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On the Number of Maximal Theta Pairs in a Finite Group

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Abstract: In [6], Bhattacharya and Mukherjee defined the notion of \( \theta \)-pair for a maximal subgroup of a finite group. They proved that for any maximal subgroup \( M \) of a finite group \( G \), there exists a \( \theta \)-pair related to \( M \). In [11], Zhao improved this result. He proved that for any maximal subgroup \( M \) of a finite group \( G \), there exists a normal maximal \( \theta \)-pair related to \( M \).

In this paper we introduce the notion of \( n\theta \)-maximal and primitive \( n\theta \)-maximal group. We show that for \( n = 1, 2 \), \( G \) is \( n\theta \)-maximal if and only if \( G \) is primitive \( n\theta \)-maximal. Also, we characterize the \( 1\theta \)-maximal group and prove some results about \( 2\theta \)-maximal groups. Finally, we introduce the notion of \( n\theta \)-pair group and prove that for all \( n \neq 2, 3 \), there exists \( n\theta \)-pair groups and for \( n = 2, 3 \) there is no \( n\theta \)-pair groups.

Key Words: Maximal \( \theta \)-pair, \( n\theta \)-maximal group, primitive \( n\theta \)-maximal group, \( n\theta \)-pair group

Mathematics Subject Classification: 1991 Mathematics Subject Classification: 20E34, 20D10

1. Introduction

In this paper all groups considered are assumed to be finite groups. For convenience we denote \( M < G \) to indicate that \( M \) is a maximal subgroup of a group \( G \). Also, \( M_G \) denotes the core of \( M \) in \( G \) and \( \Phi(G) \) is the Frattini subgroup of the group \( G \).

In [6], Mukherjee and Bhattacharya introduced the concept of \( \theta \)-pairs associated to maximal subgroups of a group, and used this concept to investigate the structure of some groups. In [2], Beidleman and Smith generalized the concept to the universe of infinite groups. The investigation on \( \theta \)-pairs are continued in [1], [2], [7], [10], [11], [12], [13] and [14].

Let us recall the definition of \( \theta \)-pair which is introduced by Mukherjee and Bhattacharya in [6].

Definition 1 [6]. Given a maximal subgroup \( M \) of a group \( G \), a \( \theta \)-pair of \( M \) is any pair \( (A, B) \) of subgroups satisfying the following conditions:

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(a) \( B < G, \ B < A \).
(b) \( <M,A> = G \ \text{ and } \ B < M \).
(c) \( | \) has no proper normal subgroup of \( ^{\circ} \).

In addition, if \( A < G \), then \( (A,B) \) is called a normal \#-pair. A \#-pair \( (A, B) \) is said to be maximal if there is no \#-pair \( (C, D) \) such that \( A < C \). The nonempty set of all 0-pairs of \( M \) in \( G \) is denoted by \( \mathcal{N}(M) \) and \( \delta(G) = \bigcup M < G \bigwedge ^{\circ} \). Similarly, \( \mathcal{O}_{\text{max}}(M) \) denotes the set of all maximal 0-pairs of \( M \) in \( G \) and \( O_{\text{max}}(G) = \bigcup_{M < G} \mathcal{O}_{\text{max}}(M) \).

**Definition 2.** A group \( G \) is called n\#-maximal if \( \mathcal{O}_{\text{max}}(G) = n \). Also, we say that \( G \) is primitive n\#-maximal, if \( A < G \) and \( N < G \) \( \delta(G) \) implies that \( | \mathcal{O}_{\text{max}}(G) | = n \).

In this paper, all notations are standard and taken mainly from [3], [4], [6] and [9].

**2. Groups with exactly \( n \) Maximal 0-pairs, \( n = 1, 2 \)**

In this section we obtain the number of maximal \#-pairs of some finite groups and prove that for any positive integer \( n \), there exists a finite group \( G \) such that \( \mathcal{O}_{\text{max}}(G) = n \). To do this, suppose \( G \) is a finite group and \( \mathcal{T}(G) \) denotes the set of all prime factors of \( |G| \). In the following simple lemma, we obtain the number of maximal \#-pairs in a finite nilpotent group.

**Lemma 1.** Let \( G \) be a nilpotent group with exactly \( n \) maximal subgroup. Then \( G \) is a primitive n\#-maximal group.

**Proof.** We first show that if \( M \) is a maximal subgroup of \( G \), then \( \mathcal{O}_{\text{max}}(M) = \{(G, M)\} \). To do this, suppose \( M \) is a maximal subgroup of \( G \), then \( | \) has a prime order and so \( (G, M) G \ \mathcal{N}(M) \). If \( (A, B) \) is another maximal 0-pair of \( M \) in \( G \), then \( A = G \) and so \( (A, 1) \) is a normal maximal 0-pair. Now, by Theorem 2.5 of [11], \( B = MQ = M \) and \( \mathcal{O}_{\text{max}}(M) = \{(G, M)\} \). Next, we assume that \( G \) is a nilpotent group with exactly \( n \) maximal subgroup, \( M_1, M_2, \cdots, M_n \). Therefore, for all \( i, 1 < i < n \), \( O_{\text{max}}(M_i) = \{(G, M_i)\} \). This shows that \( e_{\text{max}}(G) = \{(G, M_i)| 1 < i < n \} \) and \( G \) is a n\#-maximal group. We now assume that \( N < \mathcal{T}(G) \) is a normal subgroup of \( G \). Set, \( S = \{M\} \mathcal{M} \mathcal{G} \) and \( T = \{S| S < \mathcal{T} \} \). Therefore, the map from \( S \) to \( T \) that sends \( M \) to \( ^{\circ} \) is easily seen to be a one-to-one correspondence. Thus, \( \mathcal{O}_{\text{max}}(G) = n \) and the lemma is proved, \( o \)

**Corollary.** For all positive integer \( n \), there exist a primitive n\#-maximal group.

**Proof.** Let \( G \) be a cyclic group with \( |\mathcal{T}(G)| = n \). Then \( G \) has exactly \( n \) maximal subgroup and by Lemma 1, \( G \) is a primitive n\#-maximal group. \( o \)

**Lemma 2.** Let \( G \) be a finite group and \( iV \) be a normal subgroup of \( G \). Then \( \mathcal{O}_{\text{max}}(G) \) \( < \mathcal{O}_{\text{max}}(G) \).

**Proof.** By Lemma 2.1 of [6], the map \( r : \mathcal{O}_{\text{max}}(G) \rightarrow \mathcal{O}_{\text{max}}(G) \) that sends \( (C, D) \) to \( (C, D) \) is well-defined. Now, it is easy to see that the map \( r \) is one-to-one. \( o \)
Remark 1. In the definition of primitive $n\theta$-maximal group, if we omit the condition $N \leq \Phi(G)$ then there is no primitive $n\theta$-maximal group, for $n > 1$. To see this, we assume that $G$ is an arbitrary $n\theta$-maximal group, for $n > 1$. By Theorem 2.3 of [11] there is a normal maximal $\theta$-pair $(A, MG)$ of $M$, in which $M$ is a maximal subgroup. Consider $\frac{G}{A}$, then we can see that the map $\tau$, in the proof of Lemma 2, is not onto. This shows that $G$ is not primitive.

Remark 2. Let $G$ be a finite group. $G$ is $\theta$-maximal if and only if $G$ is primitive $\theta$-maximal. To see this, it is enough to show that every $\theta$-maximal group is primitive. Suppose $N \leq G$, then by Lemma 2, $|\theta_{\text{max}}(\frac{G}{N})| < |\theta_{\text{max}}(G)| = 1$. Thus, $|\theta_{\text{max}}(\frac{G}{N})| = 1$, proving the result.

In [11], Zhao proved that if $M$ is a maximal subgroup of $G$ and $(S, T)$ is a normal $\theta$-pair of $M$, then $M$ has a normal maximal $\theta$-pair $(A, B)$ such that $(S, T) \leq (A, B)$ and $\frac{A}{S} \cong \frac{B}{T}$. Furthermore, he proved that if $M \leq G$ and $(A, B)$ is a normal maximal $\theta$-pair of $\theta(M)$, then $B = MG$. We use these results to prove the following theorem:

Theorem 1. $G$ is $\theta$-maximal if and only if $\frac{G}{\Phi(G)}$ is a simple group.

Proof. Suppose $G$ is $\theta$-maximal, say $\theta_{\text{max}}(G) = \{(C, D)\}$. Suppose $C \neq G$. Then there exists a maximal subgroup $M$ of $G$ such that $C \subseteq M$. Since $\theta_{\text{max}}(M) \neq \emptyset$, hence $\theta_{\text{max}}(M) = \{(C, D)\}$. This implies that $G = \langle M, C \rangle = M$, a contradiction. Thus $C = G$ and $(C, D)$ is a normal maximal $\theta$-pair. Now, by the mentioned result of Zhao, $D = MG$. If $K$ is a maximal subgroup of $G$, then by assumption $K_G = M_G$ and so $D = M_G = \Phi(G)$. This shows that $\frac{G}{\Phi(G)}$ is a simple group.

Conversely, suppose $\frac{G}{\Phi(G)}$ is a simple group and $(C, D) \in \theta_{\text{max}}(G)$ is a maximal $\theta$-pair. Since $\frac{G}{\Phi(G)}$ is simple, hence for any maximal subgroup $K$ of $G$, $(G, \Phi(G))$ is a normal maximal $\theta$-pair of $K$. Therefore, $C = G$ and $D = \Phi(G)$. Now, by the above result of Zhao, for any maximal subgroup $K$ of $G$, $D = K_G$. Therefore, $(C, D) = (G, \Phi(G))$, proving the theorem.

Theorem 2. $G$ is $\theta$-maximal if and only if there exists a maximal subgroup $M$ of $G$ such that $\theta_{\text{max}}(M) = \theta_{\text{max}}(G)$.

Proof. Suppose $M$ is a maximal subgroup of $G$ such that $\theta_{\text{max}}(M) = \theta_{\text{max}}(G)$ and $|\theta_{\text{max}}(G)| > 1$. Let $(A, M_G)$ be a normal maximal $\theta$-pair of $G$ associated to $M$. If $A \neq G$ then $\frac{G}{A}$ contains a normal maximal $\theta$-pair $(R, T)\frac{A}{R}$ associated to a maximal subgroup $\frac{T}{A}$ of $\frac{G}{A}$. By Lemma 2.1 of [6], $(R, T_G)$ is a normal maximal $\theta$-pair of $G$ and so $(R, T_G) \in \theta_{\text{max}}(M)$. But, $A < R$ and $(A, M_G) \in \theta_{\text{max}}(M)$, a contradiction. Now for any maximal subgroup $K$ of $G$, $K_G = M_G$ and so $\Phi(G) = M_G$. This shows that $\frac{G}{\Phi(G)}$ is a simple group, which is a contradiction. Therefore, $G$ is a $\theta$-maximal group. The converse is obvious.

Lemma 3. If $(C, D) \in \theta(M)$, then for all $g \in G$, $(C^g, D) \in \theta(M^g)$.

Proof. Since, $D \triangleleft G, D \triangleleft C$ and $C \nsubseteq M$, we have $D \cap C^g \subseteq C^g$. Assume that $\frac{G}{D^g}$ properly contains a non-trivial normal subgroup $\frac{T}{D}$ of $\frac{G}{D}$. Then we have,
But, \((C,D) \in \mathcal{O}(M), a\) contradiction. Therefore, \((C^0,D) \in \mathcal{O}(M^*)\) and the lemma is proved, o

**Corollary.** Let \(M\) be a maximal subgroup of the group \(G\). Then, for all \(g \in G\), \([0(M)] \supseteq \mathcal{O}(M^*)\).

**Proof.** By Lemma 3, the map \(r : \mathcal{O}(M) \rightarrow \mathcal{O}(M^*)\) that sends \((C,D)\) to \((C^0,D)\) is well-defined. Now, it is easy to see that the map \(r\) is a one-to-one correspondence. o

In what follows, we investigate the structure of 29- maximal and primitive 29-maximal groups.

**Lemma 4.** \(G\) is 20-maximal if and only if \(G\) is primitive 2#-maximal.

**Proof.** Suppose \(G\) is a 20-maximal group and \(N\) is a normal subgroup of \(G\) such that \(TV < \mathcal{S}(G)\). By Lemma 2, \(\mathcal{O}_{max}(F) < 2\). If \(\mathcal{O}_{max}(G) = 1\) then by Theorem 1, \(\mathcal{S}\) is simple. But, \(\mathcal{A'} \subset \mathcal{S}(G)\) and so \(\mathcal{S}(\mathcal{A}) = \mathcal{A}\), this implies that \(\mathcal{A}\) is a simple group. Therefore, \(G\) is 10-maximal, which is a contradiction. o

**Lemma 5.** Let \(G\) be a 29-maximal group and \(O_{max}(G) = \{(A,B),(C,D)\}\). Then the following statements hold:

(a) \(A < G\) and \(C < G\).
(b) \(A = G\) or \(G = G\).
(c) \(|\{T \in G | T < -G\}| = 2\).

**Proof.** We can assume that \((A,B)\) is a normal maximal 0-pair. Suppose \(C\) is not normal in \(G\) and \(g \in G\). Since \(\mathcal{A} < R\), \((A,B) = (U,V)\) and \((C,D) = (U,D)\).

Therefore, \(C < U - A = C\), a contradiction. Finally, by Theorem 2, there are two maximal subgroups \(M\) and \(L\) such that \((A,B) \in \mathcal{O}_m(G)\) and \((C,D) \in \mathcal{O}_m(G)\), so by part (a), \(B = MG\) and \(D = LQ\). We now assume that if is another maximal subgroup of \(G\), then \(#_{max}(\mathcal{A})\) contains \((-4,1)\) or \((C,D)\).

Thus, \(M = MG\) and \(K = LQ\) and so \(m = |\{r_\in G | T < -G\}| < 2\). Suppose \(m = 1\) then \(S(G) = MG\) and \(LQ\) is a simple group, a contradiction. If \(4 = G\) then \((4,\mathcal{E}) = (G,MG)\) and \((C,Z) = (G,LQ)\) and so \((4, B) - (C,D)\), which is a contradiction. Therefore, \(m = 2\), as desired. o

**Theorem 3.** Suppose \(G\) satisfies the following conditions,

a) \(\mathcal{A} < G\) and \(C < G\).

b) \(^1\mathcal{S}\) is a direct product of two simple groups.

Then \(G\) is 2#-maximal.

**Proof.** By condition (a) and Theorem 1, \(\mathcal{S}\) is not simple. Hence we can assume that \(^1\mathcal{S} = \mathcal{A}_x \mathcal{A}\), in which \(^1\mathcal{S}\) and \(\mathcal{S}\) are two non-trivial simple
subgroups of $\frac{G}{\Phi(G)}$. By condition (a) there are two maximal subgroups $M$ and $L$ such that $M_G \neq L_G$ and $\Phi(G) = M_G \cap L_G$. We now assume that $T$ is a maximal subgroup of $G$ such that $K \subseteq T_G$. So, $T_G = L_G$ or $M_G$. Suppose $K \subseteq M_G$ and $K \not\subseteq L_G$, then $G = KL$, $P \not\subseteq M_G$, $(K, \Phi(G)) \in \theta(L)$ and $(P, \Phi(G)) \in \theta(M)$. Let $(U, L_G)$ be a normal maximal $\theta$-pair of $L$ such that $(K, \Phi(G)) \leq (U, L_G)$. We can see that $U = G$. Using similar argument as in above, if $(V, M_G)$ is a normal maximal $\theta$-pair of $M$ such that $(P, \Phi(G)) \leq (V, M_G)$, then $V = G$. If $(C, D)$ is another maximal $\theta$-pair of $G$ then there exists a maximal subgroup $T$ such that $(C, D) \in \theta(T)$, so $T_G = L_G$ or $T_G = M_G$. Suppose that $T_G = L_G$, then $(G, L_G), (C, D) \in \theta(T)$ and since $C \neq G$ so $(C, D) \leq (G, L_G)$, which is a contradiction. Therefore $G$ is 2$\theta$-maximal.

**Corollary.** If $\frac{G}{\Phi(G)}$ is a direct product of two simple groups with co-prime orders, then $G$ is 2$\theta$-maximal.

**Proof.** By Theorem 3, it is enough to show that $|\{M_G | M \leq G\}| = 2$. To do this, we prove that if $G = A \times B$, where $A$ and $B$ are normal simple subgroups of $G$ with co-prime orders, then $G$ has exactly four normal subgroups. Suppose $N$ is a normal subgroup of $G$ different from $A$ and $B$. We can assume that $N \cap A = N \cap B = 1$ and so $A \cong \frac{G}{N} \cong B$, a contradiction. Therefore, $|\{X_G | X < G\}| = 2$ and the proof is complete. 

### 3. Groups with exactly $n \theta$-pair

In this section we introduce the notion of $n\theta$-pair group and prove that there is no 2$\theta$- and 3$\theta$-pair group. Finally, we construct a groups with exactly $n \theta$-pairs, for $n \neq 2, 3$. To do this, we need the structure of groups with exactly one or two maximal subgroups. It is well known that if a finite group $G$ has exactly one maximal subgroup, then $|G|$ is divisible by exactly one prime number and $G$ is cyclic. It has been proved [5] that if $G$ has exactly two maximal subgroups then $|G|$ is indeed divisible by two primes and $G$ is cyclic. Throughout this section $m(G)$ denotes the number of maximal subgroups of $G$.

**Definition 3.** A group $G$ is called $n\theta$-pair, if and only if $|\theta(G)| = n$.

**Lemma 6.** A group $G$ is 1$\theta$-pair if and only if $G$ is a cyclic group of prime power order.

**Proof.** Suppose $G$ is 1$\theta$-pair. Then by Theorem 1, $\frac{G}{\Phi(G)}$ is a simple group and $\theta(G) = \{(G, \Phi(G))\}$. Suppose $m(G) > 1$. Then $\Phi(G)$ is not maximal in $G$ and for any maximal subgroup $M$ of $G$, $(M, \Phi(G))$ is a $\theta$-pair of $L$, in which $L$ is a maximal subgroup of $G$ distinct from $M$, a contradiction. This shows that $m(G) = 1$ and so $G$ is a cyclic group of prime power order.

**Lemma 7.** If there exists a maximal subgroup $M$ of $G$ such that $\theta(M) = \Phi(G)$, then $G$ is 1$\theta$-pair.

**Proof.** By Theorem 2, $G$ is 1$\theta$-maximal and so $\frac{G}{\Phi(G)}$ is a simple group. If $m(G) > 1$ then $(M, \Phi(G)) \in \theta(L)$ and $(L, \Phi(G)) \in \theta(M)$, for two distinct maximal subgroups
$M$ and $L$ of $G$, which is a contradiction. Therefore, $m(G) = 1$ and by Lemma 6, $G$ is $1\theta$-pair. ∅

**Lemma 8.** There is no $n\theta$-pair cyclic group of order $p_1^{r_1} \cdot p_2^{r_2} \cdots p_n^{r_n}$, $p_1 < p_2 < \cdots < p_n$, in which $n > 1$.

**Proof.** Suppose $\{M_1, M_2, \ldots, M_n\}$ is the set of all maximal subgroups of $G$. Then $(G, M_i), 1 \leq i \leq n$, are $n$ maximal $\theta$-pairs for $G$ and so $G$ has at least $n$ $\theta$-pair. Assume that $M$ is a maximal subgroup of index $p_1$, $A$ is a maximal subgroup of $M$ of index $p_2$ and $L$ is a maximal subgroup of $G$ of index $p_3$. Then $(M, A) \in \theta(L)$, a contradiction. ∅

**Theorem 4.** There is no $2\theta$-pair group.

**Proof.** Let $G$ be a $2\theta$-pair group. By Lemma 7, there is no maximal subgroup $M$ of $G$ such that $\theta(M) = \theta(G)$ and so $G$ is $2\theta$-maximal. Thus, $|\{X_G \mid X < G\}| = 2$. Suppose that $(C, L_G)$ and $(G, M_G)$ are two distinct maximal $\theta$-pairs of $G$ associated to maximal subgroups $L$ and $M$, respectively. We claim that $G$ has exactly two maximal subgroups. To do this, we assume that $T$ is a maximal subgroup of $G$ distinct from $M$ and $L$. If $C \subseteq T$ then $C \subseteq D$, a contradiction. Thus $(C, D) \in \theta(T)$, which is a contradiction. We now assume that $(E, F)$ is a maximal $\theta$-pair of $\theta(L_i)$ such that $(C, D) \leq (E, F)$. Therefore, $C \leq E, D = F, \frac{C}{D} \leq \frac{E}{D}$ and $\frac{C}{D} \leq \frac{G}{D}$. This shows that $(C, D)$ is a maximal $\theta$-pair of $\theta(L_i)$ and the proof is complete. ∅

**Lemma 9.** Let $G$ be a finite group such that all of maximal $\theta$-pairs of $G$ are normal and $\{M_G \mid M < G\} = \{L_1, \ldots, L_r\}$. Then $\theta_{\text{max}}(G) = \theta_{\text{max}}(L_1) \cup \cdots \cup \theta_{\text{max}}(L_r)$.

**Proof.** Suppose $(C, D)$ is an arbitrary maximal $\theta$-pair of $G$. Then $D = L_{i_1}$, for some $1 \leq i \leq r$. If $C \subseteq L_i$ then $C \subseteq D$, a contradiction. Thus $(C, D) \in \theta(L_i)$. Now we assume that $(E, F)$ is a maximal $\theta$-pair of $\theta(L_i)$ such that $(C, D) \leq (E, F)$. Therefore, $C \leq E, D = F, \frac{C}{D} \leq \frac{E}{D}$ and $\frac{C}{D} \leq \frac{G}{D}$. This shows that $(C, D)$ is a maximal $\theta$-pair of $\theta(L_i)$ and the proof is complete. ∅

**Theorem 5.** There is no $3\theta$-pair group.

**Proof.** Let $G$ be a $3\theta$-pair group. By Lemma 7, there is no maximal subgroup $M$ of $G$ such that $\theta(M) = \theta(G)$. Our main proof will consider a number of cases.

Case 1. **There are two maximal subgroups $M$ and $L$ of $G$ such that $|\theta(M)| = 2$ and $|\theta(L)| = 1$.** Assume that $(B, M_G), (C, D) \in \theta(M)$ and $(A, L_G) \in \theta(L)$. We can see that $C \leq G$ and $C \neq G$. We claim that $G$ has at least three maximal subgroups. By lemma 6, $G$ has at least two maximal subgroups. Assume that $G$ has exactly two maximal subgroups, say $M$ and $L$. Thus, by a theorem of Khazal, mentioned above, $G$ is cyclic and so $(A, L_G) = (G, L), (B, M_G) = (G, M)$. Since $\frac{G}{L}$ is a simple group, we have $(M, \Phi(G)) \in \theta(L)$, a contradiction. Therefore $G$ has at least three maximal subgroups. We now see that $M_G \neq L_G$. Thus, for any maximal subgroup $X$ of $G, X_G = L_G$ or $X_G \leq M_G$. Suppose $A = G$. If $L$ is non-normal
and $g \in G - N_G(L)$, then $(L^g, L_G) \in \theta(L)$, which is impossible. So $L \leq G$ and we can see that $(M_G, L \cap M_G) \in \theta(L)$, a contradiction. Thus $A \neq G$ and so $A \leq M_G$. Also, $C \leq L_G$ and hence $C \leq L_G \leq A \leq M_G$, which is a contradiction.

Case 2. $G$ is $\theta$-maximal and there are maximal subgroups $M$, $L$ and $K$ of $G$ such that $(A, L_G) \in \theta(L)$, $(B, K_G) \in \theta(K)$ and $(C, M_G) \in \theta(M)$. By Lemma 9 and Case 1, $\{(M_G \mid M < G)\} = 3$. We claim that one of the subgroups $A$, $B$ and $C$ is equal to $G$ and the other two are proper. To do this, suppose $A = C = G$. Then $M, L \subset G$ and $(L, M \cap L) \in \theta(M)$, which is impossible. Therefore, we can assume that $A \neq G$, $B \neq G$ and $|\theta(G/A)| = |\theta(G/B)| = 1$. Suppose $R$ and $S$ are the unique maximal subgroups of $G/A$ and $G/B$, respectively. Thus, $(G/A, R)$ and $(G/B, S) \in \theta(G/A)$ and $(G/A, R) \in \theta(G/B)$. This shows that $(G, R)$ and $(G, S)$ are $\theta$-pairs of $G$ and so $R = S$.

We can assume that $M < G$ and $A, B \leq M$. Now $(L_G, L^G), (M_G, M^G) \in \theta(G/L_G)$ and $|\theta_{max}(G/L_G)| \leq 3$. Therefore, $|\theta_{max}(G/L_G)| = 3$ and there exists another $\theta$-pair $(L^G, L^G) \in \theta(G/L_G)$. It is easy to see that $L_G \subset K_G$. Using similar argument as in above, $K_G \subset L_G$ and so $L_G = K_G$, which is a contradiction.

**Theorem 6.** There exists a group with exactly $n$ $\theta$-pair, for $n \neq 2, 3$.

**Proof.** For $n = 1$, a cyclic group of prime power order has exactly one $\theta$-pair. Suppose $n \geq 4$ and $G = \mathbb{Z}_{p^n}^q$. Then $G$ has exactly two maximal subgroups $M$ and $N$ of orders $p^n$ and $p^{n-1}q$, respectively. Suppose $A_i$ and $B_i$, $0 \leq i \leq n$, are subgroups of $G$ of order $p^i$ and $p^i q$. Now it is easy to see that $\theta(M) = \{(B_i, A_i) \mid 0 \leq i \leq n\}$ and $\theta(N) = \{(A_n, A_{n-1}), (B_n, B_{n-1})\}$. Therefore $G$ has exactly $n+3$, $\theta$-pair, proving the result.

We conclude this paper with the following open question:

**Question:** Is there a non-abelian finite group with exactly $n$ $\theta$-pairs, for a given positive integer $n \neq 2, 3$?

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