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On some modifications of two theorems of Erdős


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Abstract: If certain sums of two completely additive functions are constant or convergent, then the functions are some constant multiples of the logarithm function.

Key Words: characterization of additive functions

Mathematics Subject Classification: 11A25

In 1946 Erdős [2] proved the following theorems:

Theorem 1 (Erdős). Let $f$ be a real valued additive function. If $f(n+1) - f(n) \to 0$, then $f(n) = c \log n$ for all $n \in \mathbb{N}$.

Theorem 2 (Erdős). If a real valued additive function $f$ is monotonically increasing, then $f(n) = c \log n$.

I. Kátai [3] generalized Theorem 1 for completely additive functions using a result of E. Wirsing [6]:

Theorem 3 (Kátai). Let $f$ be a completely additive function. If $\sum_{i=1}^{m} c_i f(n + a_i) = o(\log n)$, then $f(n) = c \log n$ for all $n \in \mathbb{N}$.

P.D.T.A. Elliott [1] and the author ([4],[5]) found the following further generalizations:

Theorem 4 (Elliott, [1]). Let $f$ be an additive function, $A > 0, C > 0, B, D$ integers and $\Delta_1 = AC(AD - BC) \neq 0$. If $f(An + B) - f(Cn + D) \to c$, then $f(n) = c_1 \log n$ for all $(n, \Delta_1)$.

Theorem 5 [4]. Let $f$ be a completely additive function, $A > 0, C > 0, B, D$ integers and $\Delta_2 = AC(A + 1)(C + 1)(AD - BC) \neq 0$. If $f(An + B) + f(Cn + D) \to c$, then $f(n) = 0$ for all $(n, \Delta_2)$.

Theorem 6 [5]. Let $f$ be a completely additive function. If

$$f(2n + A) - f(n)$$

is monotonic from some number on, then $f(n) = c \log n$ with some $c \geq 0$ for all $n \in \mathbb{N}$.

In this paper I prove the following generalizations of Theorem 5 and Theorem 6:

Partially supported by the Hungarian National Foundation for Scientific Research Grant No. T 017433.
Theorem 7. Let $A > 0, C > 0, B, D$ be integers and $\epsilon \in \{1, -1\}$. If

\[(1) \quad f_1(An + B) + f_1(Cn + D) + f_2(n) \to c,\]

for the completely additive functions $f_1$ and $f_2$, then $f_i(n) = c_i \log n$ or $f_i(n) = 0$ ($i \in 1, 2$) for all $n$ coprime to $\Delta_3 = ABCD(C^2B^2 - A^2D^2)(A^2D + 1)(C^2B + 1)$.

Theorem 8. Let $A > 1, B > 0$ be integers, $\epsilon \in \{1, -1\}$ and $\alpha \in C \setminus \{0, -2\}$. If

\[(2) \quad f(An + B) + f(An - B) + \alpha f(n) \to c\]

for a completely additive function $f$, then $f(n) = 0$ for all $n \in N$.

Theorem 9. Let $f_1, f_2$ denote completely additive arithmetical functions and $\epsilon \in \{1, -1\}$. If one of the conditions

\[(3) \quad f_1(n + 2k + \epsilon) - f_1(n + 2k) + f_2(n + \epsilon) - f_2(n) = o(\log n),\]

\[(4) \quad f_1(n + 2k + \epsilon) - f_1(n + 2k) + f_2(n) - f_2(n - \epsilon) = c,\]

\[(5) \quad f_1(n + 2) - f_1(n - 1) + f_2(n - 1) - f_2(n) = c,\]

\[(6) \quad f_1(n) - f_1(n - 3) + f_2(n - 1) - f_2(n) = c,\]

\[(7) \quad f_1(n + 3) - f_1(n) + f_2(n - 1) - f_2(n) = c\]

is satisfied, then $f_i(n) = C_i \log n$ ($i = 1, 2$) for all $n \in N$.

Proofs

Proof of Theorem 7. We may assume, that $B$ and $C$ are positive. (Otherwise we replace $n$ by $n + s$ in (1) with a number $s$ big enough such that $B' = B + sA > 0$ and $D' = D + sA > 0$.) We substitute $n$ by $CBn$ and $ADn$ in (1), resp. Therefore

\[(8) \quad f_1(ACn + 1) + \epsilon f_1(C^2Bn + D) + f_2(n) \to c_1\]

and

\[(9) \quad f_1(A^2Dn + B) + \epsilon f_1(ACn + 1) + f_2(n) \to c_2.\]

The difference of (9) and (8) shows that

$$f_1(C^2Bn + D) - \epsilon f_1(A^2Dn + B) \to c_3.$$

Finally we apply Theorem 4 and Theorem 5, resp.

Proof of Theorem 8. We replace $n$ by $Bn$ in (2). So we have

\[(10) \quad f(An + 1) + f(An - 1) \to c.\]

Now we substitute $n$ by $An, (A + 1)n, A(A + 1)n, A(A + 1)n + 1$ and $A(A + 1)n - 1$ in (10). Therefore we obtain the following assertions:

\[(11) \quad f(A^2n + 1) + f(A^2n - 1) + \alpha f(n) \to c_1,\]

\[(12) \quad f(A(A + 1)n + 1) + f(A(A + 1)n - 1) + \alpha f(n) \to c_2,\]

\[(13) \quad f(A^2(A + 1)n + 1) + f(A^2(A + 1)n - 1) + \alpha f(n) \to c_3,\]
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\( f(A^2n + 1) + f(A^2(A + 1)n + A - 1) + \alpha f(A(A + 1)n + 1) \rightarrow c_4, \)

\( f(A^2(A + 1)n - A + 1) + f(A^2n - 1) + \alpha f(A(A + 1)n - 1) \rightarrow c_5. \)

By the linear combination of the equations (14)+(15)-(11)-(12) we have

\( f(A^2(A + 1)n - A + 1) + f(A^2(A + 1)n + A - 1) - (\alpha^2 + \alpha)f(n) \rightarrow c_7. \)

Then we replace \( n \) by \( (A - 1)n \) in this formula, which yields that

\( f(A^2(A + 1)n - 1) + f(A^2(A + 1)n + 1) - (\alpha^2 + \alpha)f(n) \rightarrow c_8. \)

The difference of (16) and (13) shows that \((\alpha^2 + 2\alpha)f(n) \rightarrow c_8, \) i.e. \( f = 0 \) if \( \alpha \not\in \{0, -2\}. \)

**Proof of Theorem 9.**

**Case 1.** Replacing \( n \) by \( n + \epsilon \) in (3) we have

\( f_1(n + 2k + 2\epsilon) - f_1(n + 2k + \epsilon) + f_2(n + 2\epsilon) - f_2(n + \epsilon) = o(\log n). \)

The sum of (3) and (17) yields that

\( f_1(n + 2k + 2\epsilon) - f_1(n + 2k + \epsilon) + f_2(n + 2\epsilon) - f_2(n) = o(\log n). \)

Replacing \( n \) by \( 2n \) in (18) we get that

\( f_1(n + k + \epsilon) - f_1(n + k) + f_2(n + \epsilon) - f_2(n) = o(\log n). \)

The difference of (19) and (3) shows that

\( f_1(n + 2k + \epsilon) - f_1(n + k + \epsilon) - f_1(n + k) + f_1(n + 2k) = o(\log n). \)

By Theorem 3 we have that \( f_1(n) = c_1 \log n \) or \( f_1(n) = 0. \) We substitute this result in (3) to obtain \( f_2(n) = c_2 \log n. \)

**Case 2.** We replace \( n \) by \( n + \epsilon \) in (4). Therefore

\( f_1(n + 2k + 2\epsilon) - f_1(n + 2k + \epsilon) + f_2(n + \epsilon) - f_2(n) = c. \)

The sum of (4) and (20) yields that

\( f_1(n + 2k + 2\epsilon) - f_1(n + 2k) + f_2(n + \epsilon) - f_2(n - \epsilon) = 2c. \)

We replace \( n \) by \( n - k \) in (4) and by \( 2n \) in (21). So we have

\( f_1(n + k + \epsilon) - f_1(n + k) + f_2(n - k) - f_2(n - \epsilon) = c'. \)

\( f_1(n + k + \epsilon) - f_1(n + k) + f_2(2n + \epsilon) - f_2(2n - \epsilon) = c''. \)

The difference of (23) and (22) shows that

\( f_2(2n + \epsilon) - f_2(n - k) = f_2(2n - \epsilon) - f_2(n - k - \epsilon) + c'''(\epsilon = 1 \text{or} -1), \)

i.e. \( f_2(2n + 2k + \epsilon) - f_2(n) \) is monotonic. Finally we apply Theorem 6.

**Case 3.** We replace \( n \) by \( n + 1 \) in (5), so we have

\( f_1(n + 3) - f_1(n) + f_2(n) - f_2(n + 1) = c. \)
We substitute $n$ by $2n + 1$ in the sum of (5) and (24), i.e.

$$f_1(n + 2) - f_1(n - 1) + f_1(n + 3) - f_1(n) + f_2(n - 1) - f_2(n + 1) = 2c,$$

which follows

$$(25) \quad f_1(2n + 3) - f_1(n + 3) = f_1(2n + 1) - f_1(n + 2) + c,$$

i.e. $f_1(2n + 3) - f_1(n)$ is monotonic. Finally we apply Theorem 6.

The proof of the remaining two cases is very similar.

References


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Received: February 20, 1998