Topol. Appl. **100** (2000), 143–209.

[5] Borsuk, K.: Theory of Retracts, *PWN, Warsaw*, 1967.

[6] Brown, R. F.: *On the Nielsen fixed point theorem for compact maps.* Duke. Math. J., 1968, 699–708.

[7] Brown, R. F.: *Topological identification of multiple solutions to parametrized nonlinear equations.* Pacific J. Math. **131** (1988), 51–69.

[8] Brown, R. F.: *Nielsen fixed point theory and parametrized differential equations.* In: Contemp. Math. **72**, AMS, Providence, RI, 1989, 33–46.

[9] Cecchi, M., Furi, M., Marini, M.: *About the solvability of ordinary differential equations with assymptotic boundary conditions.* Boll. U. M. I., Ser. IV, 4-C, **1** (1985), 329–345.

[10] Fečkan, M.: *Multiple solution of nonlinear equations via Nielsen fixed-point theory: a survey.* In: Nonlinear Anal. in Geometry and Topology (Th. M. Rassias, ed.), Hadronic Press, Inc., Fl., (2000), 77–97.

[11] Granas, A.: *The Leray-Schauder index and the fixed point theory for arbitrary ANRs.* Bull. Soc. Math. France **100** (1972), 209–228.

[12] Krasnoseľskij, M. A.: The Operator of Translation along Trajectories of Differential Equations. *Nauka, Moscow*, 1966 (in Russian).