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Congruence Distributivity and Modularity Permit Tolerances*

Dedicated to Béla Csákány on his seventieth birthday

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Abstract

We prove that the distributive resp. modular law holds in congruence distributive resp. congruence modular varieties even for tolerance relations.

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Let dist(x, y, z) resp. mod(x, y, z) denote the distributive law

\[ x \land (y \lor z) \leq (x \land y) \lor (x \land z) \]

resp. the modular law

\[ x \land (y \lor (x \land z)) \leq (x \land y) \lor (x \land z). \]

For an algebra A, the set of tolerances and the lattice of congruences of A will be denoted by Tol A and Con A, respectively. We say that dist(tol,tol,tol) holds

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in A if \( \Gamma \land (\Phi \lor \Psi) \subseteq (\Gamma \land \Phi) \lor (\Gamma \land \Psi) \) is valid for any \( \Gamma, \Phi, \Psi \in \text{Tol}_A \), where the meet resp. the join is the intersection resp. the transitive closure of the union.

I.e., denoting the transitive closure by * , \( \Phi \lor \Psi = (\Phi \cup \Psi)^* = \Psi^* \lor \Phi^* \) (the second join is from \( \text{Con}_A \)) for any tolerances \( \Phi \) and \( \Psi \) in the present paper throughout. The meaning of \( \text{mod}(\text{tol},\text{tol},\text{tol}) \) is analogous.

**Theorem 1** If \( \mathcal{V} \) is a congruence distributive resp. congruence modular variety then \( \text{dist}(\text{tol},\text{tol},\text{tol}) \) resp. \( \text{mod}(\text{tol},\text{tol},\text{tol}) \) holds in all algebras of \( \mathcal{V} \).

**Proof** Suppose \( \mathcal{V} \) is congruence distributive. Then we have Jónsson terms, cf. Jónsson [5], i.e. ternary \( \mathcal{V} \)-terms \( t_0, \ldots, t_n \) for some even \( n \in \mathbb{N}_0 = \{0, 1, 2, \ldots \} \) such that \( \mathcal{V} \) satisfies the identities

\[
\begin{align*}
t_0(x, y, z) &= x, \\
t_n(x, y, z) &= z, \\
t_i(x, x, y) &= t_{i+1}(x, x, y) \\
&\text{for } i \text{ even,} \\
h(x, y, y) &= t_{i+1}(x, y, y) \\
&\text{for } i \text{ odd, and} \\
U(x, y, x) &= x
\end{align*}
\]

for all \( i \). Now let \( A \in \mathcal{V} \), \( \Gamma, \Phi, \Psi \in \text{Tol}_A \) and \( (a, b) \in \Gamma \land (\Phi \lor \Psi) \). Then there is an even \( k \) and there are elements \( c_0 = a, c_1, \ldots, c_{k-1}, c_k = b \) such that \( (c_i, c_{i+1}) \in \Phi \) for even \( i \), \( (c_i, c_{i+1}) \in \Psi \) for odd \( i \) and \( (a, b) = (c_0, c_k) \in \Gamma \). Since

\[
t_i(a, u, b) = t_i(t_i(a, v, a), u, t_i(b, v, b)) \Gamma t_i(t_i(a, v, b), u, t_i(a, v, b)) = t_i(a, v, b),
\]

for all \( i \) and any \( u, v \in A \) we have

\[
(t_i(a, u, b), t_i(a, v, b)) \in \Gamma.
\]

(1)

Now we define a sequence from \( a \) to \( b \) as follows:

\[
a = t_0(a, c_0, b) = t_1(a, c_0, b) \Phi t_1(a, c_1, b) \Psi t_1(a, c_2, b) \Phi t_1(a, c_3, b) \\
\Psi \ldots \Phi t_1(a, c_k-1, b) \Psi t_1(a, c_k, b) = t_1(a, b, b) = t_2(a, b, b)
\]

\[
= t_2(a, c_k, b) \Psi t_2(a, c_k-1, b) \Phi t_2(a, c_k-2, b) \Psi \ldots \Phi t_2(a, c_0, b)
\]

\[
= t_2(a, a, b) = t_3(a, a, b) \Phi t_3(a, c_1, b) \Psi t_3(a, c_2, b) \Phi \ldots \Psi
\]

\[
t_3(a, c_k, b) = t_4(a, c_k, b) \Psi t_4(a, c_{k-1}, b) \Phi \ldots \Phi
\]

\[
t_{n-1}(a, c_k-1, b) \Psi t_{n-1}(a, c_k, b) = t_{n-1}(a, b, b) = t_n(a, b, b) = b.
\]

It follows from (1) that any two consecutive members of this series are in \( (\Gamma \land \Phi) \cup (\Gamma \land \Psi) \subseteq (\Gamma \land \Phi) \lor (\Gamma \land \Psi) \). Thus \( (a, b) \in (\Gamma \land \Phi) \lor (\Gamma \land \Psi) \), whence \( \text{dist}(\text{tol},\text{tol},\text{tol}) \) holds in \( \mathcal{V} \).

Now let \( \mathcal{V} \) be congruence modular. Then we have Day terms, i.e. quaternary \( \mathcal{V} \)-terms \( m_0, m_1, \ldots, m_k \) for some \( 0 < k \in \mathbb{N}_0 \) such that \( \mathcal{V} \) satisfies the identities

\[
m_0(x, y, u, v) = x, \quad m_k(x, y, u, v) = y
\]

\[
m_i(x, y, x, y) = m_{i+1}(x, y, x, y) \quad \text{for } i \text{ even,}
\]

\[
m_i(x, y, z, z) = m_{i+1}(x, y, z, z) \quad \text{for } i \text{ odd, and}
\]

\[
m_i(x, x, y, y) = x \quad \text{for all } i,
\]

cf. Day [3]. First we show that, for any \( A \in \mathcal{V} \) and \( \Gamma, \Phi, \Psi \in \text{Tol}_A \),

\[
(\Gamma \land (\Phi \circ (\Gamma \land \Psi) \circ \Phi)) \subseteq (\Gamma \land \Phi) \lor (\Gamma \land \Psi).
\]

(2)
Let \((a, b) \in \Gamma \cap (\Phi \circ (\Gamma \cap \Psi) \circ \Phi)\). Then there are \(c, d \in A\) with \((a, c), (d, b) \in \Phi\), \((c, d) \in \Gamma \cap \Psi\) and, of course, \((a, b) \in \Gamma\). Consider the elements \(d_i = m_i(a, b, c, d)\) for \(i = 0, 1, \ldots, k\), \(e_i = m_i(a, b, c, d) = m_{i+1}(a, b, c, d)\) for \(i\) odd, and \(e_i = m_i(a, b, c, d) = m_{i+1}(a, b, c, d)\) for \(i\) even. Then \(d_0 = a, d_k = b\), and \((d_i, e_i), (e_i, d_{i+1}) \in \Gamma \cap \Psi\) for \(i\) odd.

For \(i\) even we have \((d_i, e_i), (e_i, d_{i+1}) \in \Phi\), i.e., \((d_i, e_i) \in \Gamma \cap \Phi\). Similarly, \((e_i, d_{i+1}) \in \Gamma \cap \Phi\).

Now \((a, b) \in (\Gamma \wedge \Phi) \vee (\Gamma \wedge \Psi)\) follows from transitivity and from the fact that all the \((d_i, e_i)\) and \((e_i, d_{i+1})\) belong to \((\Gamma \wedge \Phi) \vee (\Gamma \wedge \Psi)\). This shows (2).

Now define \(\Phi_0 = \Phi\) and \(\Phi_{n+1} = \Phi_n \circ (\Gamma \cap \Psi) \circ \Phi_n\) for \(n \geq 1\). Notice that all the \(\Phi_n\) belong to \(\text{Tol} A\). We claim that, for all \(n \in \mathbb{N}_0\),

\[
\Gamma \cap \Phi_n \subseteq (\Gamma \cap \Phi) \vee (\Gamma \cap \Psi).
\]

This is evident for \(n = 0\). Assuming (3) for an arbitrary \(n\) and applying (2) we obtain \(\Gamma \cap \Phi_{n+1} = \Gamma \cap (\Phi_n \circ (\Gamma \cap \Psi) \circ \Phi_n) \subseteq (\Gamma \cap \Phi_n) \vee (\Gamma \cap \Psi) \subseteq (\Gamma \cap \Phi) \vee (\Gamma \cap \Psi) \subseteq (\Gamma \cap \Phi) \vee (\Gamma \cap \Psi)\), i.e. (3) holds for \(n+1\). Thus (3) holds for all \(n\) and we obtain \(\Gamma \wedge (\Phi \vee (\Gamma \wedge \Psi)) = \Gamma \cap \bigcup \{\Phi_n : n \in \mathbb{N}_0\} = \bigcup \{\Gamma \cap \Phi_n : n \in \mathbb{N}_0\} \subseteq (\Gamma \cap \Phi) \vee (\Gamma \cap \Psi)\). This proves Theorem 1.

**Corollary 1 (Gumm [4])** If \(V\) is a congruence modular variety then it satisfies Gumm’s Shifting Principle, i.e., for any \(A \in V\), \(\alpha, \gamma \in \text{Con} A\) and \(\Phi \in \text{Tol} A\) if \((x, y), (u, v) \in \alpha\), \((x, u), (y, v) \in \Phi\), \((u, v) \in \gamma\) and \(\alpha \cap \Phi \subseteq \gamma\) then \((x, y) \in \gamma\).

**Proof** \((x, y) \in \alpha \cap (\Phi \vee (\alpha \wedge \gamma)) \subseteq (\alpha \wedge \Phi) \vee (\alpha \wedge \gamma) \subseteq \gamma \vee \gamma = \gamma\). □

Notice that Theorem 1 also implies the Triangular Principle and the Trapezoid Principle for congruence distributive varieties, cf. [1] and [2].

Now we give an example. Consider the monounary algebra \(B = (\{0, 1, 2\}, -)\) where \(-0 = 0, -1 = 2\) and \(-2 = 1\). Then \(\alpha\) with the associated partition \(\{(0), (1, 2)\}\) is the only nontrivial congruence of \(B\), so \(\text{Con} B\) is distributive. Notice that

\[
\Phi = \{(0, 1), (1, 0), (0, 2), (2, 0), (0, 0), (1, 1), (2, 2)\}
\]

is a tolerance and \(\alpha \cap \Phi^* \not\subseteq (\alpha \cap \Phi)^*\). Hence the following statement indicates that Theorem 1 cannot be extended for single algebras.

**Proposition 1** If \(\text{mod}(\text{tol}, \text{tol}, \text{tol})\) or \(\text{dist}(\text{tol}, \text{tol}, \text{tol})\) holds in an algebra \(A\) then \(\Gamma \cap \Phi^* \subseteq (\Gamma \cap \Phi)^*\) for any \(\Gamma, \Phi \in \text{Tol} A\).

**Proof** Apply \(\text{mod}(\Gamma, \Phi, 0)\) or \(\text{dist}(\Gamma, \Phi, 0)\). □
References


