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Approximation Properties of Certain Linear Positive Operators in Exponential Weighted Spaces of Functions of Two Variables

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Abstract

We introduce certain modified Szasz–Mirakyan operators in exponential weighted spaces of functions of two variables and we study approximation properties of these operators.

The similar theorems for functions of one variable were given in [3].

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1 Introduction

1.1 Let as in [1], for $q > 0$,

$$v_q(x) := e^{-qx}, \quad x \in R_0 := [0, +\infty). \tag{1}$$

Now, for given $p, q > 0$, we define the weighted function

$$v_{p,q}(x, y) := v_p(x)v_q(y), \quad (x, y) \in R_0^2 := R_0 \times R_0, \tag{2}$$

and next the weighted space $C_{p,q}$ of all real-valued functions f continuous on R_0^2 for which $v_{p,q}f$ is uniformly continuous and bounded on R_0^2. The norm on $C_{p,q}$ is defined by the formula

$$\|f\|_{p,q} \equiv \|f(\cdot,\cdot)\|_{p,q} := \sup_{(x,y)\in R_0^2} v_{p,q}(x,y)\,|f(x,y)|\,. \tag{3}$$

The modulus of continuity of $f \in C_{p,q}$ we define as usual by the formula

$$\omega(f, C_{p,q}; t_1, t_2) := \sup_{0\le h\le t_1,\, 0\le\delta\le t_2} \|\Delta_{h,\delta} f(\cdot,\cdot)\|_{p,q}\,, \quad t_1, t_2 \ge 0, \tag{4}$$

where $\Delta_{h,\delta} f(x,y) := f(x+h, y+\delta) - f(x,y)$ and $(x+h, y+\delta) \in R_0^2$.
From (4) it follows that

$$\lim_{t_1,t_2\to 0+} \omega(f, C_{p,q}; t_1, t_2) = 0 \tag{5}$$

for every $f \in C_{p,q}$, $p, q > 0$.
Moreover let $C^1_{p,q}$ be the set of all functions $f \in C_{p,q}$ which first partial derivatives belong also to $C_{p,q}$.

1.2 In this paper we introduce the following class of operators in $C_{p,q}$.

Definition 1 *Let $r, s \in N := \{1, 2, \ldots\}$ be fixed numbers. For functions $f \in C_{p,q}$, $p, q > 0$, we define the operators $A_{m,n}(f; p, q, r, s; x, y) \equiv A_{m,n}(f; x, y)$*

$$A_{m,n}(f; x, y) :=$$

$$:= \frac{1}{g(mx+1; r)\, g(ny+1; s)} \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(mx+1)^j}{(j+r)!}\frac{(ny+1)^k}{(k+s)!} f\left(\frac{j+r}{m+p}, \frac{k+s}{n+q}\right), \tag{6}$$

for $(x,y) \in R_0^2$, $m, n \in N$, where

$$g(t; r) = \sum_{i=0}^{\infty} \frac{t^i}{(i+r)!}, \quad \textit{for } t \in R_0, \tag{7}$$

i.e.

$$g(0; r) = \frac{1}{r!}, \qquad g(t, r) = \frac{1}{t^r}\left(e^t - \sum_{j=0}^{r-1} \frac{t^j}{j!}\right) \qquad \textit{if } t > 0.$$

In [3] were examined the operators

$$A_n(f; p, r; x) := \frac{1}{g(nx+1; r)} \sum_{k=0}^{\infty} \frac{(nx+1)^k}{(k+r)!} f\left(\frac{k+r}{n+p}\right), \quad r \in N,\ p > 0, \tag{8}$$

for functions of one variable, belonging to exponential weighted spaces.
In this paper we shall give similar results for operators $A_{m,n}(f)$.

From (6)–(8) we deduce that $A_{m,n}(f)$ are well defined in every space $C_{p,q}$, $p, q > 0$. Moreover for fixed $r, s \in N$ and $p, q > 0$ we have

$$A_{m,n}(1; p, q, r, s; x, y) = 1 \quad \text{for } (x, y) \in R_0^2,\ m, n \in N, \tag{9}$$

and if $f \in C_{p,q}$ and $f(x, y) = f_1(x) f_2(y)$ for all $(x, y) \in R_0^2$, then we have

$$A_{m,n}(f; p, q, r, s; x, y) = A_m(f_1; p, r; x) A_n(f_2; q, s; y) \tag{10}$$

for all $(x, y) \in R_0^2$ and $m, n \in N$.

In this paper by $M_k(\alpha_1, \ldots, \alpha_j)$ we shall denote suitable positive constants depending only on indicated parameters $\alpha_1, \ldots, \alpha_j$.

2 Lemmas and theorems

2.1 In this section we shall give some properties of the above operators, which we shall apply to the proofs of the main theorems.

In the paper [3] were proved the following two lemmas for $A_n(f; p, r; \cdot)$ defined by (8).

Lemma 1 *Let $p > 0$ and $r \in N$ be fixed numbers. Then for all $x \in R_0$ and $n \in N$ we have*

$$A_n(1; p, r; x) = 1,$$

$$A_n(t - x; p, r; x) = \frac{1 - px}{n + p} + \frac{1}{(n + p)(r - 1)!\, g(nx + 1; r)},$$

$$A_n\left((t - x)^2; p, r; x\right) = \left(\frac{1 - px}{n + p}\right)^2 + \frac{nx + 1}{(n + p)^2} + \frac{1 - nx - 2px + r}{(n + p)^2 (r - 1)!\, g(nx + 1; r)},$$

$$A_n\left(e^{pt}; p, r; x\right) = \frac{g\left((nx + 1)e^{p/(n+p)}; r\right)}{g(nx + 1; r)} e^{pr/(n+p)},$$

$$A_n\left((t - x)^2 e^{pt}; p, r; x\right) =$$

$$= \left\{\left(\frac{nx + 1}{n + p} e^{p/(n+p)} - x\right)^2 + \frac{nx + 1}{(n + p)^2} e^{p/(n+p)}\right\} A_n\left(e^{pt}; p, r; x\right) +$$

$$+ \frac{(nx + 1)e^{p/(n+p)} - 2x(n + p) + r}{(n + p)^2 (r - 1)!\, g(nx + 1; r)} e^{pr/(n+p)}.$$

Lemma 2 *For every fixed $p > 0$ and $r \in N$ there exist positive constants $M_i \equiv M_i(p, r)$, $i = 1, 2$, such that for all $x \in R_0$, $n \in N$*

$$v_p(x)\, A_n\left(1/v_p(t); p, r; x\right) \le M_1, \tag{11}$$

$$v_p(x)\, A_n\left((t - x)^2 / v_p(t); p, r; x\right) \le M_2 \left[\left(\frac{x + 1}{n + p}\right)^2 + \frac{x + 1}{n + p}\right]. \tag{12}$$

Applying Lemma 2, we can prove the basic property of $A_{m,n}(f)$.

Lemma 3 *For fixed $p,q>0$ and $r,s\in N$ there exists a positive constant $M_3\equiv M_3(p,q,r,s)$ such that*

$$\|A_{m,n}(1/v_{p,q}(t,z);p,q,r,s;\cdot,\cdot)\|_{p,q}\le M_3 \quad \text{for } m,n\in N. \tag{13}$$

Moreover for every $f\in C_{p,q}$ we have

$$\|A_{m,n}(f;p,q,r,s;\cdot,\cdot)\|_{p,q}\le M_3\|f\|_{p,q} \quad \text{for } m,n\in N,\ r,s\in N. \tag{14}$$

The formulas (6)-(7) and the inequality (14) show that $A_{m,n}$, $m,n\in N$, defined by (6) are linear positive operators from the space $C_{p,q}$ into $C_{p,q}$.

Proof The inequality (13) follows immediately from (2), (10) and (11).
If $f\in C_{p,q}$ and $r,s\in N$, then from (6) and (3) we get

$$\|A_{m,n}(f;p,q,r,s)\|_{p,q}\le\|f\|_{p,q}\,\|A_{m,n}(1/v_{p,q};p,q,r,s)\|_{p,q}, \quad m,n\in N,$$

which by (13) implies (14). This completes the proof of Lemma 3. □

2.2 Now we shall give two theorems on the degree of approximation of functions by $A_{m,n}$ defined by (6).

Let

$$\Phi_{m,p}(x):=\left(\frac{x+1}{m+p}\right)^2+\frac{x+1}{m+p},\quad x\in R_0,\ m\in N,\ p>0, \tag{15}$$

$$\Psi_{n,q}(y):=\left(\frac{y+1}{n+q}\right)^2+\frac{y+1}{n+q},\quad y\in R_0,\ n\in N,\ q>0. \tag{16}$$

Theorem 1 *Suppose that $f\in C^1_{p,q}$ with fixed $p,q>0$. Then there exists a positive constant $M_4=M_4(p,q,r,s)$ such that for all $m,n\in N$, $r,s\in N$ and $(x,y)\in R_0^2$*

$$v_{p,q}(x,y)\,|A_{m,n}(f;p,q,r,s;x,y)-f(x,y)|_{p,q}\le$$

$$\le M_4\left\{\|f'_x\|_{p,q}\sqrt{\Phi_{m,p}(x)}+\|f'_y\|_{p,q}\sqrt{\Psi_{n,q}(y)}\right\}. \tag{17}$$

Proof Let $(x,y)\in R_0^2$ be a fixed point. Then for $f\in C^1_{p,q}$ we have

$$f(t,z)-f(x,y)=\int_x^t f'_u(u,z)\,du+\int_y^z f'_v(x,v)\,dv,\quad (t,z)\in R_0^2.$$

From this and by (9) we get

$$A_{m,n}(f(t,z);p,q,r,s;x,y)-f(x,y)=$$

$$= A_{m,n}\left(\int_x^t f'_u(u,z)du; p,q,r,s;x,y\right) + A_{m,n}\left(\int_y^z f'_v(x,v)dv; p,q,r,s;x,y\right). \tag{18}$$

By (1)–(3) we have

$$\left|\int_x^t f'_u(u,z)du\right| \leq \|f'_x\|_{p,q}\left|\int_x^t \frac{du}{v_{p,q}(u,z)}\right|$$
$$\leq \|f'_x\|_{p,q}\left(\frac{1}{v_{p,q}(t,z)} + \frac{1}{v_{p,q}(x,z)}\right)|t-x|,$$

which by (1), (2), (6) and (8)–(10) and Lemma 1 implies that

$$v_{p,q}(x,y)\left|A_{m,n}\left(\int_x^t f'_u(u,z)du; p, qr, s; x, y\right)\right| \leq$$
$$\leq v_{p,q}(x,y)A_{m,n}\left(\left|\int_x^t f'_u(u,z)du\right|; p,q,r,s;x,y\right)$$
$$\leq \|f'_x\|_{p,q}\, v_{p,q}(x,y)\left\{A_{m,n}\left(\frac{|t-x|}{v_{p,q}(t,z)}; p,q,r,s;x,y\right)\right.$$
$$\left. + A_{m,n}\left(\frac{|t-x|}{v_{p,q}(x,z)}; p,q,r,s;x,y\right)\right\}$$
$$\leq \|f'_x\|_{p,q}\, v_q(y)A_n\left(\frac{1}{v_q(z)}; q,s;y\right)\left\{v_p(x)A_m\left(\frac{|t-x|}{v_p(t)}; p,r;x\right)\right.$$
$$\left. + A_m\left(|t-x|; p,r;x\right)\right\}.$$

Applying the Hölder inequality, (7), (11)–(12) and Lemmas 1–2, we get

$$A_m\left(|t-x|; p,r;x\right) \leq \left\{A_m((t-x)^2; p,r;x)A_m(1;p,r;x)\right\}^{1/2}$$
$$\leq M_5(p,r)\left[\left(\frac{x+1}{m+p}\right)^2 + \frac{x+1}{m+p}\right]^{1/2},$$

$$v_p(x)A_m\left(\frac{|t-x|}{v_p(t)}; p,r;x\right) \leq$$
$$\leq \left\{v_p(x)A_m\left(\frac{(t-x)^2}{v_p(t)}; p,r;x\right)\right\}^{1/2}\left\{v_p(x)A_m\left(\frac{1}{v_p(t)}; p,r;x\right)\right\}^{1/2}$$
$$\leq M_6(p,r)\left[\left(\frac{x+1}{m+p}\right)^2 + \frac{x+1}{m+p}\right]^{1/2}$$

for $x \in R_0$ and $m \in N$. From this and in view of Lemma 2 we deduce that

$$v_{p,q}(x,y)\left|A_{m,n}\left(\int_x^t f'_u(u,z)du; p,q,r,s;x,y\right)\right| \leq$$

$$\leq M_7(p,q,r,s)\|f'_x\|_{p,q}\sqrt{\Phi_{m,p}(x)}, \quad m \in N.$$

Analogously we obtain

$$v_{p,q}(x,y)\left|A_{m,n}\left(\int_y^z f'_v(x,v)dv;p,q,r,s;x,y\right)\right| \leq$$

$$\leq M_8(p,q,r,s)\|f'_y\|_{p,q}\sqrt{\Psi_{n,q}(y)}, \quad n \in N.$$

Combining these estimations, we derive from (18):

$$v_{p,q}(x,y)\left|A_{m,n}(f;p,q,r,s;x,y) - f(x,y)\right| \leq$$

$$\leq M_4\left\{\|f'_x\|_{p,q}\sqrt{\Phi_{m,p}(x)} + \|f'_y\|_{p,q}\sqrt{\Psi_{n,q}(y)}\right\},$$

for all $m,n \in N$, $(x,y) \in R_0^2$. This ends the proof of (17). □

Theorem 2 *Suppose that $f \in C_{p,q}$, $p,q > 0$. Then there exists a positive constant $M_9 \equiv M_9(p,q,r,s)$ such that*

$$v_{p,q}(x,y)\left|A_{m,n}(f;p,q,r,s;x,y) - f(x,y)\right| \leq$$

$$\leq M_9\,\omega\left(f, C_{p,q}; \sqrt{\Phi_{m,p}(x)}, \sqrt{\Psi_{n,q}(y)}\right) \tag{19}$$

for all $(x,y) \in R_0^2$, $m,n \in N$ and $r,s \in N$.

Proof We apply the Steklov function $f_{h,\delta}$ for $f \in C_{p,q}$

$$f_{h,\delta}(x,y) := \frac{1}{h\delta}\int_0^h du \int_0^\delta f(x+u, y+v)\,dv, \quad (x,y) \in R_0^2,\ h,\delta > 0. \tag{20}$$

From (20) it follows that

$$f_{h,\delta}(x,y) - f(x,y) = \frac{1}{h\delta}\int_0^h du \int_0^\delta \Delta_{u,v} f(x,y)\,dv,$$

$$(f_{h,\delta})'_x(x,y) = \frac{1}{h\delta}\int_0^\delta \left(\Delta_{h,v} f(x,y) - \Delta_{0,v} f(x,y)\right) dv,$$

$$(f_{h,\delta})'_y(x,y) = \frac{1}{h\delta}\int_0^h \left(\Delta_{u,\delta} f(x,y) - \Delta_{u,0} f(x,y)\right) du.$$

This implies that $f_{h,\delta} \in C^1_{p,q}$ for $f \in C_{p,q}$ and $h,\delta > 0$. Moreover

$$\left\|f_{h,\delta} - f\right\|_{p,q} \leq \omega\left(f, C_{p,q}; h, \delta\right), \tag{21}$$

$$\left\|(f_{h,\delta})'_x\right\|_{p,q} \leq 2h^{-1}\omega\left(f, C_{p,q}; h, \delta\right), \tag{22}$$

$$\left\| (f_{h,\delta})'_y \right\|_{p,q} \le 2\delta^{-1}\omega\left(f, C_{p,q}; h, \delta\right), \tag{23}$$

for all $h, \delta > 0$.

Observe that

$$v_{p,q}(x,y)\left|A_{m,n}(f;p,q,r,s;x,y) - f(x,y)\right| \le$$
$$\le v_{p,q}(x,y)\left\{\left|A_{m,n}\left(f(t,z) - f_{h,\delta}(t,z);p,q,r,s;x,y\right)\right|\right.$$
$$+\left|A_{m,n}\left(f_{h,\delta}(t,z);p,q,r,s;x,y\right) - f_{h,\delta}(x,y)\right|$$
$$\left.+\left|f_{h,\delta}(x,y) - f(x,y)\right|\right\} := T_1 + T_2 + T_3.$$

By (3), (14) and (21) we obtain

$$T_1 \le \left\|A_{m,n}\left(f - f_{h,\delta};p,q,r,s;\cdot,\cdot\right)\right\|_{p,q} \le M_3\left\|f - f_{h,\delta}\right\|_{p,q} \le M_3\,\omega\left(f, C_{p,q}; h, \delta\right),$$

$$T_3 \le \omega\left(f, C_{p,q}; h, \delta\right).$$

Applying Theorem 1 and (22) and (23), we get

$$T_2 \le M_4\left\{\left\|(f_{h,\delta})'_x\right\|_{p,q}\sqrt{\Phi_{m,p}(x)} + \left\|(f_{h,\delta})'_y\right\|_{p,q}\sqrt{\Psi_{n,q}(y)}\right\}$$
$$\le 2M_4\omega\left(f, C_{p,q}; h, \delta\right)\left\{h^{-1}\sqrt{\Phi_{m,p}(x)} + \delta^{-1}\sqrt{\Psi_{n,q}(y)}\right\}.$$

From the above we deduce that there exists a positive constant $M_{10} \equiv M_{10}(p,q,r,s)$ such that

$$v_{p,q}(x,y)\left|A_{m,n}(f;p,q,r,s;x,y) - f(x,y)\right| \le$$
$$\le M_{10}\omega\left(f, C_{p,q}; h, \delta\right)\left\{1 + h^{-1}\sqrt{\Phi_{m,p}(x)} + \delta^{-1}\sqrt{\Psi_{n,q}(y)}\right\}, \tag{24}$$

for $(x,y) \in R_0^2$, $m,n \in N$ and $h,\delta > 0$. Now, for fixed $x,y \in R_0$, $m,n \in N$, $p,q > 0$, setting $h = \sqrt{\Phi_{m,p}(x)}$ and $\delta = \sqrt{\Psi_{n,q}(y)}$ to (24), we obtain the assertion of Theorem 2. □

From Theorem 2 and the property (5) follows

Corollary *If $f \in C_{p,q}$, $p,q > 0$ and $r,s \in N$, then*

$$\lim_{m,n\to\infty} A_{m,n}(f;p,q,r,s;x,y) = f(x,y) \tag{25}$$

for $(x,y) \in R_0^2$. Moreover (25) holds uniformly on every rectangle $0 \le x \le x_0$, $0 \le y \le y_0$.

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