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On Totally Umbilical Cosymplectic Hypersurfaces of Six-dimensional Hermitian Submanifolds of Cayley Algebra

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Abstract

It is proved that cosymplectic hypersurfaces of six-dimensional Hermitian submanifolds of the Cayley algebra are totally umbilical if and only if they are totally geodesic.

Key words: Hermitian manifold, almost contact metric structure, cosymplectic structure, totally umbilical submanifold, totally geodesic submanifold.

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1 Introduction

One of the most important properties of a hypersurface of an almost Hermitian manifold is the existence on a such hypersurface determined in a natural way an almost contact metric structure. This structure has been studied mainly in the case of Kählerian [1], [2] and quasi-Kählerian [3], [4] manifolds. In the case the embedding manifold is Hermitian comparatively little is known about the geometry of its hypersurfaces. In the present work a result obtained in this direction by using the Cartan structure equations of such hypersurfaces is given.

Let $\mathbf{O} \equiv R^8$ be the Cayley algebra. As it is well-known [5], two non-isomorphic 3-vector cross products are defined on it by

$$P_1(X,Y,Z) = -X(\overline{Y}Z) + \langle X,Y\rangle Z + \langle Y,Z\rangle X - \langle Z,X\rangle Y,$$

$$P_2(X,Y,Z) = -(X\overline{Y})Z + \langle X,Y\rangle Z + \langle Y,Z\rangle X - \langle Z,X\rangle Y,$$

where $X,Y,Z \in \mathbf{O}$, $\langle\cdot,\cdot\rangle$ is the scalar product in $\mathbf{O}$ and $X \to \overline{X}$ is the operator of conjugation. Moreover, any other 3-vector cross product in the octave algebra is isomorphic to one of the above-mentioned.

If $M^6 \subset \mathbf{O}$ is a six-dimensional oriented submanifold, then the induced almost Hermitian structure $\{J_\alpha, g = \langle\cdot,\cdot\rangle\}$ is determined by the relation

$$J_\alpha(X) = P_\alpha(X, e_1, e_2), \quad \alpha = 1, 2,$$

where $\{e_1, e_2\}$ is an arbitrary orthonormal basis of the normal space of M^6 at a point p, $X \in T_p(M^6)$ [5]. The submanifold $M^6 \subset \mathbf{O}$ is called Hermitian if the almost Hermitian structure induced on it is integrable. The point $p \in M^6$ is called general [6], if

$$e_0 \notin T_p(M^6) \quad \text{and } T_p(M^6) \subseteq L(e_0)^\perp,$$

where e_0 is the unit of Cayley algebra and $L(e_0)^\perp$ is its orthogonal supplement. A submanifold $M^6 \subset \mathbf{O}$ consisting only of general points is called a general-type submanifold [6]. In what follows all the considered M^6 are meant as general-type submanifolds.

2 Cosymplectic hypersurfaces of Hermitian $M^6 \subset \mathbf{O}$

Let N be an oriented hypersurface of a Hermitian $M^6 \subset \mathbf{O}$ and let σ be the second fundamental form of the immersion of N into M^6. As it is well-known [2], [4], the almost Hermitian structure on M^6 induces an almost contact metric structure on N. We recall [3], [4] that an almost contact metric structure on the manifold N is defined by the system $\{\Phi, \xi, \eta, g\}$ of tensor fields on this manifold, where ξ is a vector, η is a covector, Φ is a tensor of a type $(1,1)$ and g is a Riemannian metric on N such that

$$\eta(\xi) = 1, \quad \Phi(\xi) = 0, \quad \eta \circ \Phi = 0, \quad \Phi^2 = -id + \xi \otimes \eta,$$

$$\langle \Phi X, \Phi Y\rangle = \langle X, Y\rangle - \eta(X)\eta(Y), \quad X, Y \in \aleph(N).$$

The almost contact metric structure is called cosymplectic [4] if

$$\nabla\eta = \nabla\Phi = 0.$$

(Here ∇ is the Riemannian connection of the metric g). The first group of the Cartan structure equations of a hypersurface of a Hermitian manifold looks as follows [8]:

$$
\begin{aligned}
d\omega^a &= \omega^a_b \wedge \omega^b + B^{ab}{}_c \omega^c \wedge \omega_b + (\sqrt{2}B^{a3}{}_b + i\sigma^a_b)\omega^b \wedge \omega \\
&\quad + (-\frac{1}{\sqrt{2}}B^{ab}{}_3 + i\sigma^{ab})\omega_b \wedge \omega, \\
d\omega_a &= -\omega^b_a \wedge \omega_b + B_{ab}{}^c \omega_c \wedge \omega^b + (\sqrt{2}B_{a3}{}^b - i\sigma^b_a)\omega_b \wedge \omega \\
&\quad + (-\frac{1}{\sqrt{2}}B_{ab}{}^3 - i\sigma_{ab})\omega^b \wedge \omega, \qquad (1) \\
d\omega &= (\sqrt{2}B^{3a}{}_b - \sqrt{2}B_{3b}{}^a - 2i\sigma^a_b)\omega^b \wedge \omega_a + (B_{3b}{}^3 + i\sigma_{3b})\omega \wedge \omega^b \\
&\quad + (B^{3b}{}_3 - i\sigma^b_3)\omega \wedge \omega_b.
\end{aligned}
$$

Here B are Kirichencko structure tensors of the Hermitian manifold [9]; $a, b, c = 1, 2$; $\widehat{a} = a + 3$; $i = \sqrt{-1}$. Taking into account that the first group of the Cartan structure equations of the cosymplectic structure must look as follows [10]:

$$
\begin{aligned}
d\omega^a &= \omega^a_b \wedge \omega^b, \\
d\omega_a &= -\omega^b_a \wedge \omega_b, \qquad (2) \\
d\omega &= 0,
\end{aligned}
$$

we get the conditions whose simultaneous fulfilment is a criterion for the hypersurface N to be cosymplectic:

$$
1)\ B^{ab}{}_c = 0, \quad 2)\ \sqrt{2}B^{a3}{}_b + \sigma^a_b = 0, \quad 3)\ -\frac{1}{\sqrt{2}}B^{ab}{}_3 + i\sigma^a_b = 0,
$$

$$
4)\ B^{3a}{}_b - \sqrt{2}B_{3b}{}^a - 2i\sigma^a_b = 0, \quad 5)\ B^{3b}{}_3 - i\sigma^b_3 = 0 \qquad (3)
$$

and the formulas of the comlpex conjugation (we leave out writing them down).

Now, we analyse the obtained conditions. From $(3)_3$ it follows that

$$
\sigma^{ab} = -\frac{1}{\sqrt{2}}B^{ab}{}_3.
$$

By alternating of this relation we have

$$
0 = \sigma^{[ab]} = -\frac{i}{\sqrt{2}}B^{[ab]}{}_3 = -\frac{i}{2\sqrt{2}}(B^{ab}{}_3 - B^{ba}{}_3) = -\frac{i}{\sqrt{2}}B^{ab}{}_3.
$$

Therefore $B^{ab}{}_3 = 0$ and consequently $\sigma^{ab} = 0$. From $(3)_2$ we get that

$$
B^{3a}{}_b = \frac{i}{\sqrt{2}}\sigma^a_b\,.
$$

We substitute this value in $(3)_4$. As a result we have

$$
\sigma^a_b = i\sqrt{2}B_{3b}{}^a.
$$

Now, we use the relations for the Kirichenko structure tensors of six-dimensional Hermitian submanifolds of Cayley algebra [9]:

$$B^{\alpha\beta}{}_{\gamma} = \frac{1}{\sqrt{2}}\varepsilon^{\alpha\beta\mu}D_{\mu\gamma}, \qquad B_{\alpha\beta}{}^{\gamma} = \frac{1}{\sqrt{2}}\varepsilon_{\alpha\beta\mu}D^{\mu\gamma},$$

where

$$D_{\mu\gamma} = \pm T^{8}_{\mu\gamma} + iT^{7}_{\mu\gamma}, \qquad D^{\mu\gamma} = D_{\widehat{\mu}\widehat{\gamma}} = \pm T^{8}_{\widehat{\mu}\widehat{\gamma}} - iT^{7}_{\widehat{\mu}\widehat{\gamma}}.$$

Here T^{φ}_{kj} are components of the configuration tensor (in A. Gray's notation [11], or the Euler curvature tensor [12]) of the Hermitian $M^6 \subset \mathbf{O}$; $\alpha, \beta, \gamma, \mu = 1, 2, 3$; $\widehat{\mu} = \mu + 3$; $k, j = 1, 2, 3, 4, 5, 6$; $\varphi = 7, 8$; $\varepsilon^{\alpha\beta\mu} = \varepsilon^{\alpha\beta\mu}_{123}$, $\varepsilon_{\alpha\beta\mu} = \varepsilon^{123}_{\alpha\beta\mu}$ are components of the third order Kronecker tensor [13].

From $(3)_1$ we obtain

$$B^{ab}{}_{c} = 0 \Leftrightarrow \frac{1}{\sqrt{2}}\varepsilon^{ab\gamma}D_{\gamma c} = 0 \Leftrightarrow \frac{1}{\sqrt{2}}\varepsilon^{ab3}D_{3c} = 0 \Leftrightarrow D_{3c} = 0.$$

The similar reasoning can be applied in reference to the condition $B^{ab}{}_{3} = 0$ obtained above:

$$B^{ab}{}_{3} = 0 \Leftrightarrow \frac{1}{\sqrt{2}}\varepsilon^{ab\gamma}D_{\gamma 3} = 0 \Leftrightarrow \frac{1}{\sqrt{2}}\varepsilon^{ab3}D_{33} = 0 \Leftrightarrow D_{33} = 0.$$

So, $D_{3c} = D_{33} = 0$, that is $D_{3\alpha} = 0$.

From $(3)_5$ we get

$$\sigma^{b}_{3} = \sigma_{3\hat{b}} = -iB^{3b}{}_{3} = -i\frac{1}{\sqrt{2}}\varepsilon^{3b\gamma}D_{\gamma 3} = 0.$$

We have $\sigma_{ab} = \sigma_{\hat{a}\hat{b}} = \sigma_{3b} = \sigma_{3\hat{b}} = 0$. We shall compute the rest of the components of the second fundamental form using $(3)_2$:

$$\sigma_{\hat{a}b} = \sigma^{a}_{b} = i\sqrt{2}B^{a3}{}_{b} = i\sqrt{2}\frac{1}{\sqrt{2}}\varepsilon^{a3\gamma}D_{\gamma b} = i\varepsilon^{a3c}D_{cb}.$$

Then

$$\sigma_{\hat{1}1} = i\varepsilon^{13c}D_{c1} = i\varepsilon^{132}D_{21} = -iD_{21};$$
$$\sigma_{\hat{1}2} = i\varepsilon^{13c}D_{c2} = i\varepsilon^{132}D_{22} = -iD_{22};$$
$$\sigma_{\hat{2}1} = i\varepsilon^{23c}D_{c1} = i\varepsilon^{231}D_{11} = iD_{11};$$
$$\sigma_{\hat{2}2} = i\varepsilon^{23c}D_{c2} = i\varepsilon^{231}D_{12} = iD_{12};$$
$$\sigma_{1\hat{1}} = \overline{\sigma_{\hat{1}1}} = iD^{12};$$
$$\sigma_{1\hat{2}} = \overline{\sigma_{\hat{1}2}} = iD^{22};$$
$$\sigma_{2\hat{1}} = \overline{\sigma_{\hat{2}1}} = -iD^{11};$$
$$\sigma_{2\hat{2}} = \overline{\sigma_{\hat{2}2}} = -iD^{12}.$$

We obtian that the matrix of the second fundamental form of the immersion of the cosymplectic hyperspace N into $M^6 \subset \mathbf{O}$ looks as follows:

$$\sigma = \begin{pmatrix} 0 & 0 & 0 & iD^{12} & -iD^{11} \\ 0 & 0 & 0 & iD^{22} & -iD^{12} \\ 0 & 0 & \sigma_{33} & 0 & 0 \\ -iD_{12} & -iD_{22} & 0 & 0 & 0 \\ iD_{11} & iD_{22} & 0 & 0 & 0 \end{pmatrix}.$$

3 The main result

Theorem *The following statements are equivalent:*

1. The cosymplectic hypersurface of a Hermitian $M^6 \subset \mathbf{O}$ is a totally umbilical submanifold.

2. The cosymplectic hypersurface of a Hermitian $M^6 \subset \mathbf{O}$ is a totally geodesic submanifold.

Proof In accordance with the definition [10], a hypersurface of a manifold is called totally umbilical if

$$\sigma = \lambda g, \quad \lambda - const.$$

Knowing how the matrix of the Riemannian metric looks [4]:

$$g = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix},$$

we make the conclution that the conditions

$$D_{11} = D_{22} = D^{11} = D^{22} = 0$$

are necessary for a cosymplectic hypersurface of Hermitian $M^6 \subset \mathbf{O}$ to be totally umbilical. Using the identities from [9]

$$D_{11}D_{22} = (D_{12})^2, \qquad D^{11}D^{22} = (D^{12})^2,$$

we obtain that the matrix σ of a totally umbilical hypersurface of a Hermitian $M^6 \subset \mathbf{O}$ looks as follows:

$$\sigma = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_{33} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence, $\lambda = 0$, that is why $\sigma_{33} = 0$. Therefore the matrix vanishes, and as a result we have that the hypersurface is totally geodesic.

Of course, it is obvious that every totally geodesic cosymplectic hypersurface of a Hermitian $M^6 \subset \mathbf{O}$ is totally umbilical. □

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