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Divisor Class Groups of Ordered Subgroups

Jiří Močkoř, Angeliki Kontolatou

Abstract. We show that if a po-group $G$ admits a theory of quasi-divisors (strong theory of quasi-divisors, respectively), then the factor po-group $G/H$ has the same property if $H$ is an o-ideal of $G$. We introduce a notion of a divisor class group $C$ of an ordered subgroup $G$ of an $l$-group $\Gamma$ and we show some relationships between properties of $C$ and conditions under which the inclusion $G \subseteq \Gamma$ is a strong theory of quasi-divisors. Finally, we present some examples of po-groups with a strong theory of quasi-divisors.

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1. Introduction

In [13] we introduced the notion of a po-group $G$ which admits a strong theory of quasi-divisors and we investigated some relationships between the existence of this strong theory of quasi-divisors and the existence of some approximation theorems for $t$-valuations of $G$. Recall that a directed po-group $(G,+,\leq)$ has a theory of quasi-divisors if there exists an $l$-group $(\Gamma,.,\leq)$ and a map $h : G \rightarrow \Gamma$ such that

(i) $h$ is an order isomorphism from $G$ into $\Gamma$.

(ii) $(\forall \alpha \in \Gamma_+) (\exists g_1, \ldots, g_n \in G_+) \alpha = h(g_1) \wedge \ldots \wedge h(g_n)$.

Moreover, we say that $G$ has a strong theory of quasi-divisors if there exists an $o$-isomorphism $h$ from $G$ into an $l$-group $\Gamma$ such that

(iii) $(\forall \alpha, \beta \in \Gamma_+) (\exists \gamma \in \Gamma_+) \alpha \cdot \gamma = h(G), \beta \wedge \gamma = 1$.

In the theory of po-groups with a theory of quasi-divisors (or, equivalently $t$-Prüfer po-groups, see [9]) an important role has a localization of an r-system. We recall very roughly this construction (see [3],[13]).

Let $(G, x)$ be a po-group with an $r$-system $x$ of a finite character (for the notion of an r-system see e.g. [9]). Let $H$ be an o-ideal of $G$, i.e. $H$ is a directed convex subgroup of $G$, and let $\varphi : G \rightarrow G/H$ be a canonical homomorphism. Then for any lower bounded subset $A \subseteq G/H$ we may find a lower bounded subset $A \subseteq G$ such that $\{aH : a \in A\} = A$ (see [13]). Then we set $A_x := A_x/H$. According to [13], $x_H$ is an $r$-system on $G/H$ of a finite character. We also introduced a notion of an $x$-local o-ideal, where $H$ is such o-ideal if $x_H$ is a local r-system, i.e. in $(G/H)_+$ there exists the unique maximal $x_H$-ideal.

In this paper we show at first that if $G$ is a po-group with a theory of quasi-divisors (strong theory of quasi-divisors, respectively), then the same property has
the factor po-group $G/H$, where $H$ is an o-ideal. Then for a po-group $G$ and its o-isomorphism $h$ into an l-group $\Gamma$ we introduce a notion of a divisor class group $C_h$ of $h : G \rightarrow \Gamma$ and we show some relationships between properties of $C_h$ and conditions under which the inclusion $G \subseteq \Gamma$ is a strong theory of quasi-divisors. Finally, using these results we present a method for constructing examples of po-groups with strong theory of quasi-divisors by using the restricted Hahn group $H(\Delta,Z)$, where $\Delta$ is some root system.

2. Divisor class groups

We start this section with investigation of some structural properties of po-groups $G$ and $G/H$, where $H$ is an o-ideal of $G$. Recall that if $G$ is a directed po-group, then a t-ideal generated by a lower directed subset $X \subseteq G$ is defined as

$$X_t = \bigcup_{Y \subseteq X, \text{finite}} Y_o$$

where $(a)_o = \{g \in G : g \geq a\}$.

**Lemma 2.1.** Let $H$ be an o-ideal of a directed po-group $(G, x)$ with an r-system $x$ of a finite character and let $\mathcal{P}_{x,H}$ be a proper $x_H$-ideal of $(G/H)_+$. Then the following statements are equivalent:

1. $\mathcal{P}_{x,H}$ is a prime $x_H$-ideal.
2. There exists a prime $x$-ideal $Q$ of $G_+$ such that $Q/H = \mathcal{P}_{x,H}$.

**Proof:** (1) $\Rightarrow$ (2). Let $P$ be a lower directed subset of $G$ which represents $\mathcal{P}$. Then from a definition of a localization we have $\mathcal{P}_{x,H} = P_x/H$. Since $\mathcal{P}_{x,H}$ is a prime $x_H$-ideal, according to [13]:2.4, we obtain that $\mathcal{H} = [(G/H_+ \setminus \mathcal{P}_{x,H})]$ is an $x_H$-local o-ideal of $G/H$, where $[X]$ is a subgroup generated by $X$. Then there exists an o-ideal $T$ of $G$ such that $\mathcal{H} = T/H$, $H \subseteq T$. In what follows we may identify $(G/H)/\mathcal{H}$ and $G/T$ (under the map $(aH)\mathcal{H} \mapsto a.T$). Then $P_x \cap T = \emptyset$ as follows from the maximality of $\mathcal{P}_{x,H}/\mathcal{H} = P_x/T$. According to [9], there exists a prime $x$-ideal $Q$ of $G_+$ such that $P_x \subseteq Q$, $Q \cap T = \emptyset$. We show that $\mathcal{P}_{x,H} = Q/H$. In fact, since $Q/T$ is a proper $(x_H)_H$-ideal in $G/T$, we have

$$\mathcal{P}_{x,H}/\mathcal{H} = P_x/T = (P_x/H)/\mathcal{H} \subseteq (Q/H)/\mathcal{H} \subseteq P_{x,H}/\mathcal{H}$$

and $\mathcal{P}_{x,H}/\mathcal{H} = (Q/H)/\mathcal{H}$, hence $P_x/T = Q/T$. We show that $Q = G_+ \setminus T$. In fact, if $a \in T_+$, then from $a \in Q$ it follows that $T = aT \in Q/T = P_x/T$, a contradiction. Let $a \in G_+ \setminus Q$ and let us assume that $a \notin T$. Then $aT > T$ and since $P_x/T = Q/T$ is the unique maximal $x_H$-ideal in $G/H$, we have $aT \in Q/T$ (see [9]). Since $T$ is an o-ideal, there exist $q \in Q, h_1, h_2 \in T_+$ such that $ah_1 = qh_2$. Since $ah_1 \geq q$, we have $ah_1 \in Q, h_1 \notin Q$ and it follows that $a \in Q$, a contradiction. Therefore, $\mathcal{P}_{x,H} = Q/H$.

(2) $\Rightarrow$ (1). Let $Q$ be a prime $x$-ideal of $G_+$ such that $Q/H = \mathcal{P}_{x,H}$. Then $Q \cap H = \emptyset$ and if $aH, bH \geq H$, $abH \in Q/H$, we have $(t_1,a).(t_2,b)h_1 = h_2p$
for some $t_1, h_1 \in H_+, p \in Q$ and $at_1, bt_2 \geq 1$. Then since $h_1 \not\in Q$, we have $(t_1a)(t_2b) \in Q$ and it follows that $aH \in \mathcal{P}_{xH}$, or $bH \in \mathcal{P}_{xH}$.

In what follows, we denote by $\mathcal{H}_H(G,x)$ the set of all $x$-local $\sigma$-ideals $T$ of a po-group $G$ with an $r$-system $x$ such that $H \subseteq T$. If $H = \{1\}$, we write simply $\mathcal{H}(G,x)$.

**Proposition 2.2.** Let $(G,x)$ be a directed po-group with an $r$-system $x$ of a finite character and let $H_0$ be an $x$-ideal of $G$. Then there exists a bijection $\varphi$ between $\mathcal{H}_{H_0}(G,x)$ and $\mathcal{H}(G/H_0,x_{H_0})$ such that $G/H_0 \cong (G/H_0)/\varphi(H)$ for any $H \in \mathcal{H}_{H_0}(G,x)$.

**Proof:** Let $T \in \mathcal{H}(G/H_0,x_{H_0})$. Then $T = T/H_0$, where $T$ is an $x$-ideal of $G$, $H_0 \subseteq T$. We show that $T \in \mathcal{H}_{H_0}(G,x)$. According to [13], there exists $T \in \mathcal{H}_{H_0}(G,x)$ such that $T$ is a $x$-ideal of $G$ and according to 2.1, there exists a prime $x$-ideal $Q$ in $G$ such that $Q/H_0 = \mathcal{P}_{xH}$. Then $Q = G_+ \setminus T$ and according to [13], there exists a prime $x$-ideal $Q/H_0$ of $G/H_0$. Then $(G/H_0)_+ \setminus Q/H_0 = (H/H_0)_+$ and it follows that $H/H_0$ is $x$-local. Hence, $\psi(H) = H/H_0$ is the inverse of $\varphi$. □

**Proposition 2.3.** Let $(G,x)$ be an $x$-Prüfer directed po-group such that $x$ is of a finite character and let $H$ be an $x$-ideal of $G$. Then $G/H$ is a $x$-Prüfer group.

**Proof:** Let $T \in \mathcal{H}(G/H,x_H)$. Then according to 2.2, there exists $T \in \mathcal{H}_H(G,x)$ such that $(G/H)/T \cong G/T$. Then the proposition follows from [2]; Th.8. □

**Proposition 2.4.** Let $G$ be a directed po-group with a theory of quasi-divisors and let $H$ be an $x$-ideal of $G$. Then $G/H$ has a theory of quasi-divisors.

**Proof:** Since $G$ has a theory of quasi-divisors, it is a Prüfer t-group according to [2]. Then according to 2.3, $G/H$ is a $t_H$-Prüfer group and since $t_H \leq t$ in $G/H$, then according to [2]; Th.1, $G/H$ is a $t$-Prüfer group as well. Hence, $G/H$ has a theory of quasi-divisors. □

**Proposition 2.5.** Let $G$ be a directed po-group with a strong theory of quasi-divisors and let $H$ be an $x$-ideal of $G$. Then $G/H$ has a strong theory of quasi-divisors.

**Proof:** Let $h : G \rightarrow \Gamma$ be a strong theory of quasi-divisors. Since $h$ is a theory of quasi-divisors as well, (see [13]), $\Gamma$ may be identified with the Lorenzen $t$-group $\Lambda_t(G)$ of $G$ and we may assume that $h : G \rightarrow \Lambda_t(G)$ is an inclusion $x \mapsto (x)_t$. If $H$ is an $x$-ideal, then according to 2.4, $G/H$ admits a theory of quasi-divisors which then may be identified with the inclusion $h_H : G/H \rightarrow \Lambda_t(G/H)$. Since $t_H \leq t$ on $G/H$, then the composition $\varphi$ of morphisms $(G,t) \rightarrow (G/H,t_H) \rightarrow (G/GH,t)$ is a $(t,t)$-morphism. Hence, according to [2]; Th.1, there exists an $l$-epimorphism $\hat{\varphi}$ such that the diagram
commutes. The proposition then follows from the fact that \( h \) is a strong theory of quasi-divisors and \( \hat{\varphi} \) is an \( l \)-epimorphism.

Now, let \( G \) and \( \Gamma \) be ordered groups and let \( h: G \to \Gamma \) be an \( o \)-isomorphism from \( G \) into \( \Gamma \). Then the factor group \( C_h = \Gamma / h(G) \) is called a divisor class group of \( h \). We show at first that the construction of \( C_h \) has some functorial character.

**Proposition 2.6.** Let \( G \) admits a theory of quasi-divisors \( h: G \to \Gamma \) and let \( H \) be an \( o \)-ideal of \( G \). Let \( h_H: G/H \to \Gamma \) be a theory of quasi-divisors. Then there exists an \( o \)-epimorphism \( \psi: \Gamma \to \tilde{\Gamma} \) and epimorphism \( \sigma: C_h \to C_{hH} \) such that the diagram

\[
\begin{array}{ccc}
G & \xrightarrow{h} & \Gamma \\
\downarrow \psi & & \downarrow \varphi \\
G/H & \xrightarrow{h_H} & \tilde{\Gamma}
\end{array}
\]

commutes.

**Proof:** Since \( G \) admits a theory of quasi-divisors, \( G \) is a \( t \)-Prüfer group and we may identify \( \Gamma \) with the group of finitely generated \( t \)-ideals of \( G \). Analogously, \( \tilde{\Gamma} \) may be identified with the group of finitely generated \( t \)-ideals of \( G/H \). Since the canonical map \( \psi \) is a \((t, t_H)\)-morphism and \( t_H \leq t \), \( \psi \) is a \((t, t)\)-morphism as well and according to [9]; Th.1, there exists an \( o \)-epimorphism \( \hat{\psi}: \Gamma \to \tilde{\Gamma} \) such that \( \hat{\psi} : h = h_H : \psi \). Let \( \alpha = A_t \in \Gamma \), where \( A \) is a finite set in \( G \). We set \( \sigma(\varphi(A_t)) = \hat{\varphi}(\psi(A))_t \). This definition is correct. In fact, let \( A_t, B_t \in \Gamma \) be such that \( \varphi(A_t) = \varphi(B_t) \). Then there exists \( g \in G \) such that \( A_t = (gB)_t \). Let \( f \) be a bijection between \( H(G/H, t_H) \) and \( H(H, t) \) (see 2.2). Then \( (\psi(A))_t = A_t/f(T) \) for all \( T \in H(G/H, t_H) \). Hence,

\[
(\psi(A))_t/T = (gB)_t/f(T) \equiv (\psi(g).\psi(B))_tH/T.
\]

Thus, according to [13];2.8, we obtain \( (\psi(A))_t = \psi(g).\psi(B) \).

The rest is obvious.

It should be observed that

\[
\ker \sigma = \{ \varphi(A_t) \in C_h : \text{there exists } \inf_{G/H} (\psi(A)) \}.
\]

In fact, if \( \varphi(A_t) \in \ker \sigma \), then \( \sigma(\varphi(A_t)) = \hat{\varphi}(\psi(A))_t = 0 \). Then there exists \( g \in G \) such that \( (\psi(A))_t = (\psi(g)) \) and it follows that \( \psi(g) = \inf(\psi(A)) \).
Lemma 2.7. Let \( h \) be an \( o \)-isomorphism from a directed \( po \)-group \( G \) into an \( l \)-group \( \Gamma \). Then the following statements are equivalent:

1. \( h \) is a theory of quasi-divisors.
2. \((\forall \alpha \in \Gamma_+)\alpha = \inf_{\Gamma}(h(G) \cap (\alpha)_t)\).

**Proof:** (1) \( \Rightarrow \) (2). Let \( \alpha \in \Gamma_+ \). Since \( h \) is a theory of quasi-divisors, there exist \( g_1, \ldots, g_n \in h(G) \cap (\alpha)_t \) such that \( \alpha = h(g_1) \land \ldots \land h(g_n) \) in \( \Gamma \). Let \( \beta \in \Gamma \) be a lower bound of elements from \( h(G) \cap (\alpha)_t \). Then \( h(G) \cap (\alpha)_t \subseteq (\beta)_t \) and it follows that
\[
(\alpha)_t = (h(g_1) \land \ldots \land h(g_n))_t = (h(g_1)_t, \ldots, h(g_n)_t)_t \subseteq (\beta)_t.
\]

Hence, \( \alpha \geq \beta \) and \( \alpha = \inf(h(G) \cap (\alpha)_t) \).

(2) \( \Rightarrow \) (1). Let \( \alpha \in \Gamma_+ \). Since \( h(G) \cap (\alpha)_t \) is lower bounded, we have \( (h(G) \cap (\alpha)_t)_t = (\alpha)_t \). Hence, \( (h(G) \cap (\alpha)_t)_t \) is a \( t \)-invertible \( t \)-ideal and since \( t \) is an \( r \)-system of a finite character, \( (h(G) \cap (\alpha)_t)_t \) is finitely generated and its generators could be chosen from the set \( h(G) \cap (\alpha)_t \) (see [9]). Hence, there exist \( h(g_1), \ldots, h(g_n) \in h(G) \cap (\alpha)_t \) such that \( \alpha = h(g_1) \land \ldots \land h(g_n) \) and it follows that \( h \) is a theory of quasi-divisors. \( \square \)

**Proposition 2.8.** Let \( h : G \rightarrow \Gamma \) be a theory of quasi-divisors of a directed \( po \)-group \( G \), let \( C_h \) be a divisor class group of \( h \) and let \( \varphi : \Gamma \rightarrow C_h \) be a canonical map. Then for any \( \alpha \in \Gamma \), \( \alpha > 1 \), we have
\[
\varphi(\Gamma_+ \setminus (\alpha)_t) = C_h.
\]

**Proof:** Since \( h \) is a theory of quasi-divisors, for any \( \alpha \in \Gamma \) there exists \( \beta \in \Gamma_+ \) such that \( \varphi(\alpha) = \varphi(\beta) \). Hence, \( C_h = \varphi(\Gamma_+) \). Let \( \alpha \in \Gamma_+, \alpha > 1 \) and let \( \beta \in \Gamma_+ \). Then there exists \( \gamma \in \Gamma_+ \setminus (\alpha)_t \) such that \( \beta, \gamma \in h(G) \). In fact, let us assume at first that \( \alpha \) is incomparable with \( \beta \) or \( \alpha > \beta \). Then \( (h(G) \cap (\alpha)_t) \setminus (h(G) \cap (\alpha)_t) \neq \emptyset \) as follows from 2.7. Let \( h(g) \) be an element of this nonempty set. Then \( h(g) = h(\gamma) \beta \), where \( \gamma \geq 1 \) and \( \gamma \in \Gamma_+ \setminus (\alpha)_t \).

Let \( \alpha \leq \beta \). Then \( \alpha, \beta > \beta \geq \alpha \) and it follows that \( (h(G) \cap (\alpha, \beta)_t) \subset (h(G) \cap (\beta)_t) \). Let \( h(g) \in h(G) \cap (\beta)_t \) be such that \( h(g) \notin (\alpha, \beta)_t \). Then \( h(g) = h(\gamma) \beta \), where \( \gamma_1 \geq 1 \). If \( \gamma_1 \geq \alpha \), then \( h(g) = \beta \gamma_1 \geq \beta \alpha \), a contradiction. Hence, we proved that \( \varphi(\Gamma_+) \subseteq \varphi(\Gamma_+ \setminus (\alpha)_t) \). \( \square \)

Now, we say that an \( l \)-group \( \Gamma \) is finitely atomic, if for any element \( \alpha \in \Gamma, \alpha > 1 \), the set of all atoms \( \sigma \in \Gamma_+ \) such that \( \sigma \leq \alpha \) is nonempty and finite. A trivial example of a finitely atomic \( l \)-group is a group \( \mathbb{Z}^{(p)} \).

**Theorem 2.9.** Let \( h \) be an \( o \)-isomorphism from a directed \( po \)-group \( G \) into an \( l \)-group \( \Gamma \), let \( C_h \) be a divisor class group of \( h \) and let \( \varphi : \Gamma \rightarrow C_h \) be a canonical map. Let us consider the following statements:

1. \( h \) is a strong theory of quasi-divisors.
(2) If \( \alpha_1, \ldots, \alpha_n \) are elements of \( \Gamma \) such that \( \alpha_i > 1 \) for all \( i \), then \( \varphi(\Gamma_+ \setminus \{\alpha_1, \ldots, \alpha_n\}) = \mathcal{C}_\ell \).

(3) If \( \alpha_1, \ldots, \alpha_n \) are atoms in \( \Gamma_+ \), then \( \varphi(\Gamma_+ \setminus \{\alpha_1, \ldots, \alpha_n\}) = \mathcal{C}_h \).

Then (1) \( \implies \) (2) \( \implies \) (3). If \( \Gamma \) is finitely atomic, then all the statements are equivalent.

**Proof:** (1) \( \implies \) (2). Let \( \alpha_1, \ldots, \alpha_n \in \Gamma_+ \) for all \( i \). Let \( \varphi(\delta) \in \mathcal{C}_h \). Then there exists \( \alpha \in \Gamma_+ \) such that \( \delta \alpha \in h(G) \). Let \( \beta = \alpha_1 \ldots \alpha_n \). Then there exists \( \gamma \geq 1 \) such that \( \beta \land \gamma = 1 \) and \( \alpha \land \gamma \in h(G) \). Hence, \( \varphi(\alpha) + \varphi(\gamma) = 0 = \varphi(\delta) + \varphi(\alpha) \) and \( \varphi(\gamma) = \varphi(\delta) \). If \( \gamma \not\in \bigcap_i (\Gamma_+ \setminus (\alpha_i)_i) \), then there exists \( i \) such that \( \gamma \geq \alpha_i \). But, in this case we have \( \gamma \land \beta \geq \alpha_i > 1 \), a contradiction.

(2) \( \implies \) (3). Trivial.

Now, let us assume that that \( \Gamma \) is finitely atomic and let (3) hold. Let \( \alpha, \beta \in \Gamma_+ \), \( \alpha \not\in h(G) \). Since \( \mathcal{C}_h = \varphi(\Gamma_+) \), we have \( -\varphi(\alpha) \in \varphi(\Gamma_+) \) and there exists \( \delta \geq 1 \) such that \( -\varphi(\alpha) = \varphi(\delta) \). Hence, \( \alpha, \delta \in h(G) \). Now, according to the assumption we have \( \{ \sigma : \sigma \text{ is an atom in } \Gamma_+ , \sigma \leq \beta \} = \{ \sigma_1, \ldots, \sigma_n \} \) and according to (3) we have \( \varphi(\bigcap_i (\Gamma_+ \setminus (\sigma_i)_i)) = \mathcal{C}_h \). Then there exists \( \gamma \in \bigcap_i (\Gamma_+ \setminus (\sigma_i)_i) \) such that \( \varphi(\gamma) = \varphi(\delta) \). If \( \gamma \land \beta > 1 \) then there exists an atom \( \sigma \) such that \( \sigma \leq \beta \land \gamma \leq \beta, \gamma \) and it follows that \( \sigma = \sigma_i \) for some \( i \), a contradiction with \( \gamma \not\geq \sigma_i \). Hence, \( \beta \land \gamma = 1 \) and \( \alpha, \gamma \in h(G) \). Therefore, \( h \) is a strong theory of quasi-divisors.

**3. Examples**

In this part of the paper we should like to present a method for constructing examples of \( \mathfrak{p} \)-groups with a strong theory of quasi-divisors. This method is based on application of Theorem 2.9 onto a special \( l \)-group, the restricted Hahn group \( H(\Lambda, \mathbb{Z}) \) and this method generalizes in some sense a method of constructing examples of groups with divisors theory presented by L. Skula [17].

Recall that if \( \Lambda \) is a root system (i.e. \( (\Lambda, \leq) \) is a partly ordered set for which \( \{ \alpha \in \Lambda : \alpha \geq \gamma \} \) is totally ordered for any \( \gamma \in \Lambda \)), then the restricted Hahn group \( H(\Lambda, \mathbb{Z}) \) on \( \Lambda \) is the group \( \mathbb{Z}^\Lambda \) ordered in a following way:

\[
a \in H(\Lambda, \mathbb{Z}), a \geq 0 \iff a_\alpha > 0 \text{ for all } \alpha \in \text{ms}(a),
\]

where \( \text{ms}(a) \) is the maximal support of \( a \), i.e. the set of all maximal elements in \( \text{supp}(a) = \{ \alpha \in \Lambda : a_\alpha \neq 0 \} \). Then \( H(\Lambda, \mathbb{Z}) \) is an \( l \)-group (see e.g. [2]).

Now, let \( \Lambda_0 \) be the set of all minimal elements of \( \Lambda \). We say that \( \Lambda \) is atomic if for any element \( \alpha \in \Lambda \) there exists \( \beta \in \Lambda_0 \) such that \( \alpha \geq \beta \). Moreover, we say that \( \Lambda \) is finitely atomic if for any \( \alpha \in \Lambda \), the set \( \{ \sigma \in \Lambda_0 : \sigma \leq \alpha \} \) is nonempty and finite. Finally, let \( \alpha \in \Lambda \). Then by \( a^\alpha \) we denote the element of \( H(\Lambda, \mathbb{Z}) \) such that

\[
a_\beta^\alpha = \begin{cases} 
1, & \text{if } \beta = \alpha \\
0, & \text{otherwise}.
\end{cases}
\]

In the following lemma we summarize some properties of \( H(\Lambda, \mathbb{Z}) \) which would be of interest for our examples of groups with a strong theory of quasi-divisors.
Lemma 3.1. Let $\Delta$ be a root system.

(1) Let $\Delta$ be atomic and let $\alpha \in \Delta_0, b \in H(\Delta, \mathbb{Z})_+$. Then $b \geq a^\alpha$ if and only if there exists $\beta \in \mathrm{ms}(b)$ such that $\beta \geq \alpha$.

(2) If $\Delta$ is atomic, then $a \in H(\Delta, \mathbb{Z})$ is an atom if and only if $a = a^a$ for some $\alpha \in \Delta_0$.

(3) If $\Delta$ is finitely atomic, then $H(\Delta, \mathbb{Z})$ is finitely atomic.

Proof: (1). Let $b \geq a^\alpha$ for some $\alpha \in \Delta_0$. If $b = a^\alpha$, then $\alpha \in \mathrm{ms}(b)$. Let $b > a^\alpha$. Then $\mathrm{supp}(b-a^\alpha) \subseteq \mathrm{supp}(b)$ and $\alpha \in \mathrm{supp}(b)$. In fact, if $b_\alpha = 0$, then there exists $\beta \in \mathrm{ms}(b-a^\alpha)$ such that $\alpha \leq \beta$. If $\alpha = \beta$ then $\alpha \in \mathrm{ms}(b-a^\alpha)$ and it follows that $-1 = (b = a^\alpha)_\alpha > 0$, a contradiction. Hence, $\alpha < \beta$ and $\beta \in \mathrm{supp}(b)$. Then there exists $\gamma \in \mathrm{ms}(b)$ such that $\alpha < \beta \leq \gamma$.

Conversely, let $\beta \in \mathrm{ms}(b)$ be such that $\beta \geq \alpha$. Let $\beta > \alpha$ firstly and let $\gamma \in \mathrm{ms}(b-a^\alpha)$. Let us consider the two only possible cases.

(a) $\gamma = \alpha$. Since $b_\beta > 0$ and $a^\alpha_\beta = 0$ we have $\beta \in \mathrm{supp}(b-a^\alpha)$, a contradiction with the maximality of $\gamma$.

(b) $\gamma \neq \alpha$. Then $\gamma \neq \beta$ and it follows that $\gamma \in \mathrm{ms}(b)$ as follows from the minimality of $\alpha$. Then $b_\gamma - a^\alpha_\gamma = b_\gamma > 0$. Hence, if $\beta > \alpha$, we proved that $b \geq a^\alpha$.

Now, let $\beta = \alpha$ and let $\gamma \in \mathrm{ms}(b-a^\alpha)$. Let us consider again the two only possible cases.

(a) $\gamma = \alpha$. Since $b_\alpha - 1 \neq 0$ and $b_\alpha > 0$, we have $b_\alpha \geq 2$ and it follows that $(b-a^\alpha)_\alpha > 0$.

(b) $\gamma \neq \alpha = \beta$. Then from the minimality of $\alpha$ it follows that $\gamma \in \mathrm{ms}(b)$ and we have $(b-a^\alpha)_\gamma = b_\gamma > 0$. Therefore, $b \geq a^\alpha$ in this case.

(2) Let $\alpha \in \Delta_0$ and let us assume that $b \in H(\Delta, \mathbb{Z})$ be such that $a^\alpha \geq b > 0$. Then it may be proved easily that $\mathrm{ms}(a^\alpha - b) \subseteq \{a\}$. Now, if $a^\alpha > b$, we have $\mathrm{ms}(a^\alpha - b) = \{a\}$. Let $\beta \in \mathrm{supp}(b)$. Then it follows easily that $\beta \leq \alpha$. Thus, $\beta = \alpha$, a contradiction. Therefore, $a^\alpha = b$ and $a^\alpha$ is an atom. Conversely, let $b \in H(\Delta, \mathbb{Z})_+$ be an atom. Then $\mathrm{ms}(b) \neq \emptyset$ and for $\beta \in \mathrm{ms}(b)$ there exists an atom $\alpha \in \Delta_0$ such that $\alpha \leq \beta$. From (1) it follows that $b = a^\alpha$.

(3) Let $b \in H(\Delta, \mathbb{Z})_+, b > 0$. Then $\mathrm{ms}(b)$ is a finite set and according to (2) and (1), the set $\{a \in H(\Delta, \mathbb{Z})_+: a \text{ is an atom and } a \leq b\}$ equals to the set $\{a^\alpha: \alpha \in \mathrm{ms}(b)\}$ which is nonempty and finite.

Hence, $H(\Delta, \mathbb{Z})$ is finitely atomic.

Now, using the $l$-group $H(\Delta, \mathbb{Z})$, where $\Delta$ is a finitely atomic root system, we may derive examples of po-groups with a strong theory of quasi-divisors. Let us consider the following example.

Example 3.2. Let $\Delta = \{\alpha_{nj}: n \in \mathbb{N}, j = 1, 2\}$ be a root system such that

\[
\begin{array}{ccc}
\alpha_{12} & \alpha_{22} & \cdots \alpha_{n2} \\
\alpha_{11} & \alpha_{21} & \cdots \alpha_{n1}
\end{array}
\]
Let us consider a map \( \varphi : H(\Delta, Z) \rightarrow Z \) such that
\[
\varphi(a) = \sum_{n \in \mathbb{N}, j=1,2} a_{\alpha_n}(-1)^n
\]
Then \( \varphi \) is a group homomorphism and \( H(\Delta, Z) \) is finitely atomic (see 3.1). Let \( b_1, \ldots, b_n \) be atoms in \( H(\Delta, Z)_+ \). Then \( \varphi((\bigcap_{i=1}^n (H(\Delta, Z) \setminus (b_i)_1)) = Z \). In fact, according to 3.1, we may assume that
\[
b_i(\alpha) = \begin{cases} 1, & \text{if } \alpha = \alpha_{i_1} \\ 0, & \text{otherwise} \end{cases}
\]
Let \( m \in Z \). If \( m > 0 \), then there exists \( \alpha_{i_1}, i > n \), and \( i \) is even. We set
\[
a(\alpha) = \begin{cases} m, & \text{if } \alpha = \alpha_{i_1} \\ 0, & \text{otherwise} \end{cases}
\]
Then according to 3.1, \( a \in \bigcap_{i=1}^n (H(\Delta, Z) \setminus (b_i)_1) \) and \( \varphi(a) = m(-1)^i = m \). If \( m < 0 \), then there exists \( \alpha_{i_1} \) such that \( i > n \) and \( i \) is odd. We then set
\[
a(\alpha) = \begin{cases} -m, & \text{if } \alpha = \alpha_{i_1} \\ 0, & \text{otherwise} \end{cases}
\]
Then \( a \) is from the same set as in previous case and \( \varphi(a) = (-m)(-1)^i = m \). Hence,
\[
\varphi((\bigcap_{i=1}^n (H(\Delta, Z) \setminus (b_i)_1)) = Z
\]
and for the subgroup \( G = \ker \varphi \) of \( H(\Delta, Z) \) (with ordering induced from this group) the inclusion \( G \hookrightarrow H(\Delta, Z) \) is a strong theory of quasi-divisors by 2.9.

This example may be modified in a following way.

**Example 3.3.** Let \( \Delta \) be a finitely atomic root system such that \( \text{card} (\Delta) = \aleph_0 \) and let \( \sigma : \Delta \rightarrow \aleph_0 \) be a bijection. Let \( m \in Z \) and let \( \varphi_m : Z \rightarrow \mathbb{Z}/(m) \) be a canonical homomorphism. Then we may define a group homomorphism \( \varphi : H(\Delta, Z) \rightarrow \mathbb{Z}/(m) \) such that
\[
\varphi(a) = \sum_{\alpha \in \Delta} \varphi_m(a_{\alpha})(-1)^{\sigma(\alpha)} \in \mathbb{Z}/(m).
\]
Then \( \mathbb{Z}/(m) = \varphi((\bigcap_{i=1}^n (H(\Delta, Z) \setminus (b_i)_1)) \) for any finite set \( \{b_1, \ldots, b_n\} \) of atoms in \( H(\Delta, Z) \). In fact, according to 3.1, we may assume that there exist atoms \( \alpha_1, \ldots, \alpha_n \) in \( \Delta \) such that
\[
b_k(\alpha) = \begin{cases} 1, & \text{if } \alpha = \alpha_k \\ 0, & \text{otherwise} \end{cases}
\]
Let \( \varphi_m(s) \in \mathbb{Z}/(m) \). Then we may assume that \( s \geq 0 \) and then there exists \( \alpha_0 \in \Delta_0 \setminus \{\alpha_1, \ldots, \alpha_n\} \) such that \( \sigma(\alpha) \) is even. We then set

\[
a(\alpha) = \begin{cases} 
s, & \text{if } \alpha = \alpha_0 \\
0, & \text{otherwise}
\end{cases}
\]

Then \( a \in H(\Delta, \mathbb{Z}) \) and according to 3.1, \( a \not\equiv b_k, k = 1, \ldots, n \). Moreover, \( \varphi(a) = \varphi_m(a_{\alpha_0})(-1)^{\sigma(\alpha)} = \varphi_m(s) \). Hence, \( G = \ker \varphi \hookrightarrow H(\Delta, \mathbb{Z}) \) is a strong theory of quasi-divisors.

References


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