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The class number one problem for the non-normal sextic CM-fields. Part 2.

Gérard Boutteaux and Stéphane Louboutin

Abstract. We delineate the determination of all the non-normal sextic CM-fields with class number one and whose maximal totally real subfields are non-normal cubic field. There are 367 non-isomorphic such fields. Since we had already proved elsewhere that there are 19 non-isomorphic non-normal sextic CM-fields with class number one and whose maximal totally real subfields are cyclic cubic fields, we are now in a position to conclude that there are 386 non-isomorphic non-normal sextic CM-fields which have class number one.

1. Introduction

Lately, great progress have been made towards the determination of all the normal CM-fields with class number one. Due to the work of S.-H. Kwon, Y. Lefeuvre, F. Lemmermeyer, S. Louboutin, R. Okazaki, Y.-H. Park and Y.-S. Yang, all the normal CM-fields of degrees less than or equal to 48 with class number one are known (with only partial solutions in the special cases of normal fields of degree 32 and 48). In contrast, up to now the determination of all the non-normal CM-fields with class number one and of a given degree has only been solved for quartic fields (see [LO]). The present piece of work is an abridged version of half the work to be completed in [Bou] (the PhD thesis of the first author under the supervision of the second author): the determination of all the non-normal sextic CM-fields with class number one, regardless whether their maximal totally real subfield is a non-normal totally real cubic field (the situation dealt with in the present paper) or a real cyclic cubic field (the situation dealt with in [BL]).

Let $K$ be a CM-field, i.e. $K$ is a totally imaginary number field, hence of even degree $2n \geq 2$, and $K$ is a quadratic extension of its maximal totally real subfield $k$, hence $k$ is of degree $n$. We let $h_K$, $Q_k \in \{1,2\}$ and $w_K$ denote its relative class number, its Hasse unit index and the number of complex roots of unity contained.
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in \( K \), respectively. Note that \( \omega_K = \omega_A \) where \( A \) is the maximal abelian subfield of \( K \). We have (see [Wa, Chapter 4]):

\[
(1) \quad h_K = \frac{Q_K \omega_K}{(2\pi)^n} \sqrt{d_K} \text{Res}_{\mathbb{Q}(\alpha)}(\chi_K).
\]

where \( d_K \) and \( d_k \) denote the absolute values of the discriminants of \( K \) and \( k \), respectively.

2. On CM-fields of odd class numbers

**Proposition 1.** Let \( K \) be a CM-field of degree \( 2n \), \( n > 1 \) odd, and let \( \mathcal{F}_{K/k} \) denote the finite part of the conductor of the quadratic extension \( K/k \).

1. At least one prime ideal of \( k \) is ramified in the quadratic extension \( K/k \). Therefore, \( d_K > 3d_k \) and the narrow class number \( h^*_K \) of \( K \) divides the class number \( h_K \). Consequently, if \( h_K \) is odd then \( h^*_K = h_k \) and every totally positive unit of \( k \) is the square of some unit of \( k \).

2. Assume that \( h_K \) is odd. Then, exactly one prime ideal \( \mathcal{Q} \) of \( k \) is ramified in the quadratic extension \( K/k \) and \( Q_K = 1 \). Finally, if \( \mathcal{Q} \) is above an odd rational prime \( q \), then \( \mathcal{F}_{K/k} = \mathcal{Q}, q \equiv 3 \pmod{4} \) and the inertia degree \( f \) of \( \mathcal{Q} \) is odd.

**Proof.**

1. Let \( \mathcal{F}_{K/k} \) denote the finite part of the conductor of the quadratic extension \( K/k \) and let \( \chi \) be the quadratic character associated with this quadratic extension \( K/k \). According to class field theory, there exists some primitive quadratic character \( \chi_0 \) on the multiplicative group \( (A_k/\mathcal{F}_{K/k})^* \) (where \( A_k \) denotes the ring of algebraic integers of \( k \)) such that for any \( \alpha \in A_k \) we have \( \chi(\alpha) = \nu(\alpha) \chi_0(\alpha) \) where \( \nu(\alpha) \in \{ \pm 1 \} \) denotes the sign of the norm \( N_{K/k}(\alpha) \) of \( \alpha \). In particular, if \( K/k \) is unramified at all the finite places of \( k \) then for any algebraic unit \( e \) of \( k \) we must have \( \nu(e) = +1 \). Taking \( e = -1 \) for which \( \nu(e) = (-1)^n \), we obtain that \( n \) must be even.

2. Let \( t \) denote the number of primes ideals of \( k \) which are ramified in the quadratic extension \( K/k \). According to the first point of this Proposition, we have \( t \geq 1 \). Assume that \( h_K \) is odd. Since \( 2^{n-1} \) divides \( h_K \) (see [Lou3, Proposition 6]), we have \( t = 1 \). Moreover, since \( 2^n \) divides \( h^*_K \) if \( Q_K = 2 \) (see [Lou3, Proposition 6]), we have \( Q_K = 1 \). Now, if \( q \) is odd, then \( K/k \) is tamely ramified, hence \( \mathcal{F}_{K/k} = \mathcal{Q} \). Since the multiplicative group \( (A_k/\mathcal{Q})^* \) is cyclic of order \( q^f - 1 \) and since \( \nu(-1) = (-1)^n = -1 \) (by the first point), we have \( 1 = \chi((-1)) = \nu(-1) \chi_0(-1) = -\chi_0(-1) = -(q^f - 1)/2 \), which yields \( q^f = 3 \pmod{4} \). Hence, \( q \equiv 3 \pmod{4} \) and \( f \) odd. 

**Lemma 2.** Let \( K/k = k(\sqrt{\alpha})/k \) be a quadratic extension of number fields, where \( \alpha \) is an algebraic integer of \( k \). There exists some ideal \( I \) of \( k \) such that \( (4\alpha) = I^2 \mathcal{F}_{K/k} \).

**Proof.** Let \( A_k \) denote the ring of algebraic integers of \( k \) and \( \mathcal{D}_{K/k} \) denote the different of the quadratic extension \( K/k \). Since \( \mathcal{D}_{K/k} = \gcd((2\sqrt{\alpha}); \alpha \in A_k \) and \( K = \)
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for any \( a \in \mathbb{A}_k \) such that \( K = k(\sqrt[3]{a}) \) there exists an integral ideal \( \mathcal{I}_K \) of \( K \) such that \( (2\sqrt[3]{a}) = \mathcal{I}_K^2 \). Taking relative norms, we do get \( (4\alpha) = \mathcal{I}_K^2 \mathcal{P}_{K/k} = \mathcal{I}_K^2 \mathcal{F}_{K/k} \) where \( \mathcal{I} = \mathcal{N}_{K/k}(\mathcal{I}_K) \).

**Proposition 3.** Let \( K \) be a CM-field of degree \( 2n \), \( n > 1 \) odd, let \( \mathcal{F}_{K/k} \) denote the finite part of the conductor of the quadratic extension \( K/k \), assume that the class number \( h_K \) of \( K \) is odd, let \( Q \) denote the unique prime ideal \( Q \) of \( k \) which is ramified in the quadratic extension \( K/k \) (see Proposition 1) and let \( q > 2 \) denote the rational prime below \( Q \).

1. If \( q > 2 \) then \( \mathcal{F}_{K/k} = Q \).
2. If \( Q = 2 \) and \( \sqrt{-1} \not\in \mathcal{F}_{K/k} \), then there exists \( r > 1 \) odd such that \( \mathcal{F}_{K/k} = Q^r \).
3. If \( q = 2, \sqrt{-1} \in \mathcal{F}_{K/k} \) or \( q > 2 \) and \( \sqrt{-1} \not\in \mathcal{F}_{K/k} \), then \( q \) is neither totally ramified in \( k/Q \) nor inert in \( k/Q \).

**Proof.** Let \( \alpha \) be any totally positive algebraic element of \( k \) such that \( K = k(\sqrt[3]{\alpha}) \). There exists some integral ideal \( \mathcal{I} \) of \( k \) such that \( (4\alpha) = \mathcal{I}^2 \mathcal{F}_{K/k} \), by Lemma 2. Set \( h := h^+_K = h_k \), which is odd by Proposition 1.

1. \( K/k \) is tamely ramified.
2. If \( \mathcal{F}_{K/k} = Q^r \) with \( r \) even, then \( (4\alpha) = \mathcal{J}^2 \) with \( \mathcal{J} := \mathcal{I}Q^{r/2} \). Since \( h \) is odd and since both \( \mathcal{J}^2 \) and \( \mathcal{J}^h \) are principal in the narrow sense, \( \mathcal{J} \) is principal in the narrow sense and there exists some totally positive unit \( e \) of \( k \) such that \( 4\alpha = e\beta^2 \). Since \( e \) is the square of some unit of \( k \) (see Point 1 of Proposition 1), we obtain \( 4\alpha = \gamma^2 \) for some \( \gamma \in k \) and \( K = k(\sqrt{-\alpha}) = k(\sqrt{-1}), \) a contradiction.
3. Assume that \( (q) = Q^r \) is totally ramified in \( k/Q \) or that \( (q) = Q \) is inert in \( k/Q \). Since \( \mathcal{F}_{K/k} = Q^r \) for some odd \( r \) (by the previous two points) and since \( n \) is odd, we obtain \( (4\alpha) = \mathcal{J}^2 \) for some integral ideal \( \mathcal{J} \) of \( k \). As in the previous point, we obtain \( 4\alpha = \gamma^2 \) for some \( \gamma \in k \) and \( K = k(\sqrt{-\alpha}) = k(\sqrt{-q}), \) a contradiction. •

### 3. Bounds on residues

**Lemma 4.** Let \( K \) be a totally imaginary number field of degree \( 2n \geq 2 \). Set \( \lambda_n = n/(\gamma + \log(2\pi)) - 1, \) where \( \gamma = 0.577 \ldots \) denotes Euler’s constant. Then, \( \frac{1}{2} \leq \beta < 1 \) and \( \zeta_K(\beta) \leq 0 \) imply

\[
\text{Res}_{s=1}(\zeta_K) \geq (1 - \beta)d_K^{(\beta - \frac{1}{2})/2}(1 + \lambda_n(1 - \beta))(1 - 2\pi nd_K^{(\beta - 1)/2n}d_K^{-2n}e_{K/n}).
\]

**Proof.** Let \( K \) be a totally imaginary number field of degree \( 2n \geq 2 \). Assume that \( \zeta_K(\beta) \leq 0 \) for some \( \beta \) satisfying \( \frac{1}{2} \leq \beta < 1 \). According to the proof of [Lou2, Prop A], Hecke’s integral representations of Dedekind zeta functions yield

\[
\text{Res}_{s=1}(\zeta_K) \geq (1 - \beta)d_K^{(\beta - \frac{1}{2})/2}f_n(\beta)(1 - 2\pi nd_K^{(\beta - 1)/2n}).
\]

where \( f_n(\beta) = \beta(\Gamma(\beta)/(2\pi)^{\beta - 1})^{n} \) is positive and log-convex in the range \( \beta > 0 \), hence convex in the same range. Since \( f'_n(1) = (f'_n/f_n)(1) = -\lambda_n, \) we obtain \( f_n(\beta) \geq f_n(1) + (\beta - 1)f'_n(1) = 1 + \lambda_n(1 - \beta) \) in the range \( 0 < \beta < 1 \). •
To get the term \((1 - \beta) \frac{d_K^{(\beta - 1)/2}}{d_K}\) as large as possible, we would like to be allowed to choose \(\beta = 1 - 2/(\log d_K)\), and \(\zeta_K(1 - (2/\log d_K)) \leq 0\) would imply \(\text{Res}_{s=1}(\zeta_K) \geq (2 + o(1))/\log d_K\). where \(o(1)\) is an explicit error term which approaches to zero as \(d_K\) goes to infinity. Discarding this error term, we would obtain (see the proofs of Theorems 8 and 12): Let \(K\) be a sextic CM-field and assume that \(\zeta_K(1 - (2/\log d_K)) < 0\) would imply \(\text{Res}_{s=1}(\zeta_K) > 0\). Then,
\[
h_K \geq \frac{4\sqrt{d_K/d_F}}{e^{2/(\log d_F)^2}} \log d_K,
\]
and \(h_K = 1\) would imply \(d_K \leq 2 \cdot 10^7\), \(\mu_K := d_K^{1/6} \leq 7500\) and \(d_F \leq 2 \cdot 10^{11}\). Since we do not know how to prove that \(\zeta_K(1 - (2/\log d_K)) < 0\) (which would hold true if \(\zeta_K\) had no real zero in the range \(1 - (2/\log d_K) \leq s < 1\)), we will have to be a little more clever and we will obtain worse bounds (see Theorems 8 and 12 below).

Proposition 5.

1. Let \(a > 0\) be given and let \(K\) range over CM-field sextic fields

(a) There exists \(d_a\) effective such that \(d_K \geq d_a\) and \(\zeta_K(1 - (1/a \log d_K)) \leq 0\) imply
\[
\text{Res}_{s=1}(\zeta_K) \geq \frac{1}{ae^{1/2a} \log d_K}.
\]

(b) For any \(c > 1\), there exists \(d_a, c\) effective such that \(d_K \geq d_{a, c}\), \(1 - (1/a \log d_K) \leq \beta < 1\) and \(\zeta_K(\beta) \leq 0\) imply
\[
\text{Res}_{s=1}(\zeta_K) \geq \frac{1 - \beta}{ce^{1/2a}}.
\]

2. (See [Lou7]). If \(F\) is totally real cubic number field, then
\[
\text{Res}_{s=1}(\zeta_F) \leq \frac{1}{3} \log^2 d_F,
\]
and \(\frac{1}{3} \leq \beta < 1\) and \(\zeta_F(\beta) = 0\) imply
\[
\text{Res}_{s=1}(\zeta_F) \leq \frac{1 - \beta}{48} \log^3 d_F.
\]

Proof. According to Lemma 4, \(\zeta_K(1 - (1/a \log d_K)) \leq 0\) implies
\[
\text{Res}_{s=1}(\zeta_K) \geq \frac{F_a(d_K)}{e^{1/2a} \log d_K}
\]
where \(F_a(x) = (1 + (\lambda_3/a \log x))(1 - 6\pi e^{1/6}/x^{1/6})\) is clearly \(\geq 1\) for \(x \geq d_a\) large enough (notice that \(\lambda_3 = 6.245278\cdot\cdot\cdot\)). In the same way, \(1 - (1/a \log d_K) \leq \beta < 1\) and \(\zeta_K(\beta) \leq 0\) imply
\[
\text{Res}_{s=1}(\zeta_K) \geq \frac{1 - \beta}{e^{1/2a}} G_a(d_K)
\]
where \(G_a(x) = 1 - 6\pi e^{1/6}/x^{1/6}\) is \(\geq 1/c\) for \(x \geq d_{a, c} = (\frac{2\pi e^{1/6}}{c})^{1/6}\) large enough. •
Lemma 6. (See [LLO, Lemma 15].) Set $c_1 = (3 + 2\sqrt{2})/2 < 3$ and $c_2 = (2 + \sqrt{3})/4 < 1$. The Dedekind zeta function of a number field $M$ has at most one real zero in the range $1 - (1/c_1 \log d_M) \leq s < 1$ and at most two real zeros in the range $1 - (1/c_2 \log d_M) \leq s < 1$.

4. Non-normal sextic CM-fields with non-normal maximal totally real real cubic subfields

From now on, we let $K$ denote a non-normal sextic CM-field whose maximal totally real subfield $F = k$ is a non-normal cubic field. We write $K = F(y/d)$ where $S$ is a totally positive algebraic element of $F$. We let $F'$ denote the normal closure of $F$. Hence, $F'$ is a real non-abelian normal sextic field with Galois group the dihedral group of order six, and we let $L_{re}$ denote the only quadratic subfield of $F'$. Hence, $L_{re} = \mathbb{Q}(\sqrt{d_{L_{re}}})$ and $F' = F \\ L_{re} = F(y/d)$. Recall that there exists some integer $f \geq 1$ such that $d_F = d_{L_{re}} f^3$ and $d_F = d_{L_{re}} f^4 = d_{L_{re}} d_{F'}^2$ (see [Mar]). In particular, $d_{L_{re}} | d_{F'} | d_F$ and $d_{L_{re}} | d_{F'}^2$.

4.1. First case

We assume that $K$ contains an imaginary quadratic subfield $L_{im} = \mathbb{Q}(\sqrt{-d_{L_{im}}})$ of discriminant $-d_{L_{im}} < 0$. Then $w_K = w_{L_{im}}, Q_K = Q_{L_{im}} = 1$ and $K = FL_{im} = F(\sqrt{-d_{L_{im}}})$, i.e. we can choose $\delta \in \mathbb{Q}$. We set $M = L_{re}, L_{im}$ and $N = FM$. Hence, $M$ is an imaginary biquadratic bicyclic number field, $N$ is a dihedral CM-field of degree 12 and we have the following (incomplete) lattice of subfields:

\[
\begin{array}{ccc}
F & & N \\
& 2 & \\
3 & & \\
& 2 & \\
\end{array}
\]

Lemma 7. Recall that we have set $c_1 = (3 + 2\sqrt{2})/2$.

(1) Since $\zeta_N/\zeta_M = (\zeta_N/\zeta_{L_{im}})^2$, any real zero of the entire function $\zeta_N/\zeta_{L_{im}}$ is at least a double zero of $\zeta_N$, hence is less than $1 - (1/c_1 \log d_N)$. Moreover, $d_N$ divides $d_{L_{im}}^2$. Hence, $\zeta_N/\zeta_{L_{im}}(s) \geq 0$ for $1 - (1/c_1 \log d_N) \leq s < 1$.

(2) Let $K = 1$. Then $w_K = w_{L_{im}}$ and $h_{L_{im}}$ divides $h_K$. In particular, $h_K = 1$ implies $h_{L_{im}} = 1$.

(3) Assume that $h_K = 1$. Then, any real zero of $\zeta_N$ is at least a double zero of $\zeta_N$. Hence, $\zeta_N(s) \leq 0$ in the range $1 - (1/c_1 \log d_N) \leq s < 1$, by Lemma 6.

Proof. Since $d_N/d_M = (d_K/d_{L_{im}})^2$ and since $d_N$ divides $(d_{L_{im}}, d_{L_{im}})^3$, (use the conductor-discriminant formula), we deduce that $d_N | d_{L_{im}} d_{L_{im}}^3 | d_K$. Since $K/L_{im}$ is a cubic extension, then $Q_K = Q_{L_{im}} = 1$ and $h_{L_{im}} = h_{L_{im}}^2$ divides $h_K$ (see [LLO, Theorem 5]). Therefore, if $h_K = 1$ then $L_{im}$ must be one of the nine imaginary
quadratic fields of class number one (i.e. \(d_L \in \{4, 8, 3, 7, 11, 19, 43, 67, 163\}\), which implies \(\zeta_L(s) < 0\) for \(0 < s < 1\), and using \(\zeta_N = (\zeta_K / \zeta_L)^2 \zeta_M\), we obtain the last assertion. •

**Theorem 8.** (Compare with Theorem 12 below). Let \(K = FLm\) be a non-normal sextic CM-field which is a compositum of a non-normal totally real cubic field \(F\) and of an imaginary quadratic field \(Lm\). Assume that \(\zeta_{Lm}(s) < 0\) in the range \(0 < s < 1\). Then,

\[
h_K^> \geq \frac{\sqrt{d_K/d_F}}{144(\log d_F)^2 \log d_K} \quad \text{for } d_K \geq 10^{19}.
\]

Moreover, \(h_K^> = 1\) implies \(d_K \leq 2 \cdot 10^{27}\), \(\rho_K := d_K^{1/6} \leq 34500\) and \(d_F \leq 3 \cdot 10^{13}\).

**Proof.** Set \(c_1 = 3\pi^6 c_1 e^{1/6 c_1} = 287.031\ldots\) (recall that \(c_1 = (3 + 2\sqrt{2})/2\)). We first use (1) and the second point of Lemma 7 to obtain

\[
h_K^> \geq \frac{w_K}{8\pi^3} \frac{\sqrt{d_K \text{ Res}_{s=1} (\zeta_K)}}{d_F \text{ Res}_{s=1} (\zeta_F)}.
\]

We then use (3) with \(a = 3c_1\) for which \(d_a = 10^{18}\) (see the third point of Lemma 7) and we finally use (5). We obtain

\[
h_K^> \geq \frac{w_K \sqrt{d_K/d_F}}{c_1 (\log d_F)^2 \log d_K} \quad \text{for } d_K \geq 10^{19}
\]

As for the last assertion, we notice that since \(N_{F/\mathbb{Q}}(\mathcal{O}_K/F) \geq 3\) (use Proposition 1), we have \(d_K \geq 3d_F\) which implies

\[
h_K^> \geq \frac{2\sqrt{3d_F}}{c_1 (\log d_F)^2 \log(3d_F)} > 1
\]

for \(d_F \geq 3 \cdot 10^{13}\), and \(d_F \leq \sqrt{d_K/3}\) which implies

\[
h_K^> \geq \frac{8(3d_K)^{1/4}}{c_1 (\log(d_K/3))^2 \log d_K} > 1
\]

for \(d_K \geq 2 \cdot 10^{27}\). •

By using Theorem 8, necessary conditions for class numbers of CM-fields to be equal to one (see [BL], [Oka] and [Bou]) and the technique developed in [LOO, Sections 4.4 and 4.7] for computing class numbers of such non-normal sextic CM-fields, we obtain:

**Corollary 9.** There are 134 non-normal CM sextic fields \(K = FLm\) with class number one which are composita of a non-normal totally real cubic field \(F\) and of an imaginary quadratic field \(Lm\), the ones given in TABLES 1, 2 and 3 below.
4.2. Second case

We assume that $K = F(\sqrt[d]{\alpha})$ contains no imaginary quadratic field (where $\alpha$ is a totally positive algebraic integer of $F$), i.e., we assume that we cannot choose $\delta \in \mathbb{Z}$. We let $\delta_1 = \delta$, $\delta_2$ and $\delta_3$ denote the three conjugates of $\alpha$ and set $d = M \cap \mathbb{Q} = N_\mathbb{Q}(\delta)$. We then set $K = F(\sqrt[d]{\alpha})$, $K'_1 = F(\sqrt[d]{\alpha}, \delta)$, $K'_2 = F(\sqrt[d]{\alpha}, \delta_2)$, $K'_3 = F(\sqrt[d]{\alpha}, \delta_3)$. Hence, $M/F$ is bicyclic biquadratic and $K/F$, $K'/F$ and $K''/F$ are the three quadratic subextensions (with base field $F$) of the extension $M/F$.

We also set $N = F(\sqrt[3]{\alpha}, \sqrt[3]{\beta}, \sqrt[3]{\gamma})$ and $M_1 = F(\sqrt[3]{\alpha})$ (recall that $F$ denotes the normal closure of $K$ and the $M_j$'s are conjugate subfields of $N$). We have the following incomplete lattice of subfields:

We will set $M = M_1$ and $K = K'_1$.

**Lemma 10.** For $1 < i < 3$, the $M_i$ are pairwise distinct but isomorphic. Hence, $[N : F] = 8$ and $Gile(N/F) = C_2 \times C_2 \times C_2$.

**Proof.** Assume that two of them, say $M_1$ and $M_j$ are equal. Then for some $OJ \in F$ we have $O2 = a^{-1}$. Hence, $d = \sqrt[3]{\alpha} \in \mathbb{F}$ and $M_1 = F(\sqrt[3]{\alpha}, \sqrt[3]{\beta}, \sqrt[3]{\gamma})$ is normal. Since the $M_j$'s are conjugate subfields of $F$, we obtain $M - M_1 - M_j - M_3 = F(\sqrt[d]{\alpha}, \delta_2, \delta_3) = M_3$ normal since $\sqrt[d]{\alpha}$ is one of the three quadratic subextensions of $M/F$ and since $iF$ is the totally imaginary, we obtain that $K - F(\sqrt[d]{\alpha})$ of $K = F(\sqrt[d]{\alpha}, \delta_2, \delta_3)$, contrary to the hypothesis that $K$ contains no imaginary quadratic subfield.

**Lemma 11.** Recall that we have set $c_1 = (2 + \sqrt{3})/4$. We have the following factorization of Dedekind zeta functions:

\begin{align}
(8) \quad & CN/CA \quad \frac{1}{\zeta(M/F)} \\
(9) \quad & = (CM/F)(CM/0)^3
\end{align}
it

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(10)  \( \text{CM/C } F = \left( \text{C/C/CF} \right) \left( \text{CK'}/\text{CF} \right) \). Therefore, any complex zero of the entire function \( \text{CK}/\text{CF} \) is at least a triple zero of the Dedekind zeta function \( \zeta_N \), hence is less than \( 1 - \left( 1/(24c^2 \log d_K) \right) \), by Lemma 6. Moreover, \( d_N \) divides \( d_K^6 \). Hence, \( (\zeta_K/\zeta_F)(s) \geq 0 \) for \( 1 - \left( 1/(24c^2 \log d_K) \right) \leq s < 1 \).

Proof. Since \( \text{Gal}(N/F) \) is the elementary 2-group \( \text{C}_2 \times \text{C}_2 \times \text{C}_2 \), using abelian \( L \)-functions we easily obtain (8). To deduce (9), we notice that \( (M_3)_t = \text{CM}_2 = \text{CMH} \) for the \( M_t \) are pair wise isomorphic for \( 1 \leq i \leq 3 \). In the same way, since \( M/F \) is bicyclic biquadratic with quadratic subextensions \( K_j/F \), \( K_j'F \) and \( F/F \), we obtain (10). Let us finally prove that \( d_N \) divides \( d_K^6 \). To begin with, let \( f \) denote the norm of the conductor of any one the quadratic extensions \( M_i/F \) (since the \( M_t \) are pairwise isomorphic for \( 1 \leq i \leq 3 \), we have \( f_1 = f_2 = f_3 \)). Using the conductor-discriminant formula, we obtain that \( d_N \) divides \( d_K^6 (f_1 f_2 f_3)^4 = d_F^f f_1² \). Since \( d_M = d_K^6 f \), we obtain that \( d_N \) divides \( d_K^6 / d_F^f \). Now, according to [Sta, Lemma 6], the different \( D_{M/F} \) divides the different \( D_{K/F} \). Hence, \( d_M = d_K^6 N_{M/Q}(D_{M/F}) \) divides \( d_K^6 (N_{M/Q}(D_{M/F}))^{\phi} = d_F^f d_K^6 \). Hence, \( d_N \) divides \( (d_K^6 / d_F^f) \) and noticing that \( d_F \) divides \( d_F^f \), we finally obtain that \( d_N \) divides \( d_K^6 / d_F^f \), hence divides \( d_K^6 \).

Theorem 12. (Compare with Theorem 8 above). Let \( K \) be a non-normal sextic CM-field with maximal totally real subfield a non-normal cubic field \( F \) and assume that \( K \) contains no imaginary quadratic subfield. We have

\[
\begin{align*}
\text{(11)} & \quad h_K \geq \frac{\sqrt{d_K/d_F}}{355(\log d_F)^3 \log d_K} \quad \text{for } d_K \geq 10^{32}. \\
\text{Moreover, } h_K = 1 & \quad \text{implies } d_K \leq 2 \cdot 10^{29}, \quad \rho_K := d_K^{1/6} \leq 76500 \quad \text{and } d_F \leq 3 \cdot 10^{14}. \\
\text{Proof.} & \quad \text{Set } c_2 = 12 \pi^2 c_2^{1/48c_2} = 354.989 \cdots \text{ (recall that } c_2 = (2 + \sqrt{3})/4). \text{ Now, there are two cases to consider.} \\
& \quad \text{First, assume that } \zeta_K \text{ has a real zero } \beta \text{ in } [1 - (1/24c_2 \log d_K), 1]. \text{ Then, } \zeta_K(\beta) = 0 \text{ or } 0. \text{ Therefore (use (4) with } a = 24c_2 \text{ and } c = 48 \text{ for which } d_K \leq 6 \cdot 10^5), \\
& \quad \text{Res}_{s=1}(\zeta_K) \geq \frac{1 - \beta}{48^c_2^{1/48c_2}} \quad \text{for } d_K \geq 6 \cdot 10^7. \\
\text{Using (1), (6), (12) and } Q_K w_K \geq 2, \text{ we get } \\
\text{(13)} & \quad h_K \geq \frac{\sqrt{d_K/d_F}}{4\pi^2 c_2^{1/48c_2} \log d_F} \quad \text{for } d_K \geq 6 \cdot 10^7 \\
& \quad (\text{for } d_K \geq d_F). \\
& \quad \text{Second, assume that } \zeta_K \text{ has no real zero in } [1 - (1/24c_2 \log d_K), 1]. \text{ Then } \zeta_K(1 - (1/24c_2 \log d_K)) < 0 \text{ and according to Lemma 11, we conclude that } \\
& \quad \zeta_K(1 - (1/24c_2 \log d_K)) \leq 0.
\end{align*}
\]
Therefore (use (3) with \( a = 24c_2 \) for which \( d_a = 10^{22} \)),

\[
(14) \quad \text{Res}_{x=1}(\zeta_K) \geq \frac{1}{24c_2^{1/4k+1}} \log d_K \quad \text{for } d_K \geq 10^{22}.
\]

Using (1), (5), (14) and \( Q_Kw_K \geq 2 \), we get

\[
(15) \quad h_K \geq \sqrt{\frac{d_K/\delta_F}{12n^2c_2^{1/4k+2}(\log \delta_F)^2 \log d_K}} \quad \text{for } d_K \geq 10^{22}.
\]

Since the right hand side of (13) is always greater than or equal to the right hand side of (15) (for \( c_2 > 1/2 \)), the lower bound (15) always holds.

Finally, the proof of the last assertion of this Theorem is similar to the proof of the last assertion of Theorem 8.

**Lemma 13.** Assume that \( h_K \) is odd, let \( Q \) be the only prime ideal of \( F \) which is ramified in \( K/F \) (see Proposition 1) and let \( q > 2 \) be the rational prime in \( Q \).

Assume that \( \langle q \rangle = \mathbb{Q}_2/Q_2 \) is partially ramified in \( F/Q \). Then, \( \mathcal{F}_{K/F} = Q_2 \) for some odd \( r > 1 \).

**Proof.** Let \( \alpha \) be any totally positive algebraic integer of \( F \) such that \( K = F(\sqrt{-\alpha}). \)

There exists some integral ideal \( I \) of \( F \) such that \( (4\alpha) = I^2 \mathcal{F}_{K/F} \) (see Lemma 2) and \( \mathcal{F}_{K/F} = Q_1^r \) or \( Q_2^r \) for some odd \( r > 1 \) (see Proposition 3). Now, \( \mathcal{F}_{K/F} = Q_1^r \) would imply \( (4q^r \alpha) = J^2 \) with \( J = IQ_1^s \mathbb{Q}_2 \). Since the narrow class number of \( F \) is odd (by Point 1 of Proposition 1), as in the proof of Proposition 3 we would obtain \( 4q^r \alpha = \gamma^2 \) for some \( \gamma \in F \). Hence, \( K = F(\sqrt{-\alpha}) = F(\sqrt{-q^r}) = F(\sqrt{-q}) \) would contain an imaginary quadratic subfield, a contradiction.

**Lemma 14.** Let \( k \) be a number field of degree \( n \geq 1 \), let \( \mathcal{A}_k \) denote its ring of algebraic integers and let \( Q \) be a prime ideal of \( k \) above the rational prime \( q = 2 \).

Let \( e \geq 1 \) denote the ramification index of \( Q \). If there exists a primitive quadratic character \( \chi_0 \) on the multiplicative group \( (\mathcal{A}_k/Q^r)^* \), then \( e < 2e + 1 \), which implies \( 2 \leq r < 2n + 1 \).

**Proof.** If \( m = \sum_{i=0}^{n-1} a_i 2^i \) is the binary expansion of a non-negative integer \( m \geq 0 \), then it is well known that the 2-adic valuation of \( m! \) is equal to \( v_2(m) = \sum_{i=0}^{n-1} a_i \). Hence,

\[
\nu_2(2m!) = S(m) - 2m
\]

(notice that \( S(2m) = S(m) \)). Let \( k_Q \) be the \( Q \)-adic completion of \( k \) and let \( \nu_Q \) denote the associated valuation. The Taylor series expansion

\[
\sqrt{1+x} = 1 + \sum_{m \geq 1} (-1)^{m-1} \frac{(2m)!}{4^m(m!)^2} x^m
\]

is convergent if and only if

\[
\lim_{m \to +\infty} \nu_Q\left( \frac{(2m)!}{4^m(m!)^2} x^m \right) = \lim_{k \to +\infty} e(S(m) - 2m) + m\nu_Q(x) = +\infty.
\]
Gérard Boutteaux and Stéphane Louboutin

hence if and only if \( \nu_{G}(x) > 2e \). Therefore, if \( \alpha \in A_k \) satisfies \( \alpha \equiv 1 \pmod{Q^{2e+1}} \), there exists \( \beta \in A_k \) such that \( \alpha \equiv \beta^{2} \pmod{Q^{2e+1}} \), which implies \( \chi_{0}(\alpha) = +1 \). Since there must exist some \( \alpha \in A_k \) satisfying \( \alpha \equiv 1 \pmod{Q^{r-1}} \) and \( \chi_{0}(\alpha) \neq 1 \) is \( \chi_{0} \) is primitive, we do obtain that \( r \) must satisfy \( r - 1 < 2e + 1 \). Finally, since the order of the group \( (A_k/Q^r)^* \) must be even, we have \( r \geq 2 \). •

**Lemma 15.** If \( (2) = Q_1 Q_2^2 \) is partially ramified in \( F/Q \) and if there exists a primitive quadratic character \( \chi_{0} \) on the multiplicative group \( (A_k/Q^r)^* \), then \( r = 5 \).

**Proof.** Since \( 2 \leq r \leq 5 \) and \( r \) is odd, it suffices to prove that a quadratic character on the multiplicative group \( (A_k/Q^r)^* \) is not primitive. Since \( \ker((A_k/Q^r)^*) \rightarrow (A_k/Q^r)^* = \{\pm 1\} \), it suffices to prove that \( -1 \) is a square in \( (A_k/Q^r)^* \). This follows from the fact that if \( \pi \in Q_2 \setminus Q^r \), then \( \pi \in Q_2^3 \setminus Q_2^2 \), hence \( \pi^2 \equiv 2 \pmod{Q_2^2} \) and \( (1 + \pi)^2 \equiv -1 \pmod{Q_2^2} \). •

**Theorem 16.** Let \( K \) be a non normal sextic CM-field. Let \( F \) denote its totally real cubic subfield. Assume that \( F \) is non-normal, that only one prime ideal \( Q \) of \( F \) is ramified in the quadratic extension \( K/F \) and let \( q > 2 \) be the rational prime in \( Q \).

1. Assume that \( q > 2 \). Then, either
   (a) \( (q) = Q_1 Q_2 Q_3 \) splits completely in \( F \) and \( F_{K/q} = Q_1, Q_2 \) or \( Q_3 \) (and the three corresponding sextic fields may be non-isomorphic (e.g. cases 11 and 12 in Table 4 below)).
   (b) \( (q) = Q_1 Q_2^3 \) is partially ramified in \( F \) and \( F_{K/q} = Q_2 \).
   (c) \( (q) = Q_1 Q_2 \) in \( F \) with \( f_{Q_1} = 1 \) and \( f_{Q_2} = 2 \), then \( F_{K/q} = Q_1 \).
2. Assume that \( q = 2 \). Then, either
   (a) \( (2) = Q_1 Q_2 Q_3 \) splits completely in \( F \) and \( F_{K/q} = Q_1^3, Q_2^3 \) or \( Q_3^3 \) (and the three corresponding sextic fields may be non-isomorphic).
   (b) \( (2) = Q_1 Q_2^3 \) is partially ramified in \( F \) and \( F_{K/q} = Q_2^3 \).
   (c) \( (2) = Q_1 Q_2 \) in \( F \) and \( F_{K/q} = Q_1^2 \) or \( Q_2^2 \) (and the two corresponding sextic fields \( K \) are not isomorphic).

**Proof.** Use Proposition 3 and Lemmas 13 and 15. •

By using Theorem 12, necessary conditions for class numbers of CM-fields to be equal to one (see [BL], [Oka] and [Bou]), by using Theorem 16 for constructing the quadratic characters associated with the quadratic extensions \( K/F \) and by adapting the technique developed in [LOO, Sections 4.4 and 4.7] for computing class numbers of non-normal sextic CM-fields containing imaginary quadratic subfields, we obtain:

**Corollary 17.** There are 233 non-normal CM sextic fields \( K \) with class number one which contain no imaginary quadratic subfield and whose maximal totally real cubic subfields \( F \) are non-normal, the ones given in TABLES 4, 5 and 6 below.
5. Tables

### TABLE 1. $K = F_{L_{cm}}$ and $(q) = Q$ is inert in $F$

<table>
<thead>
<tr>
<th>case</th>
<th>$d_{L_{cm}}$</th>
<th>$d_F$</th>
<th>$d_L$</th>
<th>$f$</th>
<th>$F_{L}(x)$</th>
<th>$d_R$</th>
<th>$p_R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
<td>148</td>
<td>37</td>
<td>2</td>
<td>$x^3 + x^2 - 3x - 1$</td>
<td>$2^4$</td>
<td>$3^2$</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>316</td>
<td>16</td>
<td>1</td>
<td>$x^3 + x^2 - 4x - 2$</td>
<td>$2^4$</td>
<td>$3^2$</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>568</td>
<td>316</td>
<td>2</td>
<td>$x^3 - x^2 - 6x - 2$</td>
<td>$2^6$</td>
<td>$3^3$</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>940</td>
<td>940</td>
<td>1</td>
<td>$x^3 - 7x - 4$</td>
<td>$2^4$, $3^2$, $7^2$</td>
<td>47^2</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>321</td>
<td>321</td>
<td>4</td>
<td>$x^3 + x^2 - 4x - 1$</td>
<td>$2^6$, $3^2$, $10^2$</td>
<td>13.69</td>
</tr>
</tbody>
</table>

### TABLE 2. $K = F_{L_{cm}}$ and $(q) = Q^2$ is totally ramified in $F$

<table>
<thead>
<tr>
<th>case</th>
<th>$d_{L_{cm}}$</th>
<th>$d_F$</th>
<th>$d_L$</th>
<th>$f$</th>
<th>$F_{L}(x)$</th>
<th>$d_R$</th>
<th>$p_R$</th>
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<td>69</td>
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<td>$3^3$, $23$</td>
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<tr>
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<td>3</td>
<td>756</td>
<td>21</td>
<td>6</td>
<td>$x^3 - 6x - 2$</td>
<td>$2^4$, $3^2$, $7$</td>
<td>10.94</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>837</td>
<td>93</td>
<td>3</td>
<td>$x^3 - 6x - 1$</td>
<td>$3^3$, $31$</td>
<td>11.31</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>1620</td>
<td>518</td>
<td>5</td>
<td>$x^3 - 12x - 14$</td>
<td>$2^4$, $3^2$, $5$</td>
<td>14.10</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>1944</td>
<td>24</td>
<td>9</td>
<td>$x^3 - 9x - 6$</td>
<td>$2^6$, $3^3$</td>
<td>14.98</td>
</tr>
<tr>
<td>6</td>
<td>3</td>
<td>2241</td>
<td>249</td>
<td>3</td>
<td>$x^3 - 9x - 5$</td>
<td>$3^3$, $83^2$</td>
<td>15.71</td>
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<td>3</td>
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<td>12</td>
<td>15</td>
<td>$x^3 - 15x - 20$</td>
<td>$2^3$, $3$, $5$</td>
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</tr>
<tr>
<td>8</td>
<td>3</td>
<td>2808</td>
<td>312</td>
<td>3</td>
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<td>16.94</td>
</tr>
<tr>
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<td>3</td>
<td>3132</td>
<td>348</td>
<td>3</td>
<td>$x^3 - 18x - 20$</td>
<td>$2^4$, $3^3$, $29^2$</td>
<td>17.57</td>
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<tr>
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<td>60</td>
<td>9</td>
<td>$x^3 - 18x - 12$</td>
<td>$2^4$, $3^3$, $5^3$</td>
<td>20.34</td>
</tr>
<tr>
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<td>3</td>
<td>5940</td>
<td>165</td>
<td>6</td>
<td>$x^3 - 22x - 6$</td>
<td>$2^4$, $3^2$, $11^2$</td>
<td>21.74</td>
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<tr>
<td>13</td>
<td>3</td>
<td>6588</td>
<td>732</td>
<td>3</td>
<td>$x^3 - 15x - 16$</td>
<td>$2^4$, $3^3$, $67^2$</td>
<td>22.51</td>
</tr>
<tr>
<td>14</td>
<td>3</td>
<td>8289</td>
<td>921</td>
<td>3</td>
<td>$x^3 - 21x - 12$</td>
<td>$3^3$, $307^2$</td>
<td>24.30</td>
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<td>3</td>
<td>9153</td>
<td>113</td>
<td>9</td>
<td>$x^3 - 21x - 4$</td>
<td>$3^3$, $11^3$</td>
<td>25.12</td>
</tr>
<tr>
<td>16</td>
<td>3</td>
<td>11880</td>
<td>1320</td>
<td>3</td>
<td>$x^3 - 27x - 34$</td>
<td>$2^6$, $3^3$, $11^2$</td>
<td>27.40</td>
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<td>17</td>
<td>3</td>
<td>12744</td>
<td>1416</td>
<td>3</td>
<td>$x^3 - 21x - 30$</td>
<td>$2^6$, $3^3$, $59^2$</td>
<td>28.05</td>
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<tr>
<td>18</td>
<td>3</td>
<td>20835</td>
<td>2265</td>
<td>3</td>
<td>$x^3 - 33x - 48$</td>
<td>$3^3$, $5^2$, $151^2$</td>
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<tr>
<td>19</td>
<td>4</td>
<td>148</td>
<td>37</td>
<td>2</td>
<td>$x^3 + x^2 - 3x - 1$</td>
<td>$2^4$, $37^2$</td>
<td>8.39</td>
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<tr>
<td>20</td>
<td>4</td>
<td>494</td>
<td>101</td>
<td>2</td>
<td>$x^3 + x^2 - 5x - 1$</td>
<td>$2^5$, $101^2$</td>
<td>11.73</td>
</tr>
<tr>
<td>21</td>
<td>4</td>
<td>564</td>
<td>141</td>
<td>2</td>
<td>$x^3 + x^2 - 5x - 3$</td>
<td>$2^3$, $3^2$, $47^2$</td>
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</tr>
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<td>22</td>
<td>4</td>
<td>756</td>
<td>21</td>
<td>6</td>
<td>$x^3 - 6x - 2$</td>
<td>$2^5$, $3^2$, $7^2$</td>
<td>14.46</td>
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<td>23</td>
<td>4</td>
<td>1524</td>
<td>361</td>
<td>2</td>
<td>$x^3 + x^2 - 7x - 1$</td>
<td>$2^4$, $3^3$, $127^2$</td>
<td>18.26</td>
</tr>
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<td>4</td>
<td>3124</td>
<td>781</td>
<td>2</td>
<td>$x^3 - 16x - 12$</td>
<td>$2^6$, $11^2$, $71^2$</td>
<td>33.20</td>
</tr>
<tr>
<td>25</td>
<td>8</td>
<td>148</td>
<td>37</td>
<td>2</td>
<td>$x^3 + x^2 - 3x - 1$</td>
<td>$2^5$, $37^2$</td>
<td>11.87</td>
</tr>
<tr>
<td>26</td>
<td>11</td>
<td>1573</td>
<td>143</td>
<td>11</td>
<td>$x^3 + x^2 - 7x - 2$</td>
<td>$11^4$, $13^2$</td>
<td>17.34</td>
</tr>
</tbody>
</table>
TABLE 3. $K = F_{13^k}$ and $(q) = \mathcal{O}_K^+$ partially ramified in $F$

<table>
<thead>
<tr>
<th>Case</th>
<th>$d_{13^k}$</th>
<th>$d_x$</th>
<th>$d_y$</th>
<th>$f(y(x))$</th>
<th>$P_K$</th>
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<tr>
<td>1</td>
<td>321</td>
<td>321</td>
<td></td>
<td>$x^3 + x^2 + 4x - 1$</td>
<td>$3^3$ $167^2$ 8.22</td>
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<tr>
<td>2</td>
<td>3764</td>
<td>141</td>
<td>2</td>
<td>$x^3 + x^2 + 5x - 3$</td>
<td>$2^4$ $3^4$ $47^2$ 9.92</td>
</tr>
<tr>
<td>3</td>
<td>983</td>
<td>993</td>
<td>1</td>
<td>$x^3 + x^2 + 6x - 3$</td>
<td>$3^2$ $331^2$ 11.98</td>
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<td>4</td>
<td>31191</td>
<td>1101</td>
<td>1</td>
<td>$x^3 + x^2 + 9x - 12$</td>
<td>$3^3$ $367^2$ 12.40</td>
</tr>
<tr>
<td>5</td>
<td>3425</td>
<td>57</td>
<td>5</td>
<td>$x^3 - x^2 - 8x - 3$</td>
<td>$3^3$ $5^4$ 13.51</td>
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<td>31524</td>
<td>381</td>
<td>2</td>
<td>$x^3 + x^2 + 7x - 1$</td>
<td>$2^4$ $3^1$ $17^2$ 13.82</td>
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<td>37856</td>
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<td>1</td>
<td>$x^3 - x^2 - 10x - 5$</td>
<td>$3^5$ $5^2$ 16.31</td>
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<td>3144</td>
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<td>$x^3 - x^2 - 16x - 8$</td>
<td>$2^3$ $3^2$ $13^2$ 17.59</td>
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<td>36222</td>
<td>813</td>
<td>2</td>
<td>$x^3 + x^2 - 9x - 3$</td>
<td>$2^4$ $3^1$ $7^2$ 17.79</td>
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<td>35480</td>
<td>805</td>
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<td>$x^3 - x^2 + 15x - 15$</td>
<td>$2^3$ $5^2$ $59^2$ 18.30</td>
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<td>35756</td>
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<td>$x^3 - x^2 + 17x - 3$</td>
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44 & 8156 & 8336 & 1 & \textit{x}^3 + \textit{x}^2 - 27\textit{x} - 51 & 2^6 \cdot 3^3 \cdot 23 \cdot 31 & 25.76 & \\
45 & 9676 & 9676 & 1 & \textit{x}^3 - 27\textit{x} - 12 & 2^6 \cdot 41 \cdot 59^2 & 26.84 & \\
46 & 11884 & 11884 & 1 & \textit{x}^3 - 19\textit{x} - 24 & 2^6 \cdot 2971^2 & 28.75 & \\
47 & 14336 & 14336 & 1 & \textit{x}^3 - \textit{x}^2 - 27\textit{x} - 21 & 2^6 \cdot 3^2 \cdot 131^2 & 30.59 & \\
48 & 14956 & 14956 & 1 & \textit{x}^3 + \textit{x}^2 - 32\textit{x} - 6 & 2^6 \cdot 3739^2 & 31.06 & \\
49 & 16944 & 16944 & 1 & \textit{x}^3 + \textit{x}^2 - 36\textit{x} - 54 & 2^6 \cdot 3^3 \cdot 7^2 \cdot 191 & 31.77 & \\
50 & 18604 & 18604 & 1 & \textit{x}^3 - \textit{x}^2 - 36\textit{x} - 18 & 2^6 \cdot 4651^2 & 33.38 & \\
51 & 21224 & 21224 & 1 & \textit{x}^3 + \textit{x}^2 - 44\textit{x} - 84 & 2^6 \cdot 3^2 \cdot 177^2 & 34.93 & \\
52 & 4469 & 4469 & 1 & \textit{x}^3 + \textit{x}^2 - 52\textit{x} - 4^2 & 2^6 \cdot 7^2 \cdot 67^2 & 10.74 & \\
53 & 7756 & 7756 & 1 & \textit{x}^3 - 6\textit{x} - 2 & 2^6 \cdot 7^2 \cdot 31^2 & 12.59 & \\
54 & 72177 & 72177 & 1 & \textit{x}^3 + \textit{x}^2 - 8\textit{x} - 5 & 2^6 \cdot 11^3 \cdot 71 & 17.92 & \\
55 & 72233 & 72233 & 1 & \textit{x}^3 + \textit{x}^2 - 8\textit{x} - 1 & 2^6 \cdot 11^3 \cdot 79^2 & 18.67 & \\
56 & 4193 & 4193 & 1 & \textit{x}^3 - \textit{x}^2 - 12\textit{x} - 7 & 2^6 \cdot 11^3 \cdot 79^2 & 22.30 & \\
57 & 4641 & 4641 & 1 & \textit{x}^3 + \textit{x}^2 - 14\textit{x} - 21 & 3^2 \cdot 7^3 \cdot 13 \cdot 17^2 & 23.67 & \\
58 & 5089 & 5089 & 1 & \textit{x}^3 - \textit{x}^2 - 14\textit{x} - 11 & 2^6 \cdot 7^2 \cdot 23 & 23.78 & \\
59 & 5639 & 5639 & 1 & \textit{x}^3 - 17\textit{x} - 23 & 2^6 \cdot 7^3 \cdot 13^2 & 24.21 & \\
60 & 6153 & 6153 & 1 & \textit{x}^3 - \textit{x}^2 - 12\textit{x} - 3 & 2^6 \cdot 7^3 \cdot 151 & 25.34 & \\
61 & 6601 & 6601 & 1 & \textit{x}^3 - 13\textit{x} - 9 & 2^6 \cdot 23 \cdot 41^2 & 25.94 & \\
62 & 7665 & 7665 & 1 & \textit{x}^3 - 27\textit{x} - 19 & 3^2 \cdot 5^3 \cdot 7 \cdot 73 & 27.27 & \\
63 & 8113 & 8113 & 1 & \textit{x}^3 - 13\textit{x} - 5 & 2^6 \cdot 7^2 \cdot 61 & 27.79 & \\
64 & 8505 & 8505 & 1 & \textit{x}^3 - 27\textit{x} - 51 & 2^6 \cdot 5^3 \cdot 7 \cdot 23 & 28.23 & \\
65 & 9905 & 9905 & 1 & \textit{x}^3 - 17\textit{x} - 19 & 2^6 \cdot 7^3 \cdot 283 & 29.70 & \\
66 & 10353 & 10353 & 1 & \textit{x}^3 + \textit{x}^2 - 22\textit{x} - 43 & 3^2 \cdot 7^3 \cdot 17^2 \cdot 29 & 30.14 & \\
67 & 14385 & 14385 & 1 & \textit{x}^3 - \textit{x}^2 - 36\textit{x} - 69 & 3^2 \cdot 7^3 \cdot 17^2 \cdot 37^2 & 33.63 & \\
68 & 16737 & 16737 & 1 & \textit{x}^3 - \textit{x}^2 - 36\textit{x} - 27 & 3^2 \cdot 7^3 \cdot 17^2 \cdot 37^2 & 35.37 & \\
69 & 20545 & 20545 & 1 & \textit{x}^3 + \textit{x}^2 - 40\textit{x} - 67 & 5^2 \cdot 7 \cdot 587 & 37.88 & \\
70 & 25249 & 25249 & 1 & \textit{x}^3 - 19\textit{x} - 9 & 2^6 \cdot 3691^2 & 40.57 & \\
71 & 30849 & 30849 & 1 & \textit{x}^3 + \textit{x}^2 - 42\textit{x} - 45 & 3^2 \cdot 7^2 \cdot 11^3 \cdot 113 & 43.37 & \\
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\caption{TABLE 3 (continued)}
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TABLE 3 (continued).

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TABLE 5. $K$ does not contain any imaginary quadratic field and $(q) = Q_1 Q_2$ in $F$

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**TABLE 5 (continued).**

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