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Fermat and Wilson Quotients for p-Adic Integers

Ladislav Skula

Abstract: Using the p-adic limit, the notions of Fermat and Wilson quotients for composite moduli are transferred to those for p-adic integers. Some theorems on these quotients are presented which in particular are analogous to results of Eisenstein, Lerch, Friedmann and Tamarkin.

Key Words: Fermat quotient, Euler quotient, Wilson quotient, p-adic numbers.

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1. Introduction

Let $p$ be a prime and $a$ an integer not divisible by $p$. As it is well-known the Fermat quotient of $p$ with base $a$ is the integer

$$ q(p,a) = \frac{a^{p-1} - 1}{p} $$

The first general statements on this quotient are due to Eisenstein ([E], 1850):

(E1) If $p$ is odd, then

$$ 2g(2,p) = \prod_{n=1}^{p-1} (\sim 1) \equiv 1 \mod p. $$

(E2) If $u,v$ are integers and $p \nmid uv$, then

$$ q(uv,p) = q(u,p) + q(v,p) \mod p $$

(the "logarithmic property").

(E3) If $u,v \in \mathbb{Z}$ and $p \nmid u$, then

$$ q(u + pv,p) = q(u,p) \frac{v}{u} \mod p. $$
As a corollary of (E3) we have for integers \( a, b, p \nmid ab \):

\[
a \equiv b \pmod{p^2} \implies q(a, p) \equiv q(b, p) \pmod{p}.
\]

Thus by (E2) we can consider the function \( q(\cdot, p) \) as a homomorphism from the multiplicative group \( ((\mathbb{Z}/p^2\mathbb{Z})^*, \cdot) \) into the additive group \( (\mathbb{Z}/p\mathbb{Z}, +) \) of the respective residue class rings.

According to Euler's well-known theorem generalizing Fermat's little theorem we can define for relatively prime integers \( m \geq 2 \) and \( a \) the **Euler quotient** (or the **generalized Fermat quotient for composite moduli**) of \( m \) with base \( a \) by

\[
q(a, m) = \frac{a^{\varphi(m)} - 1}{m}.
\]

For this quotient similar laws are satisfied as (E1) - (E3). In [ADS1] the Fermat quotient for composite moduli \( m \) is investigated in more detail. Some formulas presented there for the case \( m = p^n \) directly invite to use a limit process and to transfer this notion to the \( p \)-adic case. This is established in Section 3 in greater detail by means of the projective limit. In Section 4 Lerch's expression of the Fermat quotient is transferred to the \( p \)-adic case and in Section 6 the Friedmann-Tamarkine congruence is presented for the Fermat quotient for \( p \)-adic integers.

Similarly, the notion of the Wilson quotient is transferred to the \( p \)-adic case by means of the \( p \)-adic limit in Section 5. Here a theorem (Theorem 5.7) is derived presenting this “\( p \)-adic" Wilson quotient by means of the \( p \)-adic limit of expressions containing certain Bernoulli numbers.

The reader is referred for the basic facts on \( p \)-adic numbers to the book [BS] and for the theory of projective systems to the book [K].

### 2. Notations and Fundamental Assertions

Throughout this paper we will use the following notations:

- \( p \) a prime,
- \( n \) a positive integer,
- \( \mathbb{Z} \) the ring of (rational) integers,
- \( \mathbb{Z}(n) \) the additive group of the ring of residue classes mod \( p^n \), thus \( \mathbb{Z}(n) = (\mathbb{Z}/p^n\mathbb{Z}, +) \),
- \( \mathbb{Z}(n)^* \) the multiplicative group of the invertible elements of the ring of residue classes mod \( p^n \), thus \( \mathbb{Z}(n)^* = ((\mathbb{Z}/p^n\mathbb{Z})^*, \cdot) \),
- \( \varphi_n \) the canonical (ring) homomorphism from the ring \( (\mathbb{Z}/p^{n+1}\mathbb{Z}, +, \cdot) \) onto the ring \( (\mathbb{Z}/p^n\mathbb{Z}, +, \cdot) \); this homomorphism will be also considered as (group) homomorphism from the group \( \mathbb{Z}(n+1) \) onto the group \( \mathbb{Z}(n) \) or from the group \( \mathbb{Z}(n+1)^* \) onto the group \( \mathbb{Z}(n)^* \),
- \( (\mathbb{Z}_p, +, \cdot) \) the ring of \( p \)-adic integers with \( p \)-adic topology,
- \( \mathbb{Z}_p \) the additive group of the ring \( (\mathbb{Z}_p, +, \cdot) \),
- \( \mathbb{Z}_p^* \) the multiplicative group of the invertible elements of the ring \( (\mathbb{Z}_p, +, \cdot) \),
- \( \psi_n \) the canonical (ring) homomorphism from the ring \( (\mathbb{Z}_p, +, \cdot) \) onto the ring \( (\mathbb{Z}/p^n\mathbb{Z}, +, \cdot) \), also considered as the group homomorphism from the group \( \mathbb{Z}(n)^* \) into the multiplicative group \( \mathbb{Z}(n) \) of the ring of residue classes mod \( p^n \).
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$\mathbb{Z}_p$ onto the group $\mathbb{Z}(n)$ or from the group $\mathbb{Z}_p^*$ onto the group $\mathbb{Z}(n)^*$, thus for

$$\alpha = \sum_{i=0}^{\infty} a_i p^i \in \mathbb{Z}_p \ (a_i \in \mathbb{Z}, \ 0 \leq a_i < p)$$

we have $\psi_n(\alpha) = \sum_{i=0}^{n-1} a_i p^i p^n \mathbb{Z}$,

$v(\alpha)$ the $p$-adic exponent of a $p$-adic integer $\alpha$,

$$\lim_{k \to \infty} \alpha_k$$

the $p$-adic limit for $p$-adic integers $\alpha_k$,

similarly all topological notions (continuity, convergence, infinite series, etc.) concern the $p$-adic topology,

$q(\alpha, p^n)$ the Fermat quotient of (composite moduli) $p^n$ (the Euler quotient of $p^n$) with base $\alpha$ ($\alpha \in \mathbb{Z}, p \nmid a$) (see, e.g. [ADS1]), thus

$$q(\alpha, p^n) = \frac{a^{p^{n-1}(p-1)} - 1}{p^n}.$$ 

Since for each $p$-adic integer $\alpha$ not divisible by $p$ we have

$$\alpha^{p^{n-1}(p-1)} \equiv 1 \pmod{p^n},$$

the $p$-adic number $q(\alpha, p^n) = \frac{a^{p^{n-1}(p-1)} - 1}{p^n}$ is $p$-adic integer. In this way the former function $q(\ , p^n)$ is extended to all $p$-adic integers not divisible by $p$.

**Proposition 2.1.** (a) The function $q(\ , p^n)$ is a uniformly continuous mapping from $\mathbb{Z}^*$ into $\mathbb{Z}_p$.

If we assume that $\alpha, \beta \in \mathbb{Z}_p^*$, then we have:

(b) $q(\alpha, p^n) \equiv q(\beta, p^n) \pmod{p^n}$

provided that $\alpha \equiv \beta \pmod{p^{n+1}},$

(c) $q(\alpha \beta, p^n) \equiv q(\alpha, p^n) + q(\beta, p^n) \pmod{p^n}.$

**Proof.** For $\alpha, \beta \in \mathbb{Z}_p^*$ there exists $\gamma \in \mathbb{Z}_p$ such that $\alpha^{p^{n-1}(p-1)} - \beta^{p^{n-1}(p-1)} = (\alpha - \beta)\gamma$, hence

$$v(q(\alpha, p^n) - q(\beta, p^n)) \geq v(\alpha - \beta) - n.$$ 

This proves part (a). Part (b) is obvious and part (c) follows from (b) and the logarithmic property for the Fermat quotient of $p^n$.

**Notation.** Let $A = a + p^{n+1} \mathbb{Z} \in \mathbb{Z}(n+1)^*$, $a \in \mathbb{Z}, p \nmid a$. Put $q_n(A) = q(a, p^n) + p^n \mathbb{Z} \in \mathbb{Z}(n)$. Using Proposition 2.1 (b), (c) we get that $q_n$ is a group homomorphism from the group $\mathbb{Z}(n+1)^*$ into the group $\mathbb{Z}(n)$.

**Proposition 2.2.** (a) Let $p$ be an odd prime or $p = 2$ and $n = 1$. Then $q_n$ is surjective and for $A \in \mathbb{Z}(n+1)^*$ we have $q_n(A) = 0$ if and only if $A^{p^{-1}} = 1$.

(b) Let $p = 2$ and $n \geq 2$. Then $q_n(\mathbb{Z}(n+1)^*) = 2\mathbb{Z}(n)$ and for $A \in \mathbb{Z}(n+1)^*$ we have $q_n(A) = 0$ if and only if $A = \pm 1$.

(The symbols $0$ and $1$ denote the zero element and the unity in the rings of residue classes mod $p^n$ and mod $p^{n+1}$, respectively.)

**Proof.** In case (a) we have, by property (E3), $q(1 + p, p^n) \equiv -1 \pmod{p^n}$, hence $p \nmid q(1 + p, p^n)$. Using the logarithmic property of the Fermat quotient of $p^n$ and the existence of a primitive root mod $p^n$ we get that $q_n$ is surjective and $q_n(A) = 0$ for an element $A \in \mathbb{Z}(n+1)^*$ if and only if $A^{p^{-1}} = 1$.

For $n \geq 2$ we have $q(5, 2^n) = 2k$ for an odd integer $k$. Let $a \in \mathbb{Z}, 2 \nmid a$. Then there exists an integer $x$ such that $0 \leq x \leq 2^{n-1} - 1$ and $a \equiv \pm 5^x \pmod{2^{n+1}}$, which implies $q(a, 2^n) \equiv xq(5, 2^n) \pmod{2^n} = 2kx$, and we are done.
Proposition 2.3. With exception of the case \( p = 2 \) and \( n = 1 \) we have
\[
q(\alpha, p^{n+1}) \equiv q(\alpha, p^n) \pmod{p^n}
\]
for each \( \alpha \in \mathbb{Z}_p^* \).

Proof. Since \( 2 | q(\alpha, 2^n) \) for each odd \( \alpha \) and \( n \geq 2 \), we get the proposition from Proposition 4.1 of ([ADS1]) for \( \alpha \in \mathbb{Z} (p \nmid \alpha) \). Using Proposition 2.1 (b) we obtain the general case. \( \square \)

Immediately from this proposition we get:

Proposition 2.4. If \( p \) is odd or \( p = 2 \) and \( n \geq 2 \), then the following diagram is commutative:

\[
\begin{array}{ccc}
Z(n+1)^* & \overset{q_n}{\longrightarrow} & Z(n) \\
\varphi_{n+1} \downarrow & & \downarrow \varphi_n \\
Z(n+2)^* & \overset{q_{n+1}}{\longrightarrow} & Z(n+1)
\end{array}
\]

3. Fermat Quotient for \( p \)-Adic Integers

Let \( \mathcal{I} \) be the set of all positive integers in the case where \( p \) is odd and the set of all integers \( \geq 2 \) in the case \( p = 2 \). For \( m, n \in \mathcal{I}, m \geq n \) denote by \( \varphi_n^m \) the (group) homomorphism \( \varphi_n^m = \varphi_n \circ \varphi_{n+1} \circ \cdots \circ \varphi_m \). Then \( \{\mathcal{I}, \mathbb{Z}(n), \varphi_n^m\} \) is a projective system whose projective limit is given by the family of (group) homomorphisms \( \{\varphi_n : \mathbb{Z}_p \rightarrow \mathbb{Z}(n) | n \in \mathcal{I}\} \). According to Proposition 2.4 the following diagram is commutative for each \( n \in \mathcal{I} \):

\[
\begin{array}{ccc}
\mathbb{Z}_p^* & \overset{q_n \circ \psi_{n+1}}{\longrightarrow} & \mathbb{Z}(n) \\
\varphi_n \downarrow & & \downarrow \varphi_n \\
\mathbb{Z}(n+1) & \overset{q_{n+1} \circ \psi_{n+2}}{\longrightarrow} & \mathbb{Z}(n+1)
\end{array}
\]

Using the properties of the projective limit we can state:

Theorem 3.1. There exists a unique continuous homomorphism \( q \) from the group \( \mathbb{Z}_p^* \) into the group \( \mathbb{Z}_p \) such that the following diagram is commutative for each \( n \in \mathcal{I} \):

\[
\begin{array}{ccc}
\mathbb{Z}(n+1)^* & \overset{q_n}{\longrightarrow} & \mathbb{Z}(n) \\
\psi_{n+1} \downarrow & & \downarrow \psi_n \\
\mathbb{Z}_p^* & \overset{q}{\longrightarrow} & \mathbb{Z}_p
\end{array}
\]

Definition 3.1 The mapping \( q \) in Theorem 3.1 will be called the Fermat quotient for the \( p \)-adic integers, or simply \( p \)-adic Fermat quotient.

Using the definition of \( q \) and Proposition 2.2 we can derive the following theorem:
Theorem 3.2. (a) For \( p \neq 2 \) the mapping \( q \) is surjective and for \( \alpha \in \mathbb{Z}_p^* \) we have \( q(\alpha) = 0 \) if and only if \( \alpha^{p-1} = 1 \).

(b) If \( p = 2 \), then \( q(\mathbb{Z}_2^*) = 2\mathbb{Z}_2 \) and for \( \alpha \in \mathbb{Z}_2^* \) we have \( q(\alpha) = 0 \) if and only if \( \alpha = \pm 1 \).

Theorem 3.3. (a) We have for each \( \alpha \in \mathbb{Z}_p^* \) and each \( n \in I \)

\[ q(\alpha) \equiv q(\alpha, p^n) \pmod{p^n}. \]

(b) The sequence of mappings \( \{q(\alpha, p^n)\}_{n=1}^{\infty} \) converges uniformly to the mapping \( q \).

(c) The mapping \( q \) is uniformly continuous.

Proof. According to Proposition 2.3 there exists \( \lim_{\nu \to \infty} q(\alpha, p^\nu) \) for each \( \alpha \in \mathbb{Z}_p^* \), which will be denoted by \( f(\alpha) \). By Proposition 2.1 (c) \( f \) is a homomorphism from the group \( \mathbb{Z}_p^* \) into the group \( \mathbb{Z}_p \).

Assume that \( n \in I \) and \( \alpha \in \mathbb{Z}_p^* \). Then there exists an integer \( m \geq n \) such that \( v(f(\alpha) - q(\alpha, p^m)) \geq n \). Using Proposition 2.3 we get

\[ v(f(\alpha) - q(\alpha, p^n)) \geq \min\{v(f(\alpha) - q(\alpha, p^m)), v(q(\alpha, p^m) - q(\alpha, p^n))\} \geq n, \]

from which we obtain that the sequence \( \{q(\alpha, p^n)\}_{n=1}^{\infty} \) converges uniformly to \( f \) and \( f(\alpha) \equiv q(\alpha, p^n) \pmod{p^n} \).

Since the \( p \)-adic valuation \( v \) is non-Archimedean ([BS], Chapt. 1, Sec. 4, Ex. 4) and \( q(\alpha, p^n) \) are uniformly continuous (Proposition 2.1 (a)), the mapping \( f \) is uniformly continuous.

It is easy to see (Proposition 2.1 (b)) that for each \( n \in I \) the following diagram is commutative:

\[
\begin{array}{ccc}
\mathbb{Z}(n+1)^* & \xrightarrow{q_n} & \mathbb{Z}(n) \\
\uparrow \psi_{n+1} & & \uparrow \psi_n \\
\mathbb{Z}_p^* & \xrightarrow{f} & \mathbb{Z}_p
\end{array}
\]

The result follows from the uniqueness of \( q \).

In the following theorem we use the symbol \( \log \) for the \( p \)-adic logarithm and we apply Leopoldt's formula ([Lp],(0))

\[ \log H = \lim_{n \to \infty} \frac{H p^n - 1}{p^n} \]

to the \( p \)-adic integer \( H = \alpha^{p-1} \), where \( \alpha \in \mathbb{Z}_p^* \):

Theorem 3.4. If \( \alpha \in \mathbb{Z}_p^* \), then

\[ q(\alpha) = \frac{\log \alpha^{p-1}}{p}. \]
4. Lerch's Expression for the Fermat Quotient

In his paper [Lr1] in 1905, Lerch presented the following expression for the Fermat quotient of an odd prime with base \( a \) (\( a \in \mathbb{Z}, p \nmid a \)):

\[
aq(a, p) = \sum_{x=1}^{p-1} \frac{ax}{p} \pmod{p}.
\]

This form was generalized by Lerch in [Lr2] (1906) for Fermat quotients of composite moduli \( m \) (\( m \in \mathbb{Z}, m \geq 2 \)) for base \( a \) (\( a \in \mathbb{Z}, (m, a) = 1 \)):

\[
aq(a, m) = a \frac{a^{e(m)} - 1}{m} \equiv \sum_{x=1}^{m} \frac{ax}{m} \pmod{m}
\]

(see [ADS1], Theorem 2.3 and Historical remarks, p.34).

To state an analogous formula for the Fermat quotient \( q \) we will define for a \( p \)-adic number \( \xi = \sum_{i=-\infty}^{\infty} x_ip^i \) (\( x_i \in \mathbb{Z}, 0 \leq x_i \leq p-1, m \in \mathbb{Z}, m \geq 0 \)) the integral part \([\xi]_p\) of \( \xi \) with respect to \( p \) by

\[
[\xi]_p = \sum_{i=0}^{\infty} x_ip^i \in \mathbb{Z}_p.
\]

Clearly, if \( \omega \in \mathbb{Z} \), then \([\frac{\omega}{p^n}]_p = [\frac{\omega}{p^n}]_p\).

**Theorem 4.1.** If \( \alpha \in \mathbb{Z}_p^* \), then

\[
aq(\alpha) = \lim_{n \to \infty} \sum_{x=1}^{p^n} \frac{1}{x} \left[ \frac{\alpha x}{p^n} \right]_p.
\]

**Proof.** Assume that \( \alpha \in \mathbb{Z}_p^*, \beta \in \mathbb{Z} \) and \( \alpha \equiv \beta \pmod{p^{2n}} \). Using Proposition 2.1 (b) and (L2) we get

\[
aq(\alpha, p^n) \equiv \sum_{x=1}^{p^n} \frac{1}{x} \left[ \frac{\beta x}{p^n} \right]_p \pmod{p^n}.
\]

Since \( \alpha x \equiv \beta x \pmod{p^{2n}} \) for each rational integer \( x \), there exists \( \gamma = \gamma(x) \in \mathbb{Z}_p \) such that \( \frac{\alpha x}{p^n} = \frac{\beta x}{p^n} + p^n \gamma \), therefore \( \left[ \frac{\alpha x}{p^n} \right]_p \equiv \left[ \frac{\beta x}{p^n} \right]_p \pmod{p^n} \) and

\[
aq(\alpha, p^n) \equiv \sum_{x=1}^{p^n} \frac{1}{x} \left[ \frac{\alpha x}{p^n} \right]_p \pmod{p^n}.
\]

The result follows. \( \square \)
Notation. For integers $N, k$ ($N \geq 1, p \nmid N, 0 \leq k \leq N - 1$), put
\[
s(k, N, n) = \sum_{\frac{p^k}{p^{k+1}} < x < \frac{p^k}{p^l}} \frac{1}{x}.
\]
Then Lerch's formula (L2) for Fermat quotient of $p^n$ for base $N$ can be expressed in the following way:
\[
Nq(N, p^n) \equiv \sum_{k=0}^{N-1} ks(k, N, n) \pmod{p^n}.
\]
Thus we can state:

**Theorem 4.2.** If $N$ is a positive integer ($p \nmid N$), then
\[
Nq(N) = \lim_{\nu \to \infty} \sum_{k=0}^{N-1} ks(k, N, \nu).
\]

**Corollary 4.3.** Let $N \in \{1, 2, 3, 4, 6\}, 0 \leq k \leq N - 1$ ($k \in \mathbb{Z}$). Then there exists $\lim s(k, N, \nu) = s(k, N)$ and we have
\[(a) \ s(0, 1) = 0,
(b) \ s(1, 2) = -s(0, 2) = 2q(2),
\ s(3, 4) = -s(0, 4) = 3q(2),
\ s(1, 4) = -s(2, 4) = q(2), \text{ for } p \neq 2,
(c) \ s(2, 3) = -s(0, 3) = \frac{3}{2}q(3),
\ s(1, 3) = 0, \text{ for } p \neq 3,
(d) \ s(5, 6) = -s(0, 6) = 2q(2) + \frac{3}{2}q(3),
\ s(1, 6) = -s(4, 6) = 2q(2),
\ s(3, 6) = -s(2, 6) = 2q(2) - \frac{3}{2}q(3), \text{ for } p \geq 5.
\]

**Proof.** The result follows from the congruence
\[
s(k, N, n) \equiv -s(N - 1 - k, N, n) \pmod{p^n}
\]
and from Theorem 4.2.

**Remark.** For $N = 5$ or $N \geq 7$ ($N \in \mathbb{Z}$) the question which sequences
\[
\{s(k, N, \nu)\}_{\nu=1}^{\infty}
\]
are convergent ($0 \leq k \leq N - 1$) remains an open problem.

**Lemma 4.4.** Let $N$ be a positive integer, $p \nmid N$ and suppose there exists
\[
\lim_{n \to \infty} s(0, N, n) = \sigma.
\]
Then

$$\sum_{\nu=0}^{1} \left( \sum_{\frac{p\nu}{N} < \frac{Np^\nu + 1}{Np} \nu < \frac{p\nu + 1}{p}} \frac{1}{x} \right) = \sigma.$$ 

Proof. Put

$$\sigma_{\nu} = \sum_{\frac{p\nu}{N} < \frac{Np^\nu + 1}{Np} \nu < \frac{p\nu + 1}{p}} \frac{1}{x}$$

for each non-negative integer \(\nu\). Then the \(n\)th partial sum of the series \(\sum_{\nu=0}^{1} \sigma_{\nu}\) equals

$$\sum_{\nu=0}^{n-1} \sigma_{\nu} = \sum_{1 \leq x < \frac{N}{p}} \frac{1}{x} = s(0, N, n)$$

and the result follows. \(\square\)

**Corollary 4.5.**

(a) If \(p \neq 2\), then

$$2q(2) = -\sum_{\nu=0}^{\infty} \left( \sum_{\frac{p\nu}{2} < \frac{p^\nu + 1}{2} \nu < \frac{p\nu + 1}{2}} \frac{1}{x} \right), \quad 3q(2) = -\sum_{\nu=0}^{\infty} \left( \sum_{\frac{p\nu}{4} < \frac{p^\nu + 1}{4} \nu < \frac{p\nu + 1}{4}} \frac{1}{x} \right).$$

(b) If \(p \neq 3\), then

$$3q(3) = -2 \sum_{\nu=0}^{\infty} \left( \sum_{\frac{p\nu}{3} < \frac{p^\nu + 1}{3} \nu < \frac{p\nu + 1}{3}} \frac{1}{x} \right).$$

(c) If \(p \geq 5\), then

$$q(2^4 \cdot 3^3) = 4q(2) + 3q(3) = -2 \sum_{\nu=0}^{\infty} \left( \sum_{\frac{p\nu}{6} < \frac{p^\nu + 1}{6} \nu < \frac{p\nu + 1}{6}} \frac{1}{x} \right).$$
5. Wilson Quotients for the $p$-Adic Case

**Definition 5.1.** Let $m \geq 2$ be an integer and $\varepsilon_m = -1$ if $m = 2, 4, p^a$ or $2p^a$ ($p$ an odd prime and $\alpha$ a positive integer) and $\varepsilon_m = 1$ otherwise.

The integer

$$W(m) = \frac{1}{m} \left[ \prod_{j=1}^{m} j - \varepsilon_m \right]$$

is called the *generalized Wilson quotient of $m$* (see [ADS2], Definition 2.1).

According to [ADS2], Propositions 3.1 and 3.2, we have

$$W(p^{n+1}) \equiv W(p^n) \pmod{p^{n-1}},$$

hence there exists $\lim_{n \to \infty} W(p^n)$.

**Definition 5.2.** Set

$$W = W_p = \lim_{n \to \infty} W(p^n)$$

and call the $p$-adic integer $W_p$ the *Wilson quotient for the $p$-adic case*, or simply *$p$-adic Wilson quotient*.

**Proposition 5.1.** $v(W - W(p^n)) \geq n - 1$.

**Proof.** According to (5.1) we get for each integer $m > n$ the inequality $v(W(p^m) - W(p^n)) \geq n - 1$. There exists an integer $m > n$ such that $v(W - W(p^m)) \geq n - 1$, therefore

$$v(W - W(p^n)) = v((W - W(p^m)) + (W(p^m) - W(p^n))) \geq \min\{v(W - W(p^m)), v(W(p^m) - W(p^n))\} \geq n - 1$$

and we are done. \qed

**Notation.** For an integer $m \geq 2$ set

$$\sigma_1(m) = \sum_{a=1}^{m} q(a,m), \sigma_2(m) = \sum_{a=1}^{m} \sum_{b=a+1}^{m} q(a,m)q(b,m).$$

Further let

$$\varepsilon_p = \begin{cases} -1 & \text{if } p \text{ is odd} \\ 1 & \text{if } p = 2 \end{cases} (= \varepsilon_p^3) \text{ and } c(n) = \varphi(p^n) = p^{n-1}(p - 1).$$

As usual the $n$-th *Bernoulli number* will be denoted by $B_n$ ($B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_3 = 0, \ldots$).

In the following proposition the congruence mod $m$ in Proposition 2.1 of [ADS2] is extended to mod $m^2$ using the same method of the proof.
Proposition 5.2. For integers $m \geq 3$ we have
\[ \varepsilon_m \varphi(m)W(m) + \left( \frac{\varphi(m)}{2} \right) mW(m)^2 \equiv \sigma_1(m) + m\sigma_2(m) \pmod{m^2}. \]

Proof. The result follows from observing that
\[
\left( \prod_{j=1 \atop (j,m)=1}^{m} \frac{\varphi(m)}{j} \right)^m = (\varepsilon_m - mW(m))^{\varphi(m)} \equiv \varepsilon_m^{\varphi(m)} + \varphi(m)e^{\varphi(m)-1}mW(m) +
\]
\[ + \left( \frac{\varphi(m)}{2} \right) \varepsilon_m^{\varphi(m)-2}m^2W(m)^2 \pmod{m^3} =
\]
\[ = 1 + \varepsilon_m \varphi(m)mW(m) + \left( \frac{\varphi(m)}{2} \right) m^2(W(m))^2 \]

and also, by the definition of $q(a,m)$,
\[
\left( \prod_{j=1 \atop (j,m)=1}^{m} \frac{\varphi(m)}{j} \right) = \prod_{a=1 \atop (a,m)=1}^{m} \left( 1 + mq(a,m) \right) \equiv
\]
\[ \equiv 1 + m\sigma_1(m) + m^2\sigma_2(m) \pmod{m^3}. \]

Proposition 5.3. \( \lim_{n \to \infty} \sigma_1(p^n) = 0. \)

Proof. Using Proposition 5.2 we get \( v(\sigma_1(p^n)) \geq n - 1 \) and the result follows.

Theorem 5.4. \( \sum_{\nu=1}^{\infty} \left( \sum_{p \mid a}^{p^n} \varphi(p^{\nu-1}+1)q(a) \right) = 0. \)

Proof. Let \( n \geq 2 \). According to Theorem 3.3 (a) we have for each integer \( a \) \( (p \nmid a) \)
\( v(q(a) - q(a,p^n)) \geq n, \) therefore
\[ v \left( \sum_{p \mid a}^{p^n} (q(a) - \sigma_1(p^n)) \right) = v \left( \sum_{a=1 \atop p \mid a}^{p^n} [q(a) - q(a,p^n)] \right) \geq
\]
\[ \geq \min\{v(q(a) - q(a,p^n)) : 1 \leq a \leq p^n, a \in \mathbb{Z}, p \nmid a \} \geq n. \]

The result follows from Proposition 5.3.

Proposition 5.5.
\[ W_p = \varepsilon_p \frac{p}{p-1} \lim_{n \to \infty} \left( \sigma_1(p^n) + \sigma_2(p^n) \right) = \varepsilon_p \frac{p}{p-1} \lim_{n \to \infty} \left( \sigma_1(p^n) + p\sigma_2(p^n) \right). \]

Proof. If we substitute for \( m \) the power \( p^n \) \( (n \geq 3) \) in the congruence of Proposition 5.2, we get
\[ \varepsilon_p(p-1)W(p^n) \equiv \sigma_1(p^n) + p\sigma_2(p^n) \pmod{p^n-1}. \]

Using Proposition 5.1 we get the result.

For the proof of Theorem 5.7 we will need the following lemma:
Lemma 5.6. Let $t$ be a positive integer and $n \geq 5$. Then
\[ \sum_{a=1 \atop p \nmid a}^{p^n} a^{tc(n)} \equiv B_{tc(n)}p^n \pmod{p^{3n-1}}. \]

Proof. For the sake of simplicity put $c = c(n)$ and $m = p^n$. Then by a well-known identity for Bernoulli numbers,
\[ \sum_{a=1}^{m-1} a^{tc} = \frac{1}{tc+1} \sum_{k=0}^{tc} \binom{tc+1}{k} B_k m^{tc+1-k}. \]
Since for $0 \leq k \leq tc - 2$ the inequality $v(B_k m^{tc+1-k}) \geq 3n - 1$ is satisfied by the von Staudt-Clausen theorem, we get
\[ \sum_{a=1}^{m-1} a^{tc} \equiv \frac{1}{tc+1} \left( \binom{tc+1}{1} B_{tc} p^n + \binom{tc+1}{2} B_{tc-1} p^{2n} \right) \pmod{p^{3n-1}}. \]
The integer $tc - 1$ is odd and greater than 3, hence $B_{tc-1} = 0$. If $a$ is an integer divisible by $p$, then $v(a^{tc}) \geq tc \geq 2^{n-1} \geq 3n + 1$. The result follows.

Theorem 5.7.
\[ W_p = -\frac{1}{2} \log \lim_{n \to \infty} \frac{1}{p^n} \left( B_{2c(n)} - 4B_{c(n)} + \frac{3(p-1)}{p} \right). \]

Proof. Put $\gamma(n) = \sum_{a=1 \atop p \nmid a}^{p^n} q(a, p^n)^2$ and $c = c(n)$. According to Lemma 5.6 we have for $n \geq 5$
\[ \gamma(n) = \frac{1}{p^{2n}} \sum_{a=1 \atop p \nmid a}^{p^n} (a^c - 2a^c + 1) = \frac{1}{p^n} \left( B_{2c} - 2B_c + \frac{p-1}{p} \right) + x_n p^{n-1} \]
and
\[ \sigma_1(p^n) = \frac{1}{p^n} \sum_{a=1 \atop p \nmid a}^{p^n} (a^c - 1) = B_c - \frac{p-1}{p} + y_n p^{2n-1}, \]
where $x_n$ and $y_n$ are (rational) integers.

Further
\[ \sigma_1(p^n)^2 = \gamma(n) + 2\sigma_2(p^n), \]
therefore
\[ \frac{\sigma_1(p^n)}{p^n} + \sigma_2(p^n) = \frac{1}{p^n} \left( B_c - \frac{p-1}{p} \right) + y_n p^{n-1} + \frac{\sigma_1(p^n)^2}{2} - \frac{\gamma(n)}{2} = \]
\[ = \frac{1}{2p^n} \left( 2B_c - \frac{2(p-1)}{p} - 2B_c + 2B_c - \frac{p-1}{p} \right) + (y_n - \frac{x_n}{2}) p^{n-1} + \frac{\sigma_1(p^n)^2}{2} = \]
\[ = -\frac{1}{2p^n} \left( B_{2c} - 4B_c + \frac{3(p-1)}{p} \right) + (y_n - \frac{x_n}{2}) p^{n-1} + \frac{\sigma_1(p^n)^2}{2}. \]
Since $\lim_{n \to \infty} (y_n - x_n) p^{n-1} = 0$ and $\lim_{n \to \infty} \frac{\sigma_1(p^n)^2}{2} = 0$ (by Proposition 5.5),
the proof is complete according to Proposition 5.5.
6. Friedmann-Tamarkine Congruence

In their paper [FT] (1909) Friedmann and Tamarkine proved for an odd prime $p$ and an integer $m$ ($3 \leq m \leq p - 2$) the following congruence:

\[(FT) \quad \sum_{a=1}^{p-1} a^m q(a,p) \equiv -\frac{1}{m} B_m \pmod{p}.\]

Note that the congruences of this kind were given by Lerch ([Lr1], 1905) for special $m$.

\[(4), \ m = 0 : \quad \sum_{a=1}^{p-1} q(a,p) \equiv W(p) \pmod{p},\]

\[(17), \ m = 1 : \quad \sum_{a=1}^{p-1} aq(a,p) \equiv \frac{1}{2} \pmod{p},\]

\[(24), \ m = 2 : \quad \sum_{a=1}^{p-1} a^2 q(a,p) \equiv -\frac{1}{12} \pmod{p}, (p \neq 3),\]

\[(18), \ m = \frac{p-1}{2} : \quad \sum_{a=1}^{p-1} \left(\frac{a}{p}\right) q(a,p) \equiv 0 \pmod{p}, (p \equiv 3 \pmod{4}),\]

\[(21), \ m = \frac{p-1}{2} : \quad \sum_{a=1}^{p-1} \left(\frac{a}{p}\right) q(a,p) \equiv 2B_{\frac{p-1}{2}} \pmod{p}, (p \equiv 1 \pmod{4}),\]

\[(22^1), \ m = \frac{p+1}{2} : \quad \sum_{a=1}^{p-1} \left(\frac{a}{p}\right) q(a,p) \equiv 0 \pmod{p}, (p \equiv 1 \pmod{4}),\]

\[(22^2), \ m = \frac{p+1}{2} : \quad \sum_{a=1}^{p-1} \left(\frac{a}{p}\right) q(a,p) \equiv C\ell(-p) \pmod{p}, (p \equiv 3 \pmod{4}),\]

where $C\ell(-p)$ is the number of divisor classes of the quadratic field $\mathbb{Q}(\sqrt{-p})$ and $C\ell(-p) \equiv -2B_{\frac{p+1}{2}} \pmod{p}$ for $p > 3$ ([BS], Chap. 5, Sec. 8, Problem 4).

The aim of this section is to transfer the congruence (FT) to the $p$-adic case for the Fermat quotient $q$. For $m = 0$ the modified relation was expressed by Theorem 5.4.

Further we will assume that $p$ is an odd prime and for a positive integer $N$ set

\[S_N(n) = 1^N + 2^N + \cdots + (n-1)^N.\]

To prove the main theorem of this section we will state some lemmas.
Lemma 6.1. If \( \mu, \nu \) are positive integers, \( \nu \geq \nu(\mu + 1) \), then
\[
S_\mu(p^\nu) \equiv B_\mu p^\nu \pmod{p^{2\nu-1}}.
\]

Proof. Put \( x = \nu(\mu + 1) \). Since \( \nu(B_k) \geq -1 \) by the von Staudt-Clausen theorem, we have for \( 0 \leq k \leq \mu - 2 \) (\( k \) an integer):
\[
v \left( \frac{1}{\mu + 1} \binom{\mu + 1}{k} B_k p^\nu(\mu + 1 - k) \right) \geq -x - 1 + 3\nu \geq 2\nu - 1,
\]
therefore, as in the proof of Lemma 5.6,
\[
S_\mu(p^\nu) = \frac{1}{\mu + 1} \sum_{k=0}^{\mu} \binom{\mu + 1}{k} B_k p^\nu(\mu + 1 - k) \equiv 1 \left( \frac{1}{\mu + 1} \right) B_\mu p^\nu \pmod{p^{2\nu-1}} \equiv B_\mu p^\nu \pmod{p^{2\nu-1}}.
\]

In Lemmas 6.2 and 6.3 we assume that \( m \) is a positive integer, \( m \not\equiv 0 \pmod{p-1} \), \( \nu \) is an integer, \( \nu \geq \nu(m + 1) + 2 \) and \( M = m + p^\nu(m-1) \).

Lemma 6.2. \( \frac{1}{p^\nu}(S_M(p^\nu) - S_m(p^\nu)) \equiv -p^{m-1}B_m \pmod{p^{\nu-1}}. \)

Note that \( \nu \geq \nu(m+1) = \nu(M+1) \), hence the numbers \( \frac{1}{p^\nu}S_M(p^\nu) \) and \( \frac{1}{p^\nu}S_m(p^\nu) \) are integers by Lemma 6.1.

Proof. We will use Kummer's congruence for the Bernoulli numbers modulo a prime power ([W], Corollary 12.3, p. 241):
If \( m \equiv M \pmod{p^{\nu-1}(p-1)} \), and \( m \not\equiv 0 \pmod{p-1} \), then
\[
(1 - p^{m-1}) \frac{B_m}{m} \equiv (1 - p^{M-1}) \frac{B_M}{M} \pmod{p^{\nu}}.
\]

Since \( M-1 \geq p^{\nu-1}(p-1) \geq 3^{\nu-1} - 2 \geq \nu \), we have \( (1 - p^{m-1}) \frac{B_m}{m} \equiv \frac{B_M}{m} \pmod{p^{\nu-1}} \), therefore
\[
B_M \equiv (1 - p^{m-1})B_m \pmod{p^{\nu-1}}.
\]

Using Lemma 6.1 we get
\[
\frac{1}{p^\nu}(S_M(p^\nu) - S_m(p^\nu)) \equiv B_M - B_m \equiv -p^{m-1}B_m \pmod{p^{\nu-1}}. \]

Lemma 6.3.
\[
\sum_{\substack{a=1 \\ p \nmid a}}^{p^\nu} a^m q(a, p^\nu) \equiv 0 \pmod{p^{\nu-2}}.
\]
Proof. For an integer $a$ divisible by $p$ we have $v(w^M) - M > p^{v-1}(p-1) > 3^{v-1} \cdot 2 > 2v$, hence
\[
\sum_{a=1 \atop p \nmid a}^{p^v} a^M \equiv S_M(p^v) \pmod{p^{2v}}
\]
and there exists $A \in \mathbb{Z}$ such that
\[
(6.1) \quad \sum_{a=1 \atop p \nmid a}^{p^v} a^M = S_M(p^v) + A \cdot p^{2v}.
\]

Since $S_m(p^v) = \sum_{a=1 \atop p \nmid a}^{p^v} a^m + p^m \sum_{a=1}^{p^{v-1}-1} a^m$, we have
\[
(6.2) \quad \sum_{a=1 \atop p \nmid a}^{p^v} a^m = S_m(p^v) - p^m S_m(p^{v-1}),
\]
and using Lemma 6.1 we get $S_m(p^{v-1}) \equiv B_m p^{v-1} \pmod{p^{2v-3}}$. Therefore there exists a $p$-adic integer $C$ such that
\[
(6.3) \quad p^m S_m(p^{v-1}) = p^{m+v-1} B_m + p^{m+2v-3} C.
\]

Summarizing (6.1) - (6.3) we obtain
\[
\sum_{a=1 \atop p \nmid a}^{p^v} a^m q(a, p^v) = \frac{1}{p^v} \left( \sum_{a=1 \atop p \nmid a}^{p^v} a^M - \sum_{a=1 \atop p \nmid a}^{p^v} a^m \right) =
\]
\[
= \frac{1}{p^v} (S_M(p^v) + Ap^{2v} - S_m(p^v) + p^{m+v-1} B_m + p^{m+2v-3} C) \equiv
\]
\[
\equiv \frac{1}{p^v} (S_M(p^v) - S_m(p^v)) + p^{m-1} B_m \pmod{p^{v-2}} \equiv
\]
\[
\equiv 0 \pmod{p^{v-2}}
\]
according to Lemma 6.2. \hfill $\Box$

**Theorem 6.4.** If $p$ is an odd prime and $m$ a positive integer, $m \not\equiv 0 \pmod{p-1}$, then
\[
\sum_{\nu=1}^{\infty} \left( \sum_{a=p^{\nu-1}+1}^{p^\nu} a^m q(a) \right) = 0.
\]

**Proof.** For a positive integer $\nu$ put
\[
A(\nu) = \sum_{a=1 \atop p \nmid a}^{p^\nu} a^m q(a) \quad \text{and} \quad B(\nu) = \sum_{a=1 \atop p \nmid a}^{p^\nu} a^m q(a, p^\nu).
\]

According to Theorem 3.3 (a) we have $v(A(\nu) - B(\nu)) \geq \nu$.

If $\nu \geq v(m+1)+2$, then by Lemma 6.3 $v(B(\nu)) \geq \nu - 2$, therefore $v(A(\nu)) = v(A(\nu) - B(\nu) + B(\nu)) \geq \min\{v(A(\nu) - B(\nu)), v(B(\nu))\} \geq \nu-2$ (for $\nu \geq v(m+1)+2$).

This proves \( \lim_{\nu \to \infty} A(\nu) = 0 \) and the proof is complete. \hfill $\Box$
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References


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