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Abstract. The paper deals with the transcendency and algebraic independence of a special infinite series. The proofs use Lindemann's theorem and a certain combinatorial identity.

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1 Introduction

There is a lot of papers concerning the transcendency and algebraic independence of exponentials. In 1873 Hermite [4] proved the transcendency of the number e and in 1882 Lindemann [5] proved the transcendency of the number π. Up to this day we know about hundred different proofs concerning the transcendency of these two numbers. One of them we can find in [3]. In 1882 Lindemann [5] proved

Theorem A. *Let n be a natural number and $\alpha_1, \dots, \alpha_n$ $(\alpha_i = \alpha_j \Leftrightarrow i = j)$, $\delta_1, \dots, \delta_n$ ($\delta_i \neq 0$ for every $i = 1, \dots, n$) be algebraic numbers. Then*

$$\sum_{i=1}^{n} \delta_i e^{\alpha_i} \neq 0.$$

A lot of results concerning this theory we can find in the books [2] and [6]. This paper deals with the special applications of the Lindemann's theorem and proves criteria for the transcendency and algebraic independence of certain infinite series.

2 Main Theorems

Theorem 1. *Let n be a natural number, $P_s(y) = \sum_{m=0}^{N} a_{s,m} y^m$ $(s = 1, 2, \dots, r)$ be polynomials with integer coefficients and $\alpha_1, \dots, \alpha_r$ be linear independent algebraic numbers such that α_s $(s = 1, 2, \dots, r)$ isn't the root of the polynomial*

$$\sum_{j=0}^{N} x^j \sum_{m=j}^{N} a_{s,m} \sum_{i=0}^{j} (-1)^{j-i} \frac{i^m}{i!(j-i)!} \tag{1}$$

for every $s = 1, \dots, r$. Then the numbers

$$X_s = \sum_{n=1}^{\infty} \frac{P_s(n)\alpha_s^n}{n!} \tag{2}$$

are algebraically independent.

Examples. The numbers $\sum_{n=1}^{\infty}\frac{n^3}{n!}2^{n/5}$, $\sum_{n=1}^{\infty}\frac{n^6}{n!}3^{n/8}$ and $\sum_{n=1}^{\infty}\frac{n^2}{n!}5^{n/3}$ are algebraically independent.

Theorem 2. *Let α be a non-zero algebraic number and $P(x)=\sum_{m=0}^{N}a_m x^m$ be a polynomial with integer coefficients. Then the number*

$$X=\sum_{n=1}^{\infty}\frac{P(n)\alpha^n}{n!}$$

is rational iff α is the root of the polynomial

$$\sum_{j=0}^{N}x^j\sum_{m=j}^{N}a_m\sum_{i=0}^{j}(-1)^{j-i}\frac{i^m}{i!(j-i)!}.$$

Otherwise X is the transcendental number.

Theorem 2 is an immediatelly consequence of Theorem 1 for $r=1$.

Examples. The numbers $\sum_{n=1}^{\infty}\frac{n^7}{n!}7^{n/9}$, $\sum_{n=1}^{\infty}\frac{n^2}{n!}6^{n/5}$ and $\sum_{n=1}^{\infty}\frac{n^3}{n!}5^{n/4}$ are transcendental.

Theorem 3. *Let $P(y)$ be a non-zero polynomial with algebraic coefficients, $\deg P=N<q$, q be a positive integer and α be a non-zero algebraic number. Then the number*

$$X=\sum_{n=1}^{\infty}\frac{P(qn)\alpha^{qn}}{(qn)!}$$

is transcendental.

Examples. The numbers $\sum_{n=1}^{\infty}\frac{n^4}{(5n)!}2^n$, $\sum_{n=1}^{\infty}\frac{n^3}{(6n)!}3^n$ and $\sum_{n=1}^{\infty}\frac{n^5}{(7n)!}4^{7n/8}$ are transcendental.

To prove these theorems we need following three lemmas.

Lemma 1. *Let n be a natural number. Then*

$$x^m=\sum_{k=0}^{m}S_{m,k}\prod_{j=0}^{k-1}(x-j),$$

where $S_{m,k}$ is the so-called Stirling number and

$$S_{m,k}=\frac{1}{k!}\sum_{i=0}^{m}(-1)^{k-i}\binom{k}{i}i^m.$$

Proof of this lemma we can find in [1] (page 110-121 of the russian edition).

Lemma 2. *Let r be a positive integer and $\alpha_1, \ldots, \alpha_r$ be linear independent algebraic numbers. Then the numbers $e^{\alpha_1}, \ldots, e^{\alpha_r}$ are algebraically independent.*

Proof of this lemma we can find in [2] page 27.

Lemma 3. *Let n and k be two natural numbers and M_n be the set of all complex roots of the equation $x^n = 1$. Then*

$$\sum_{x \in M_n} x^k = \begin{cases} n & \text{if } k \text{ is divided by } n \\ 0 & \text{otherwise} \end{cases}$$

PROOF OF LEMMA 3: The roots of the equation $x^n = 1$ we can write in the form $1, e^{2i\pi/n}, e^{4i\pi/n}, \ldots, e^{2(n-1)i\pi/n}$. If k is divided by n, then we have

$$\sum_{x \in M_n} x^k = \sum_{j=0}^{n-1} (e^{2ji\pi/n})^k = \sum_{j=0}^{n-1} 1 = n.$$

If k is not divided by n, then the roots create the geometric sequence and

$$\sum_{x \in M_n} x^k = \sum_{j=0}^{n-1} (e^{2ij\pi/n})^k = \sum_{j=0}^{n-1} e^{2ijk\pi/n} = \frac{e^{2ink\pi/n} - 1}{e^{2ik\pi/n} - 1} = 0.$$

□

PROOF OF THEOREM 1: Using Lemma 1 we can write

$$\begin{aligned} P_s(n) &= \sum_{m=0}^{N} a_{s,m} n^m = \sum_{m=0}^{N} a_{s,m} \sum_{k=0}^{m} S_{m,k} \prod_{j=0}^{k-1} (n-j) = \\ &= \sum_{k=0}^{N} \prod_{j=0}^{k-1} (n-j) \sum_{m=k}^{N} a_{s,m} S_{m,k}, \end{aligned}$$

where $\prod_{x \in \emptyset} F(x) = 1$. This, Lemma 1, (1) and (2) follows

$$\begin{aligned} X_s &= \sum_{n=1}^{\infty} \frac{P_s(n)\alpha_s^n}{n!} = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{k=0}^{N} \prod_{j=0}^{k-1} (n-j) \sum_{m=k}^{N} a_{s,m} S_{m,k} \alpha_s^n = \\ &= \sum_{k=0}^{N} \sum_{m=k}^{N} a_{s,m} S_{m,k} \sum_{n=1}^{\infty} \frac{\prod_{j=0}^{k-1} (n-j) \alpha_s^n}{n!} = \\ &= \beta_s + \sum_{k=0}^{N} \sum_{m=k}^{N} a_{s,m} S_{m,k} \sum_{n=0}^{\infty} \frac{\alpha_s^{n+k}}{n!} = \\ &= \beta_s + \sum_{k=0}^{N} \sum_{m=k}^{N} a_{s,m} S_{m,k} \alpha_s^k e^{\alpha_s} = \\ &= \beta_s + e^{\alpha_s} \sum_{k=0}^{N} \alpha_s^k \sum_{m=k}^{N} a_{s,m} \frac{1}{k!} \sum_{i=0}^{m} (-1)^{k-i} \binom{k}{i} i^m, \end{aligned}$$

where β_s is a suitable algebraic number. If α_s is the root of the polynomial (1), then $X_s = \beta_s$ is an algebraic number and the numbers X_s $(s = 1, \dots, r)$ are algebraically dependent. If not, then we can write

$$X_s = \beta_s + \gamma_s e^{\alpha_s},$$

where γ_s is a suitable nonzero algebraic number too. This and lemmma 2 implies that the numbers X_s $(s = 1, \dots, r)$ are algebraically independent. □

PROOF OF THEOREM 3: Similary like in the proof of Theorem 1 we have

$$\begin{aligned} P(n) &= \sum_{m=0}^{N} a_m n^m = \sum_{m=0}^{N} a_m \sum_{k=0}^{m} S_{m,k} \prod_{j=0}^{k-1} (n-j) = \\ &= \sum_{k=0}^{N} \prod_{j=0}^{k-1} (n-j) \sum_{m=k}^{N} a_m S_{m,k}, \end{aligned}$$

where a_m $(m = 0, \dots, N)$ are algebraic numbers, $a_N \neq 0$. This, lemma 1 and lemma 3 implies

$$\begin{aligned} X &= \sum_{q/n} \frac{P(n)\alpha^n}{n!} = \sum_{q/n} \frac{1}{n!} \sum_{k=0}^{N} \prod_{j=0}^{k-1} (n-j) \sum_{m=k}^{N} a_m S_{m,k} \alpha^n = \\ &= \sum_{k=0}^{N} \sum_{m=k}^{N} a_m S_{m,k} \sum_{q/n} \frac{1}{n!} \prod_{j=0}^{k-1} (n-j) \alpha^n = \\ &= \sum_{k=0}^{N} \sum_{m=k}^{N} a_m S_{m,k} \alpha^k \sum_{q/n} \frac{\alpha^{n-k}}{(n-k)!} = \\ &= \sum_{k=0}^{N} \sum_{m=k}^{N} a_m S_{m,k} \alpha^k \sum_{n=1}^{\infty} \frac{\alpha^{nq-k}}{(nq-k)!} = \\ &= \frac{1}{q} \sum_{k=0}^{N} \sum_{m=k}^{N} a_m S_{m,k} \alpha^k \sum_{j=0}^{q-1} e^{2ijk\pi/q} e^{\alpha e^{2ij\pi/q}} = \\ &= \sum_{j=0}^{q-1} \frac{1}{q} \sum_{k=0}^{N} \sum_{m=k}^{N} a_m S_{m,k} \alpha^k e^{2ijk\pi/q} e^{\alpha e^{2ij\pi/q}}. \end{aligned}$$

Let us denote

$$\delta_j = \frac{1}{q} \sum_{k=0}^{N} \sum_{m=k}^{N} a_m S_{m,k} \alpha^k e^{2ijk\pi/q}$$

for every $j = 0, \dots, q-1$. Then we have

$$X = \sum_{j=0}^{q-1} \delta_j e^{\alpha e^{2ij\pi/q}}. \tag{3}$$

Now we prove that there is $j \in \{0, \ldots, q-1\}$ such that $\delta_j \neq 0$. Because of $S_{N,N} \neq 0$, the polynomial

$$\delta(x) = \frac{1}{q} \sum_{k=0}^{N} \sum_{m=k}^{N} a_m S_{m,k} x^k$$

has degree N. The number of elements of the set $\{\alpha e^{2ij\pi/q}, j = 0, \ldots, q-1\}$ is equal $q > N$. Thus there is $J \in \{0, .., q-1\}$ such that $\delta_J = \delta(\alpha e^{2iJ\pi/q}) \neq 0$. Finally this, (3) and Lindemann's Theorem implies, that the number X is transcendental. □

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